# Towards the QCD Phase Diagram

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in collaboration with

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ECT$^*$ workshop on

'Confinement: connecting the light- and heavy-quark domains'

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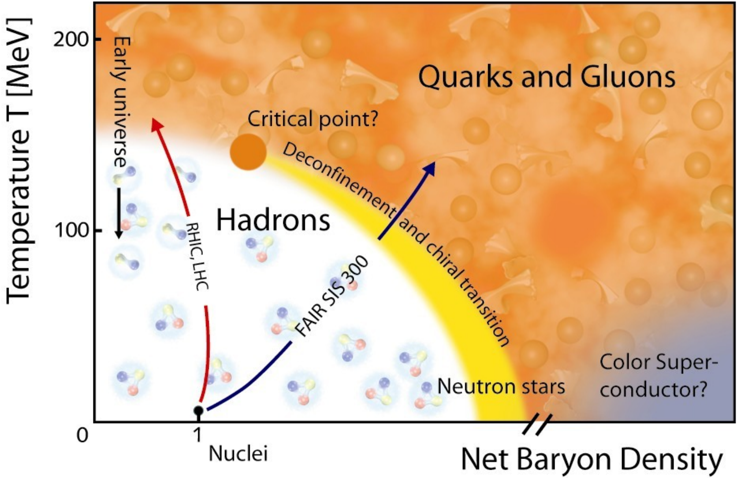
Trento, Italy

# Outline

1. Motivation: The QCD phase diagram
2. Two flavor quark-meson model
   - Mean field analysis
   - RG analysis
3. The critical region of the (tri)critical point
4. Synthesis: Polyakov loop and a chiral effective quark-meson model

# The conjectured QCD Phase Diagram




from GSI, Darmstadt

QCD: two important limits:

1. chiral limit: $m_q \to 0$

   $\mathcal{L}_{\text{QCD}}$ inv. under global chiral rotations

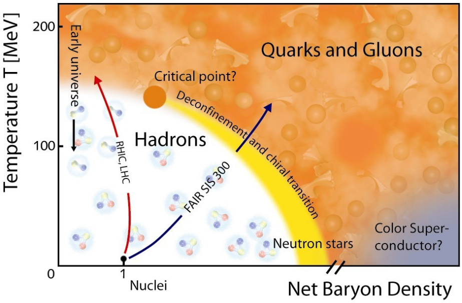
   chiral symmetry spontaneously broken

   $SU_L(N_f) \times SU_R(N_f) \to SU_{L+R}(N_f)$

   $\Rightarrow N_f^2 - 1$ massless Goldstone bosons

   order parameter: chiral condensate $\langle \bar{q}q \rangle$

   $$\langle \bar{q}q \rangle \begin{cases} > 0 \Leftrightarrow \text{symmetry broken}, & T < T_c \\ = 0 \Leftrightarrow \text{symmetric phase}, & T > T_c \end{cases}$$




experiment FAIR @ GSI, Darmstadt

chiral transition: spontaneous restoration of global $SU_L(N_f) \times SU_R(N_f)$ at high $T$

# The conjectured QCD Phase Diagram

QCD: two important limits:

1. chiral limit: $m_q \to 0$
2. heavy-quark limit: $m_q \to \infty$

   $\Rightarrow$ pure gauge theory w/ static sources

   gauge action invariant

   under global $Z(N_c)$-symmetry

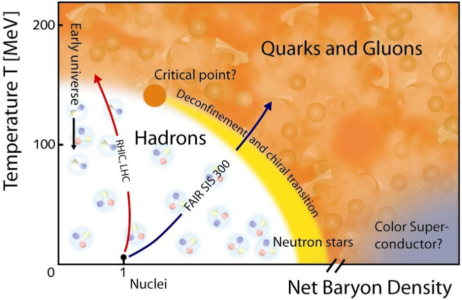
   order parameter for confinement:

   Polyakov loop variable: $\phi = \langle \mathrm{tr}_c \mathcal{P} \rangle / N_c$

$$\phi \begin{cases} = 0 \Leftrightarrow \text{confined phase,} & T < T_c \\ > 0 \Leftrightarrow \text{deconfinend phase,} & T > T_c \end{cases}$$

   free energy of static quark antiquark pair:

$$\exp\left(-\frac{F_{\bar{q}q}(r,T)}{T}\right) = \langle \mathrm{tr}_c \mathcal{P}(x) \mathrm{tr}_c \mathcal{P}^\dagger(y) \rangle / N_c^2$$




experiment FAIR @ GSI, Darmstadt

# The conjectured QCD Phase Diagram

QCD: two important limits:

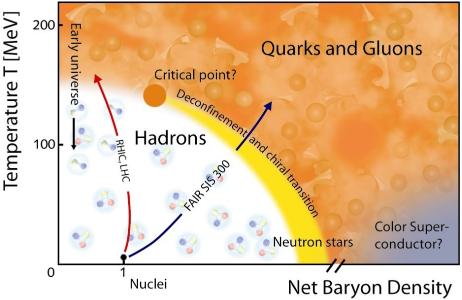
1. chiral limit: $m_q \to 0$
2. heavy-quark limit: $m_q \to \infty$

physical QCD

$\Rightarrow$ breaks both symmetries explicitly

$\Rightarrow$ but displays confinement

and light Goldstone bosons (pions)




experiment FAIR @ GSI, Darmstadt

**symmetries incorporated in effective models:**

1. Quark-meson model (or other models e.g.NJL)
2. Polyakov–quark-meson model (or PNJL models)

# The chiral $N_f = 2$ quark-meson model

- Lagrangian:

$$\mathcal{L}_{\text{qm}} = \bar{q}[i\gamma_\mu \partial^\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \vec{\pi})^2 + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

1. Mean field analysis (ignoring fluctuations)
2. Renormalization Group analysis (taking fluctuations into account)

- Lagrangian:

$$\mathcal{L}_{\text{qm}} = \bar{q}[i\gamma_\mu\partial^\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

1. Mean field analysis (ignoring fluctuations)

$\sigma \to \langle\sigma\rangle \equiv \phi, \pi \to \langle\pi\rangle = 0$, integrate quark/antiquarks

**Grand canonical potential**

$$\Omega(T,\mu) \quad = \quad -\frac{T\ln\mathcal{Z}}{V} = \frac{\lambda}{4}(\langle\sigma\rangle^2 - v^2)^2 - c\langle\sigma\rangle + \Omega_{\bar{q}q}(T,\mu)$$

with

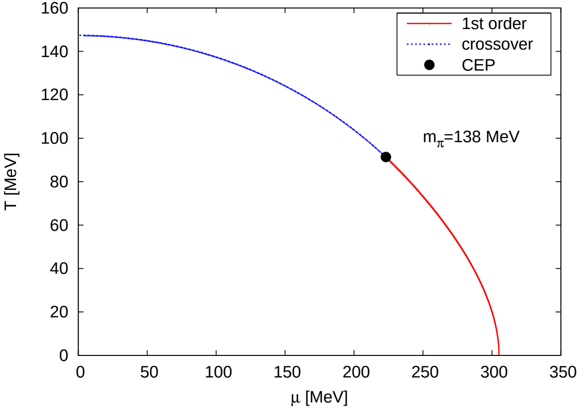
$$\Omega_{\bar{q}q}(T,\mu) \;=\; -2N_cN_fT\int\frac{d^3k}{(2\pi)^3}\left\{\ln(1+e^{-(E_q-\mu)/T}) + \ln(1+e^{-(E_q+\mu)/T})\right\}$$

[Scavenius et al. '01]

- Lagrangian:

$$\mathcal{L}_{\text{qm}} = \bar{q}[i\gamma_\mu\partial^\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

1. Mean field analysis



- Lagrangian:

$$\mathcal{L}_{\text{qm}} = \bar{q}[i\gamma_\mu\partial^\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

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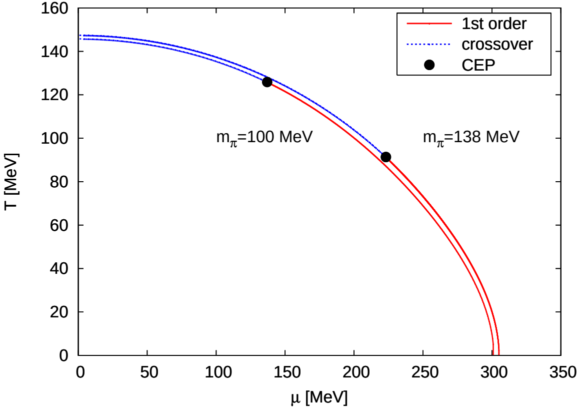

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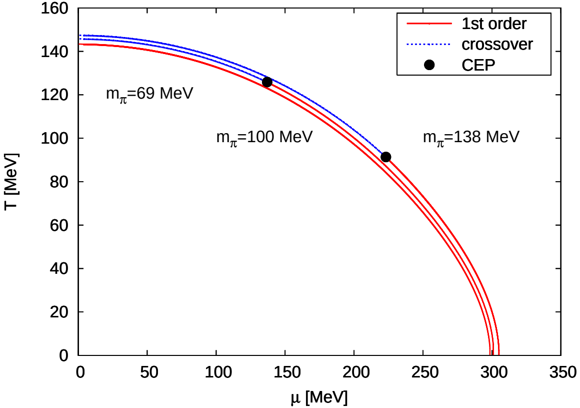
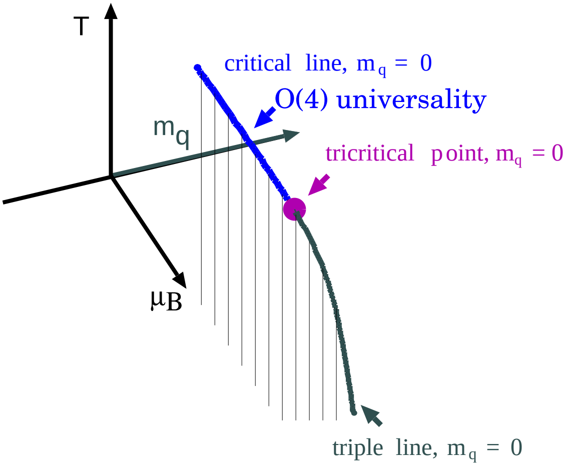
1. Mean field analysis



Chiral limit: $(m_q = 0)$ $SU(2) \times SU(2) \sim O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$

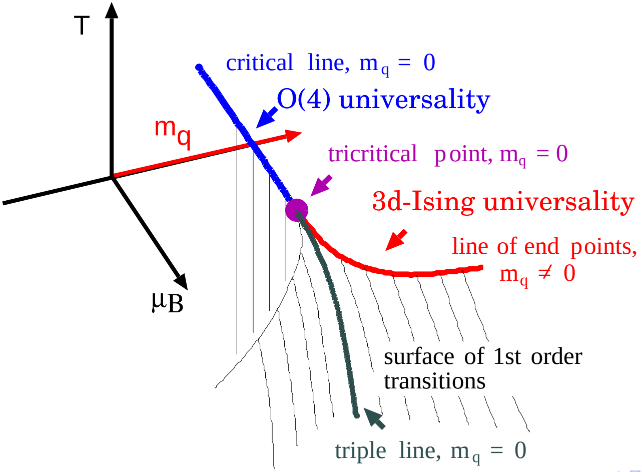



General properties

- chiral limit
  tricritical point
  (Gaussian fixed point)

Chiral limit: $(m_q = 0)$ $SU(2) \times SU(2) \sim O(4)$-symmetry $\longrightarrow$ 4 modes critical $\sigma, \vec{\pi}$

$m_q \neq 0$ : no symmetry remains $\longrightarrow$ only one critical mode $\sigma$ (Ising) ($\vec{\pi}$ massive)




General properties

- chiral limit
  tricritical point
  (Gaussian fixed point)
- finite $m_q$
  critical endpoints
  (3D-Ising class)

- Lagrangian:

$$\mathcal{L}_{\text{qm}} = \bar{q}[i\gamma_\mu\partial^\mu - g(\sigma + i\vec{\tau}\vec{\pi}\gamma_5)]q + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\vec{\pi})^2 + \frac{\lambda}{4}(\sigma^2 + \vec{\pi}^2 - v^2)^2 - c\sigma$$

2. Renormalization Group analysis (here: Proper-Time RG)

$\Rightarrow$ ansatz for $\Gamma_k$ at UV: $\Gamma_{k=\Lambda} = \int d^4x \mathcal{L}_{\text{qm}}$

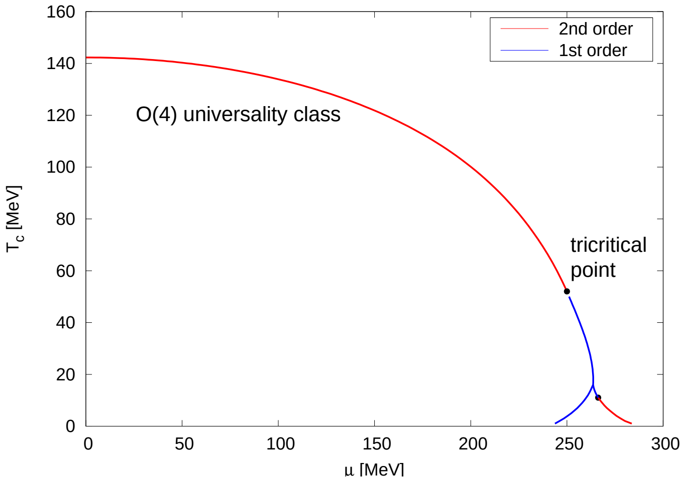
**flow for grand canonical potential** BJS, J.Wambach

$$\partial_t\Omega_k(T,\mu;\phi) = \frac{k^4}{12\pi^2}\left[\frac{3}{E_\pi}\coth\left(\frac{E_\pi}{2T}\right) + \frac{1}{E_\sigma}\coth\left(\frac{E_\sigma}{2T}\right) - \frac{2N_cN_f}{E_q}\left\{\tanh\left(\frac{E_q-\mu}{2T}\right) + \tanh\left(\frac{E_q+\mu}{2T}\right)\right\}\right]$$

$$E_\pi^2 = 1 + 2\Omega'_k/k^2\ , \qquad E_\sigma^2 = 1 + 2\Omega'_k/k^2 + 4\phi^2\Omega''_k/k^2\ , \qquad E_q^2 = 1 + g^2\phi^2/k^2$$

$$\phi \sim \langle\bar{q}q\rangle\ , \qquad \Omega'_k = \partial\Omega_k/\partial\phi \qquad \text{etc}$$

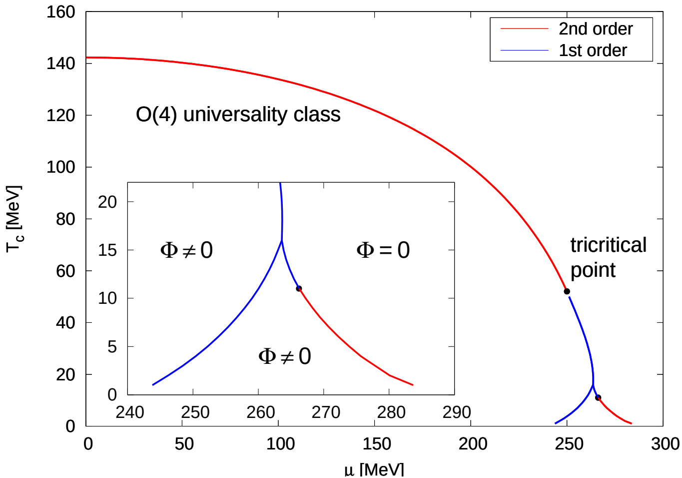
$O(4) \sim SU(2) \times SU(2)$ chiral limit



# Chiral Phase Diagram $N_f = 2$ & $m_q \sim 280$ MeV

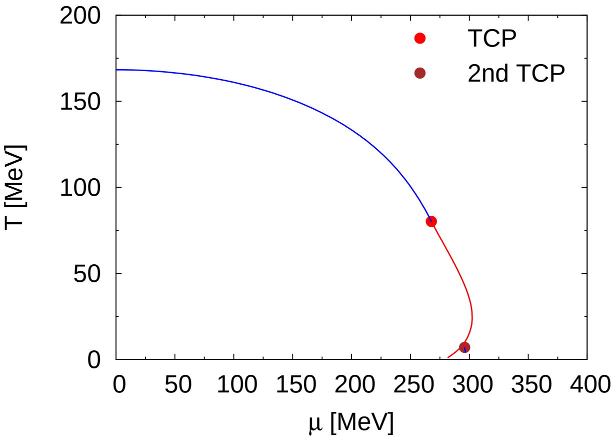
$O(4) \sim SU(2) \times SU(2)$ chiral limit no spinodal lines!

artefact of truncation?



TCP: $T_c \sim 80.2$ MeV

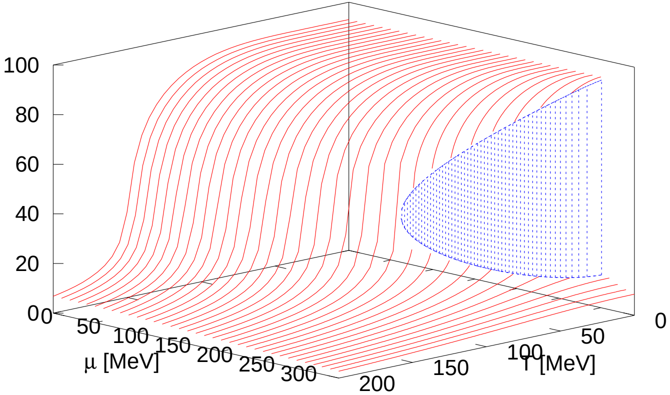
2. 'TCP': $T_c \sim 8$ MeV




[BJS,J. Wambach '05]

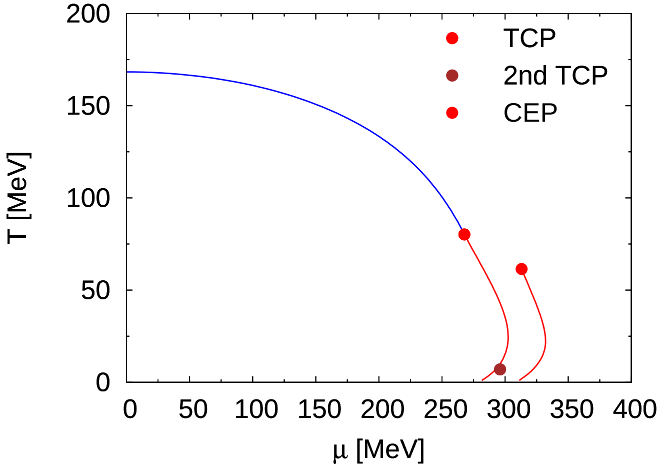
- 2nd-order transition → crossover
- shift of "$T_c$"
- shift tricritical point → critical

order parameter: $\phi(T, \mu)$



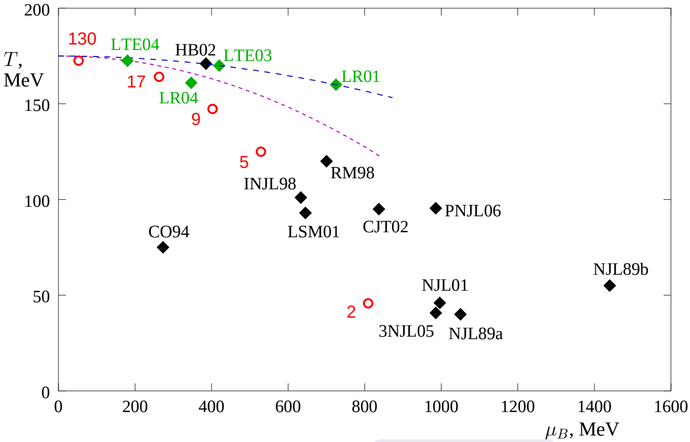
# Chiral Phase Diagram $N_f = 2$ & $m_q \sim 370$ MeV

TCP: $T_c \sim 80.2$ MeV    2. 'TCP': $T_c \sim 8$ MeV    CEP: $T_c \sim 61.5$ MeV



model studies vs. lattice simulations

Black points: models    Lines & green points: lattice    Red points: Freezeout points for HIC




Stephanov '05 & '07

lattice methods:

- reweighting
- imaginary $\mu_B$
- Taylor expansion around $\mu_B = 0$

large $m_q$ sensitivity?

if $m_s \to 0$
$\Rightarrow$ 1st-order

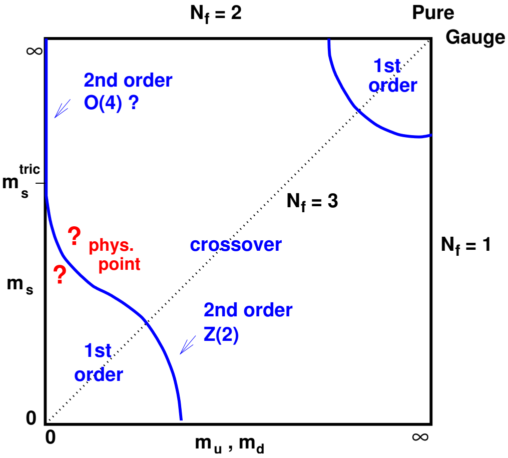
Columbia plot:

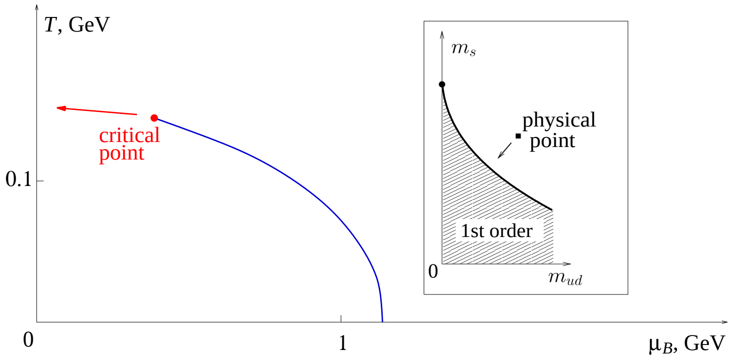
[Brown et al. '90]

$T_\chi^{N_f=2} \sim 175$ MeV

$T_d \sim 270$ MeV

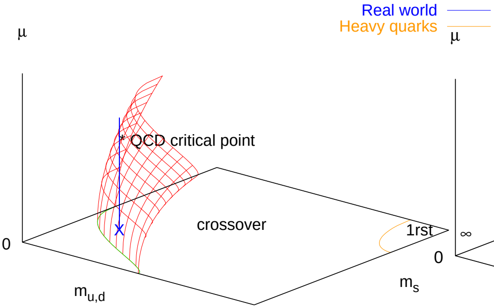
$T_\chi^{N_f=3} \sim 155$ MeV



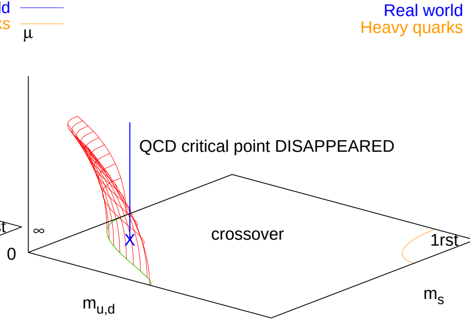



- expected direction of motion of CEP as $m_q$ are decreased
- only valid for standard scenario i.e. positive $m_c(\mu)$ slope
- universality arguments predict for $N_f = 3$ & chiral limit: first-order

# Mass Sensitivity (lattice, $N_f = 3, \mu_B \neq 0$)




standard scenario: $m_c(\mu)$ increasing




non-standard scenario: $m_c(\mu)$ decreasing

[de Forcrand, Philipsen '05]

# Outline

# Susceptibilities

estimate critical region around CEP$\rightarrow$ use susceptibilities:

- quark number density: $n_q(T,\mu) = -\dfrac{\partial\Omega(T,\mu)}{\partial\mu}$
- quark number susceptibility: $\chi_q(T,\mu) = -\dfrac{\partial^2\Omega(T,\mu)}{(\partial\mu)^2}$

  - near critical point: $\chi_q \sim |g-g_c|^{-\epsilon} \quad ; \quad g = T,\mu$
  - isothermal compressibility $\kappa_T \equiv -\dfrac{1}{V}\left(\dfrac{\partial V}{\partial P}\right)\Big|_{T,N} = \dfrac{\chi_q}{n_q^2}$

  $\rightarrow$ if $\chi_q$ is large then system is easy to compress

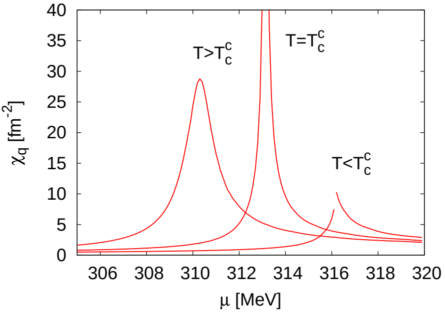
  $\Rightarrow$ interaction attractive (or weakly repulsive)

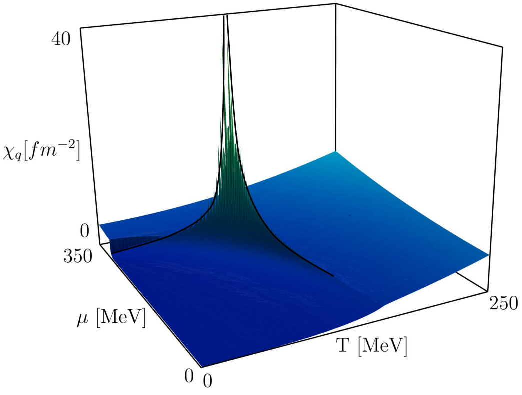
- scalar susceptibility: $\chi_\sigma(T,\mu) = 1/m_\sigma^2(T,\mu)$

  - zero-momentum projection of scalar propagator
  - encodes all fluctuations of order parameter

# Quark-number susceptibility $\chi_q(T, \mu)$

- diverges only at CEP in equilibrium
- finite everywhere else
- height decreases for decreasing $\mu$ towards $T$-axis
- For $T$ below CEP: discontinuous $\rightarrow$ 1st order

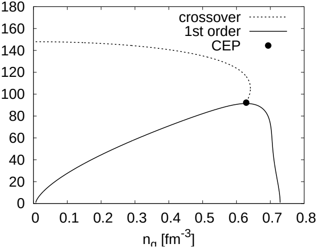


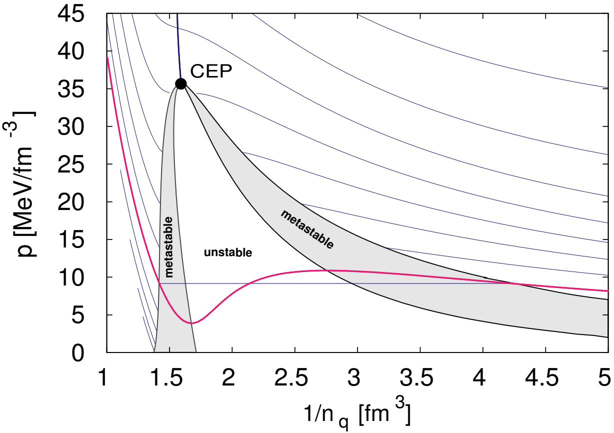

define ratio: $R_q := \chi_q / \chi_q^{\text{free}}$

$\chi_q$: massless free quark gas

e.g. $R_q = 3$ or $R_q = 5$




pressure vs. V

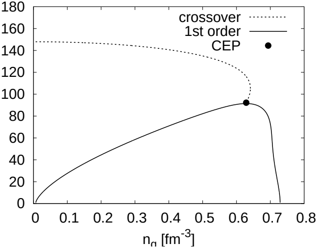



isothermal spinodale lines:

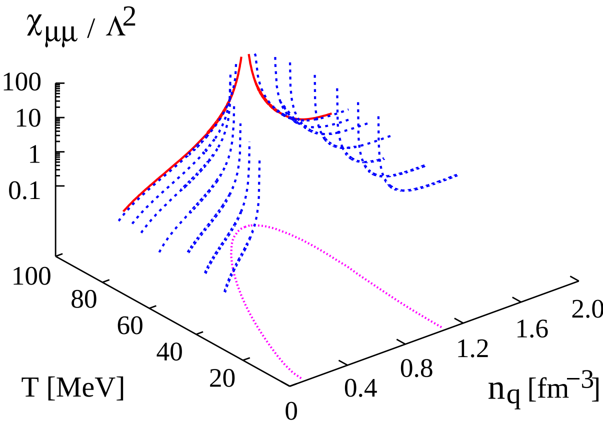
$$\left(\frac{\partial P}{\partial V}\right)_T = 0$$

isentropic spinodale lines:

$$\left(\frac{\partial P}{\partial V}\right)_S = 0$$




net quark number susceptibility $\chi_{\mu\mu}$
diverges at spinodal lines




isothermal spinodale lines:

$$\left(\frac{\partial P}{\partial V}\right)_T = 0$$

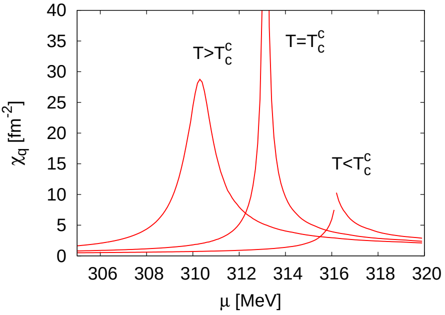
isentropic spinodale lines:

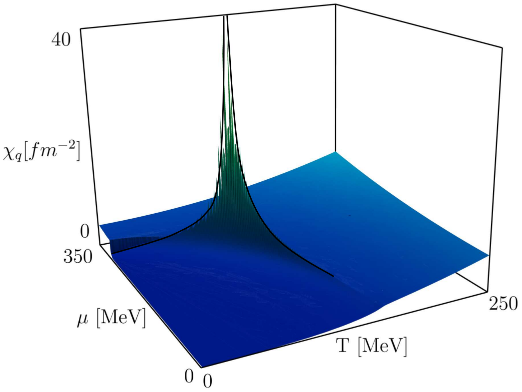
$$\left(\frac{\partial P}{\partial V}\right)_S = 0$$

[Sasaki, Friman, Redlich '07]

# Quark-number susceptibility $\chi_q(T,\mu)$

- diverges only at CEP (in equilibrium)
- finite everywhere else
- height decreases for decreasing $\mu$ towards $T$-axis
- For $T$ below CEP: discontinuous $\rightarrow$ 1st order




for estimation of the critical region at CEP

define ratio: $R_q := \chi_q / \chi_q^{\text{free}}$

$\chi_q$: massless free quark gas

e.g. $R_q = 3$ or $R_q = 5$

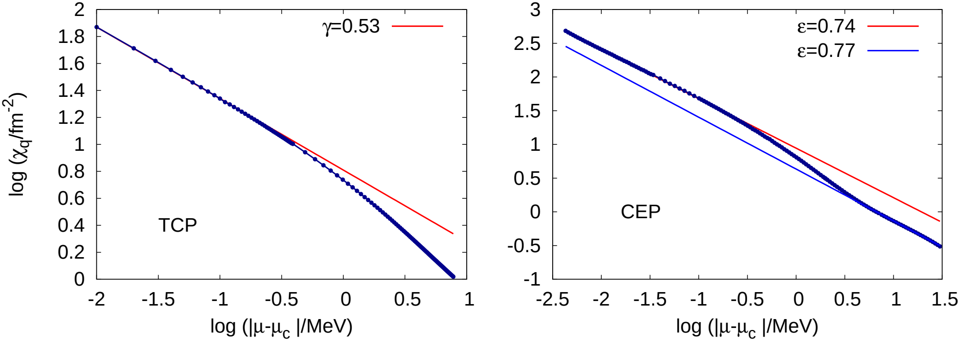
# Critical Exponents

$$\chi_q \sim |\mu - \mu_c|^{-\gamma}$$

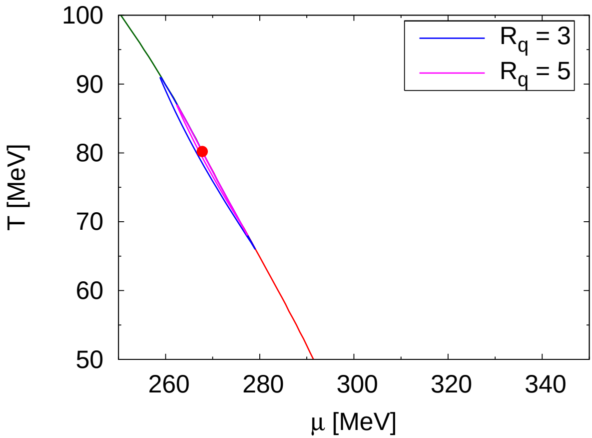
TCP: $\gamma = 0.5$ (Gaussian)

Mean field: $\epsilon = 2/3$

CEP: $\epsilon = 0.78$ (3D Ising)



- size of crit. region shrinks as $m_q \to 0$ [cf. Hatta, Ikeda '03]



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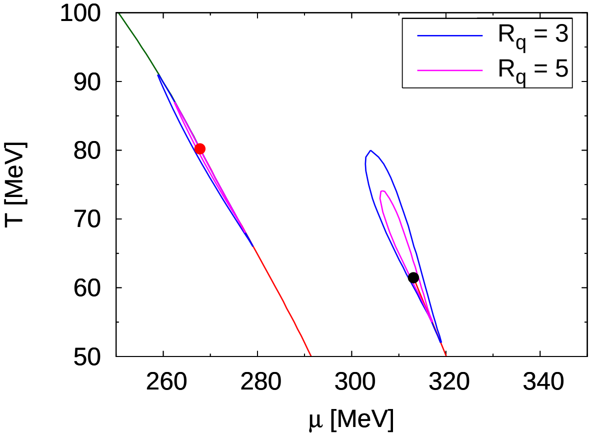
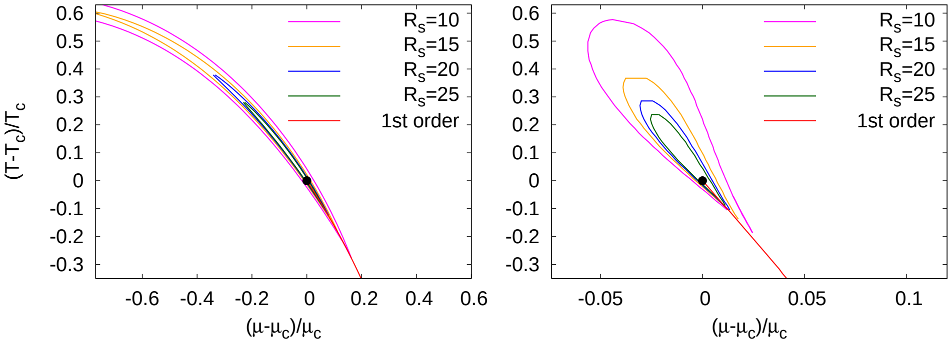


$$R_s = m_\sigma^2(0,0)/m_\sigma^2(T,\mu)$$

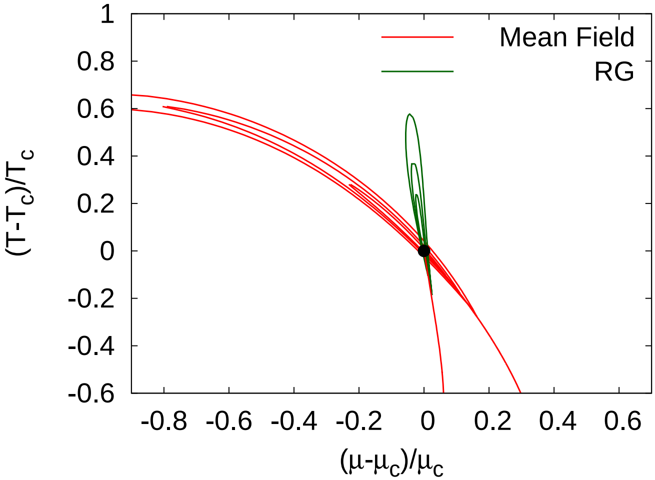
Mean field | RG




⇒ critical region with RG more compressed

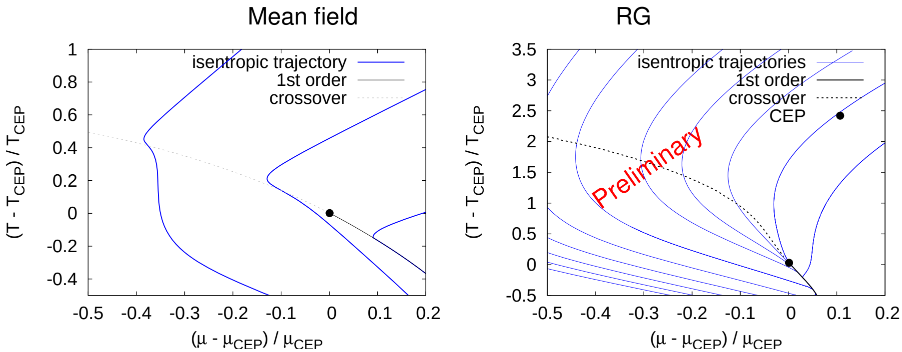
cf. [Brouzakis, Tetradis '04]

[BJS, J. Wambach '06]



same conclusion for isentropic trajectories:

$$s/n = \text{const.}$$




⇒ critical region with RG more compressed

# Outline

... is a combination of

1. the chiral quark-meson model (here $N_f = 2$)

2. with Polyakov loop dynamics: including now color dof's. $\rightarrow$ confinement

It's a new model (cf. PNJL)

- Polyakov loop variable — Polyakov 1978

$$\phi(\vec{x}) = \frac{1}{N_c} \mathrm{Tr}_c \left\langle \mathcal{P} \exp\left( i \int_0^\beta d\tau A_0(\vec{x}, \tau) \right) \right\rangle_\beta$$

- Polyakov loop potential: — Pisarski 2000

$$\frac{\mathcal{U}(\phi,\bar\phi)}{T^4} = -\frac{b_2(T,T_0)}{2}\phi\bar\phi - \frac{b_3}{6}\left(\phi^3 + \bar\phi^3\right) + \frac{b_4}{16}\left(\phi\bar\phi\right)^2$$

parameters $b_i \quad i = 2, ..., 4$ fixed to pure YM

Ratti, Weise et al. 2004
Dumitru, Pisarski 2004
Friman, Redlich, Sasaki 2006

$\Rightarrow$ first-order transition at $T_0 = 270$ MeV

remark: $b_2$ depends on gauge coupling $b_2(T,T_0) = b_2(\alpha(T,T_0))$

# Polyakov quark-meson model

- quarks couple to a background gauge field:

$$\partial^\mu \to D^\mu = \partial^\mu - iA^\mu \quad \text{with} \quad A^\mu = \delta_0^\mu \, g_s A_a^0 \frac{\lambda_a}{2}$$

$$\mathcal{L}_{\text{pol}} = -\bar{q}\gamma_0 A_0 q - \mathcal{U}(\phi, \bar{\phi})$$

$\Rightarrow$ Polyakov-quark-meson model: $\mathcal{L}_{\text{PQM}} = \mathcal{L}_{\text{qm}} + \mathcal{L}_{\text{pol}}$

- in presence of dynamical quarks: $\alpha = \alpha(N_f)$

  leads to modification in $b_2$:

Pawlowski, BJS, Wambach, in preparation

$$\Rightarrow \; T_0 = T_0(N_f):$$

| $N_f$ | $0$ | $1$ | $2$ | $2+1$ | $3$ |
|---|---|---|---|---|---|
| $T_0$ [MeV] | 270 | 240 | 208 | 187 | 178 |

# Non-vanishing chemical potential

$$\Rightarrow \phi^* \neq \bar\phi \qquad \phi\bar\phi \to \tfrac{1}{2}(\phi\bar\phi + (\phi\bar\phi)^*) \quad \text{or} \quad \tfrac{1}{2}(|\bar\phi|^2 + |\phi|^2)$$

- assume same potential at $\mu \neq 0$:

$$\frac{\mathcal{U}(\phi,\bar\phi)}{T^4} = -\frac{b_2(T, T_0(N_f))}{4}\left(|\phi|^2 + |\bar\phi|^2\right) - \frac{b_3}{6}(\phi^3 + \bar\phi^3) + \frac{b_4}{16}\left(|\phi|^2 + |\bar\phi|^2\right)^2$$

$$\Rightarrow \phi^* \neq \bar\phi \qquad \phi\bar\phi \to \tfrac{1}{2}(\phi\bar\phi + (\phi\bar\phi)^*) \quad \text{or} \quad \tfrac{1}{2}(|\bar\phi|^2 + |\phi|^2)$$

- assume same potential at $\mu \neq 0$:

$$\frac{\mathcal{U}(\phi,\bar\phi)}{T^4} = -\frac{b_2(T,T_0(N_f;\mu))}{4}\left(|\phi|^2+|\bar\phi|^2\right) - \frac{b_3}{6}(\phi^3+\bar\phi^3) + \frac{b_4}{16}\left(|\phi|^2+|\bar\phi|^2\right)^2$$

- analogously to $N_f$: implement additional $\mu$-dependence
  $\Rightarrow$ this potential has a minimum and no saddle point

determinant of Hessian:

positive $\to$ minimum

contrast : PNJL works

negative susceptibilities

$\to$ logarithmic potential

- grand canonical potential:

$$\Omega(T,\mu) = \mathcal{U}(\phi,\bar\phi) + V_{\text{renorm}}(\langle\sigma\rangle,\vec{0}) + \Omega_{\bar q q}(T,\mu)$$

with fermi contribution:

$$\Omega_{\bar q q} = -2N_f T \int \frac{d^3p}{(2\pi)^3} \left\{ \ln\left[1+3(\phi+\bar\phi e^{-(E_p-\mu)/T})e^{-(E_p-\mu)/T} + e^{-3(E_p-\mu)/T}\right] + \ln\left[1+3(\bar\phi+\phi e^{-(E_p+\mu)/T})e^{-(E_p+\mu)/T} + e^{-3(E_p+\mu)/T}\right]\right\}$$

$$E_p = \sqrt{p^2+m_q^2}$$

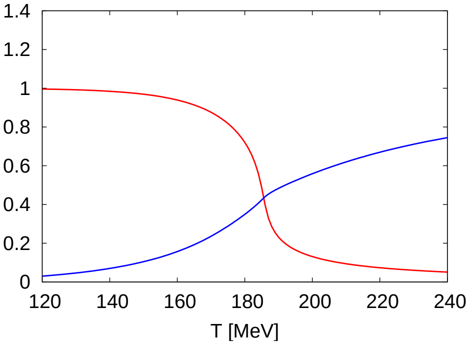
confined phase $\phi = 0$: 1q– & 2q–states suppressed, only 3 quark states
deconfined phase $\phi \neq 0$: 1q–, 2q– & 3q–states

- three EoM: $$\frac{\partial\Omega}{\partial\sigma} = 0\,, \qquad \frac{\partial\Omega}{\partial\phi} = 0\,, \qquad \frac{\partial\Omega}{\partial\bar\phi} = 0\,.$$

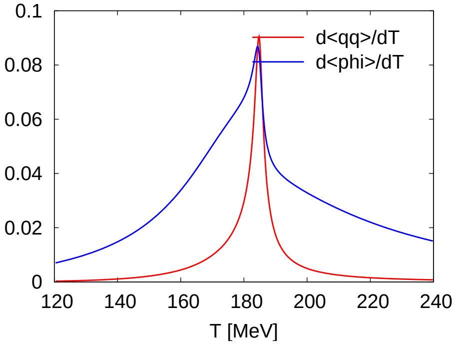
# Finite temperature and $\mu = 0$

Numerical results:

order parameters



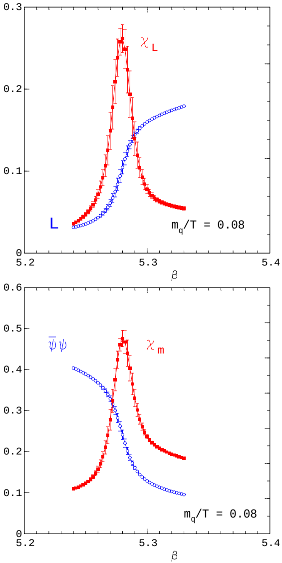

$T$-derivatives of order parameters



- here: $N_f = 2$ and $\mu = 0$
- blue line: order parameter
- red line: susceptibilities
- both transitions at same $\beta$

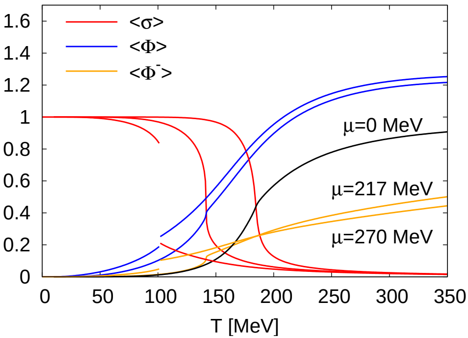
**Lattice:**

$\Rightarrow T_{\text{deconf}} \sim T_{\text{chiral}}$

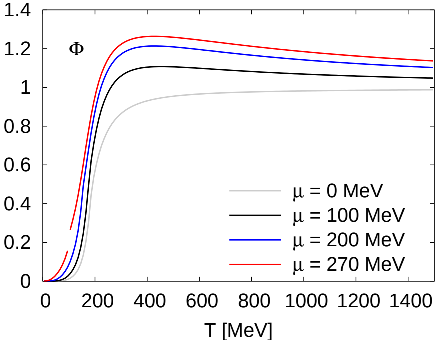


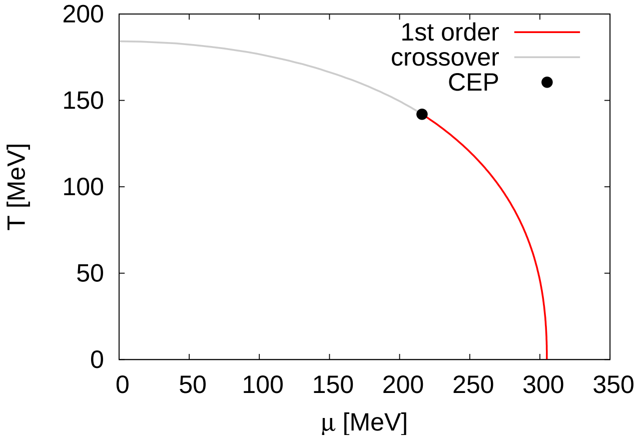
without $\mu$-modifications in Polyakov potential:

order parameters

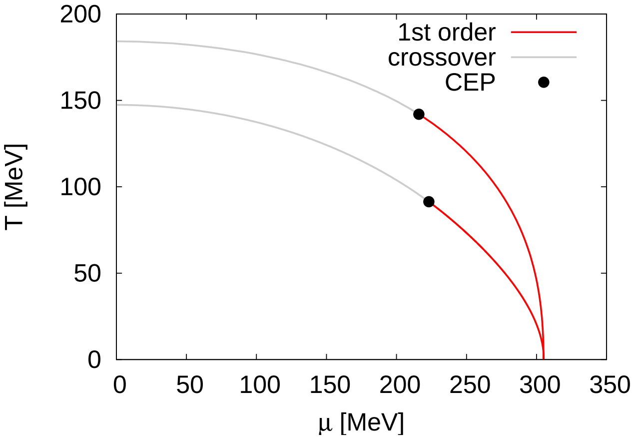



$\Phi$ for large-$T$

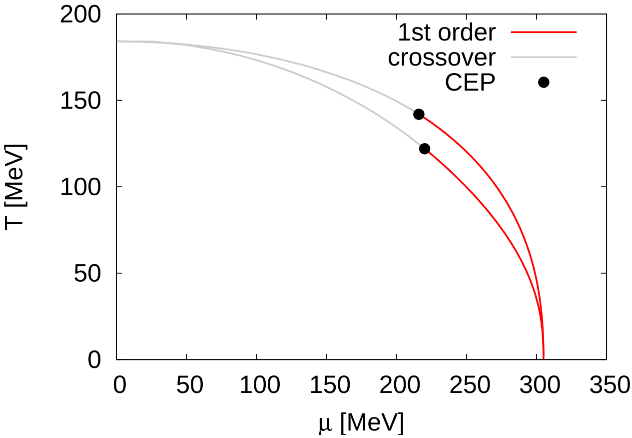





- for PQM model




- for PQM model
- for QM model
  (lower lines)

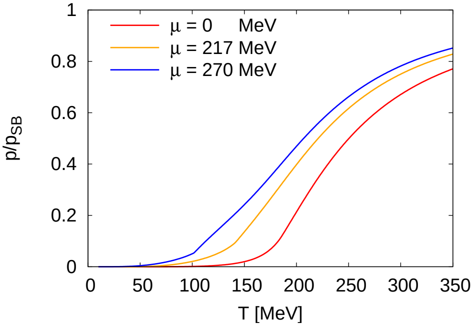


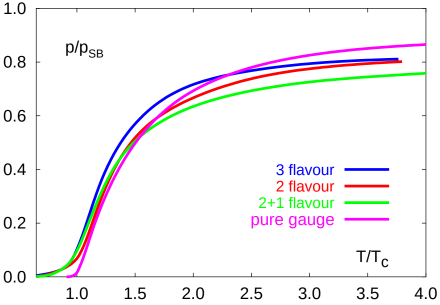

- for PQM model
- for PQM model with $\mu$-modification in Polyakov loop potential (lower lines)

- perturbative pressure of QCD with $N_f$ massless quarks

$$\frac{p}{T^4} = (N_c^2 - 1)\frac{\pi^2}{45} + N_f\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu}{T}\right)^4\right] .$$

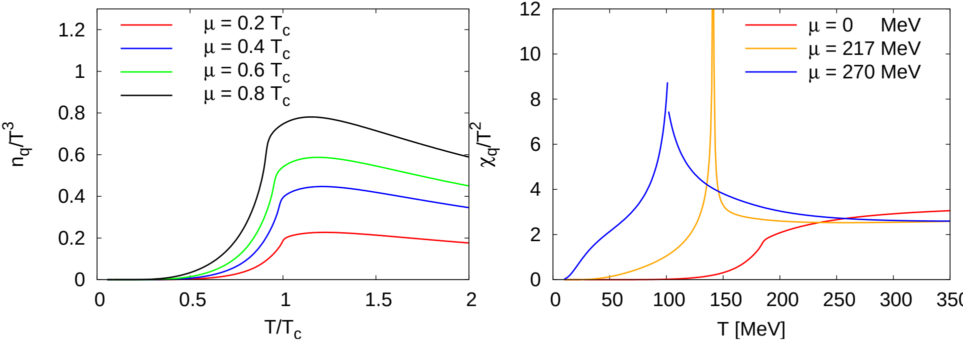
- $N_f = 2$:



# Quark number density $n_q$ & chiral susceptibility $\chi_q$

$$n_q = -\frac{\partial \Omega(T,\mu)}{\partial \mu} \qquad\qquad \chi_q = \frac{\partial n_q}{\partial \mu}$$



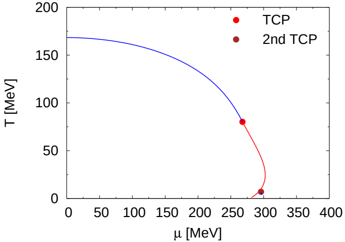

$T_c(\mu = 0) \sim 184$ MeV

# Summary & Outlook

# Summary

- ▷ MF versus proper-time RG $\rightarrow$ transparent physics, analytical threshold fkts.. . .
- ▷ in phase diagram two TCP's (chiral limit) and CEP found
- ▷ size of critical region via susceptibilities $\chi_q(T,\mu)$ and $\chi_\sigma(T,\mu)$

  $\longrightarrow$ "compressed" with fluctuations



# Summary

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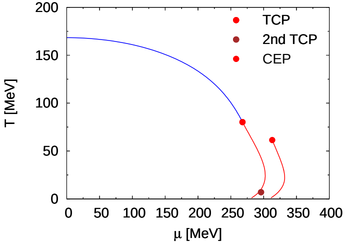


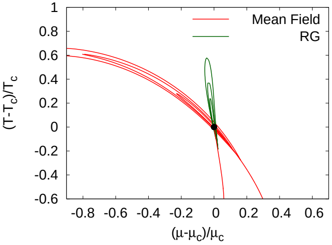


- ▷ critical exponents consistent w/ $3d$ Ising universality class at CEP

# Summary

- ▷ parameter in Polyakov loop potential: $T_0$

  pure gauge: $T_0 \sim 270$ MeV $\rightarrow T_0 \sim 210$ MeV for $N_f = 2$

  $$\Rightarrow T_0(N_f, \mu)$$

- ▷ quark-meson model is renormalizable $\rightarrow$ no UV cutoff parameter

  (cf. PNJL model)

- ▷ mean-field approximation encouraging

## Outlook

- ▷ include quark-meson dynamics with RG
- ▷ include glue dynamics with RG $\rightarrow$ full QCD
  (step by step)