The QCD Potential

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Summary

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4.1 Lattice results 2006

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Reviews of Modern Physics 77 (2005) 1423 arXiv:hep-ph/0410047.

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Heavy Quarkonium Physics
CERN Yellow Report, CERN-2005-005, (CERN, Geneva, 2005) 487 p.
arXiv:hep-ph/0412158.

1. Definition

The potential is what to write in a Schrödinger equation

$$E\,\phi = \left(\frac{p^2}{m} + V(r)\right)\,\phi$$

In a full theory, V must come from a double expansion:

- a non-relativistic expansion $\sim p/m,\ rm$: $V \to V^{(0)} + V^{(1)}/m + \cdots$;
- an expansion in $E\,r$, since V is a function of r (or p at h.o. in the non-relativistic expansion): $V \to V +$ energy-dependent effects (e.g. Lamb-shift).

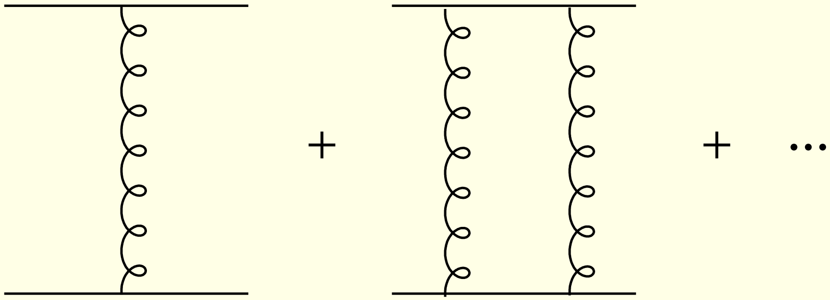
A potential V describes the interaction of a non-relativistic bound state, $p \sim mv$, $E \sim mv^2$, $v \ll 1$, once the expansions in mv/m and mv^2/mv have been exploited.

Non-relativistic scales in QCD

Near threshold:

$$E \approx 2m + \frac{p^2}{m} + \ldots \qquad \text{with} \quad v = \frac{p}{m} \ll 1$$

- The perturbative expansion breaks down when $\alpha_s \sim v$:



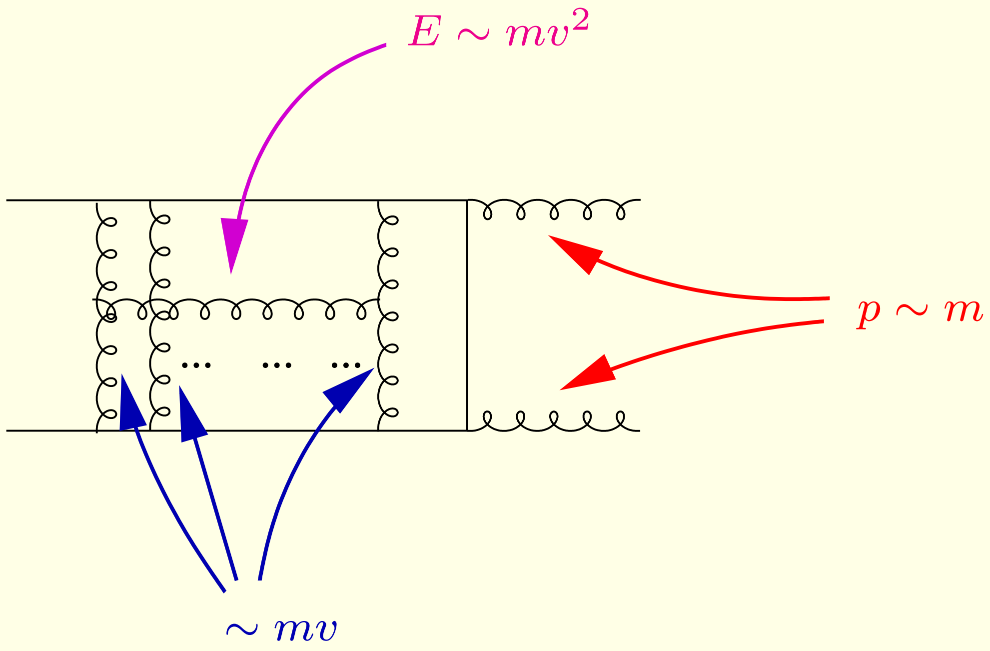
$$\approx \frac{1}{E - \left(\frac{p^2}{m} + V\right)}$$

$$\alpha_s \left(1 + \frac{\alpha_s}{v} + \ldots\right)$$

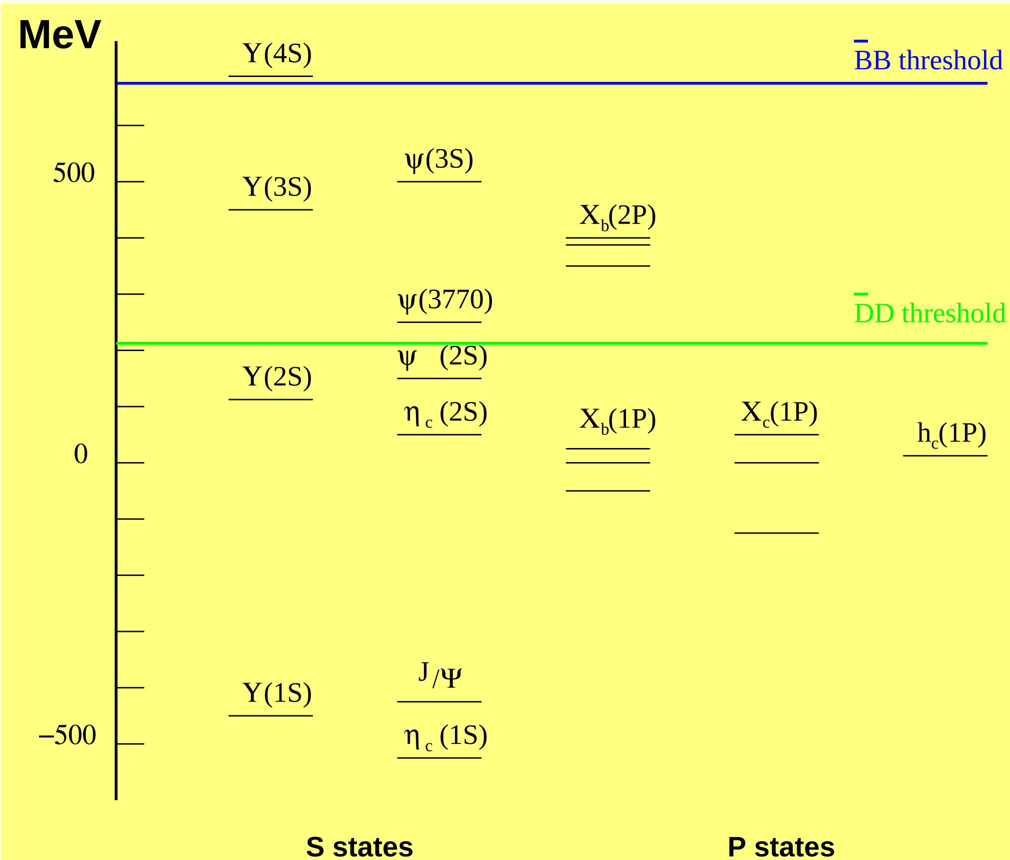
- The system is non-relativistic : $p \sim mv$ and $E = \dfrac{p^2}{m} + V \sim mv^2$.

Non-relativistic scales in QCD

Scales get entangled.



Quarkonium scales



Normalized with respect to $\chi_b(1P)$ and $\chi_c(1P)$

The mass scale is perturbative:

$m_b \simeq 5$ GeV, $m_c \simeq 1.5$ GeV

The system is non-relativistic:

$\Delta_n E \sim mv^2$, $\Delta_{\rm fs} E \sim mv^4$

$v_b^2 \simeq 0.1$, $v_c^2 \simeq 0.3$

Non-relativistic bound states are characterized

by at least three energy scales

$m \gg mv \gg mv^2 \quad v \ll 1$

Effective Field Theories

Whenever a system H, described by $\mathcal{L}_{\rm QCD}$, is characterized by 2 scales $\Lambda \gg \lambda$, observables may be calculated by expanding one scale with respect to the other.
An effective field theory makes the expansion in λ/Λ explicit at the Lagrangian level.

The EFT Lagrangian, $\mathcal{L}_{\rm EFT}$, suitable to describe H at scales lower than Λ is defined by
(1) a cut off $\Lambda \gg \mu \gg \lambda$;
(2) by some degrees of freedom that exist at scales lower than μ

$\Rightarrow$ $\mathcal{L}_{\rm EFT}$ is made of all operators O_n that may be built from the effective degrees of freedom and are consistent with the symmetries of $\mathcal{L}$.

Effective Field Theories

$$\mathcal{L}_{\rm EFT} = \sum_n c_n(\Lambda/\mu) \frac{O_n(\mu, \lambda)}{\Lambda^n}$$

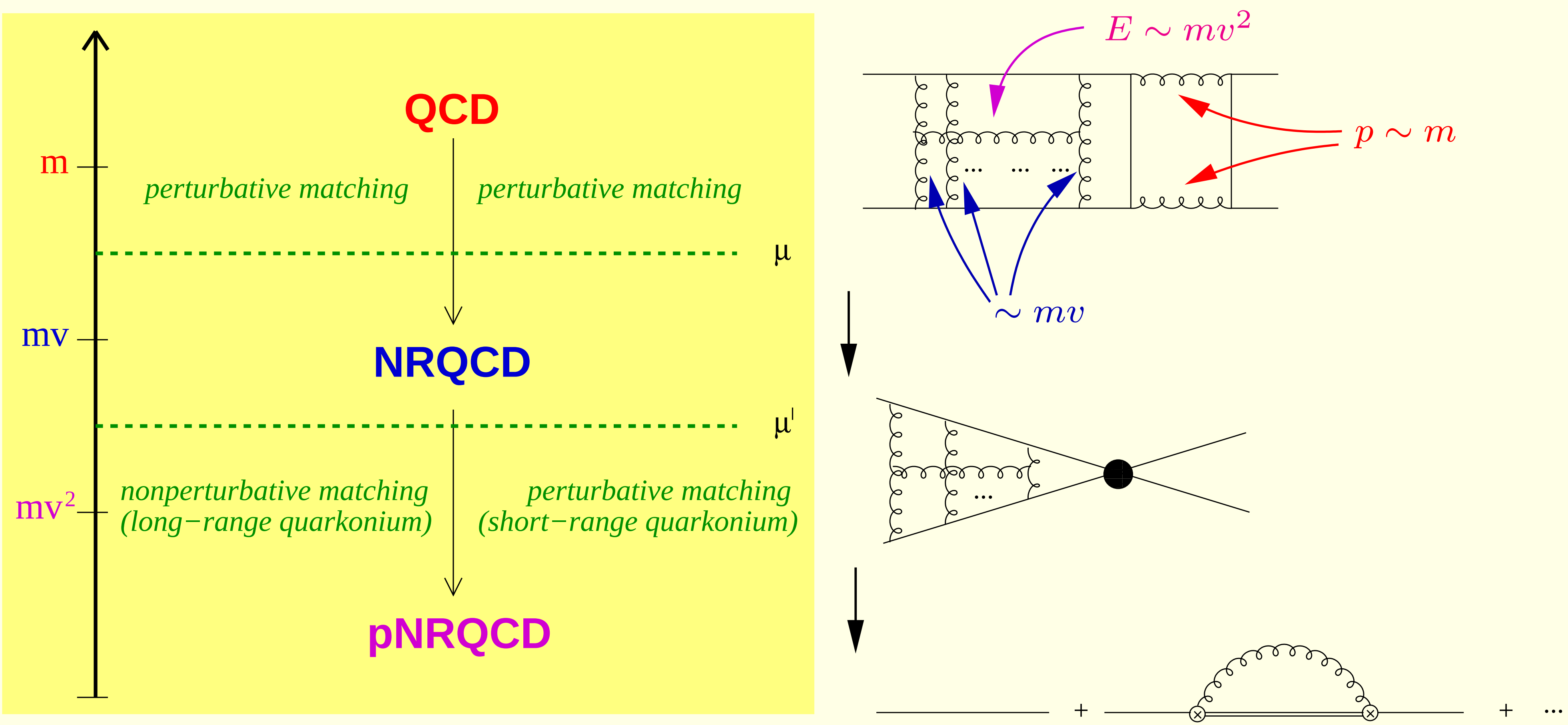
- Since $\langle O_n \rangle \sim \lambda^n$ the EFT is organized as an expansion in λ/Λ.
- The EFT is renormalizable order by order in λ/Λ.
- The matching coefficients $c_n(\Lambda/\mu)$ encode the non-analytic behaviour in Λ. They are calculated by imposing that $\mathcal{L}_{\rm EFT}$ and $\mathcal{L}$ describe the same physics at any finite order in the expansion: matching procedure.
- If $\Lambda \gg \Lambda_{\rm QCD}$ then $c_n(\Lambda/\mu)$ may be calculated in perturbation theory.

Effective Field Theories

Examples:

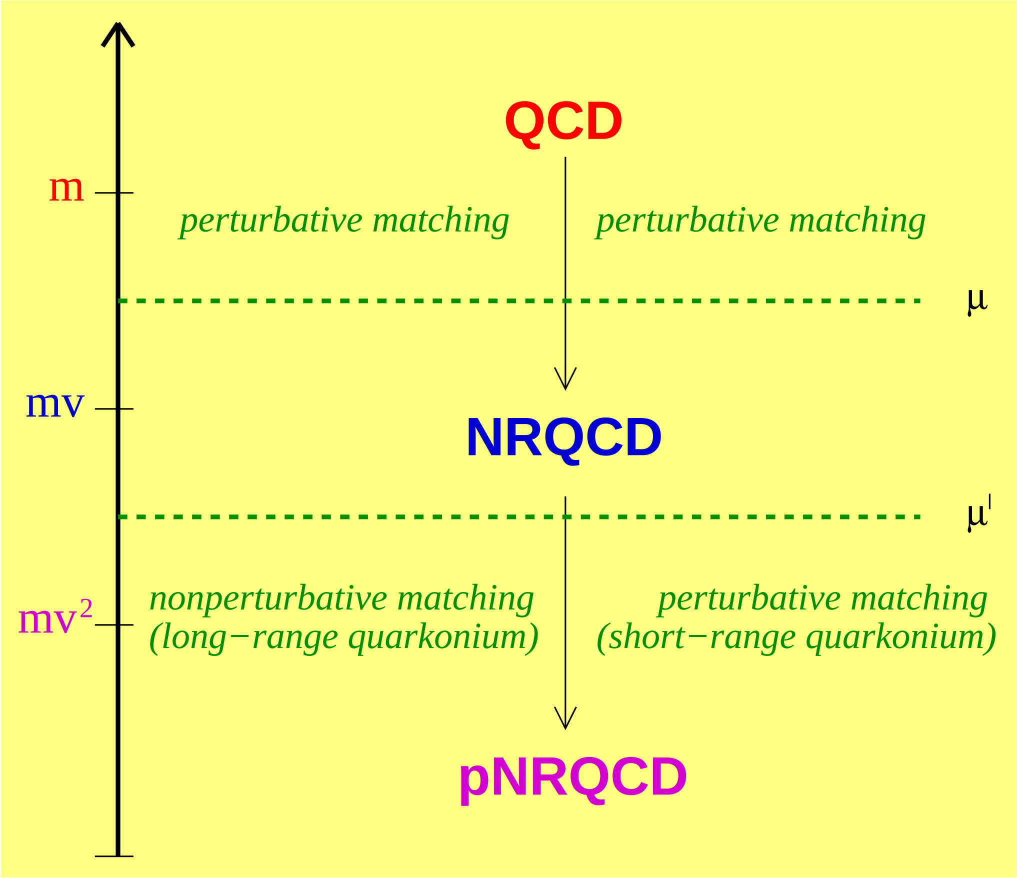
- Fermi theory of weak interactions: $\frac{\lambda}{\Lambda} = \frac{p}{M_W}$
- Chiral effective theory: $\frac{\lambda}{\Lambda} = \frac{p}{\Lambda_\chi}$
- Heavy quark effective theory (HQET): $\frac{\lambda}{\Lambda} = \frac{\Lambda_{\mathrm{QCD}}}{m}$
- Soft collinear effective theory (SCET):
  (involves different expansions over different momentum regions)
- $\ldots$

EFTs for systems made of two heavy quarks



- They exploit the expansion in v/ factorization of low and high energy contributions.
- They are renormalizable order by order in v.
- In perturbation theory (PT), RG techniques provide resummation of large logs.

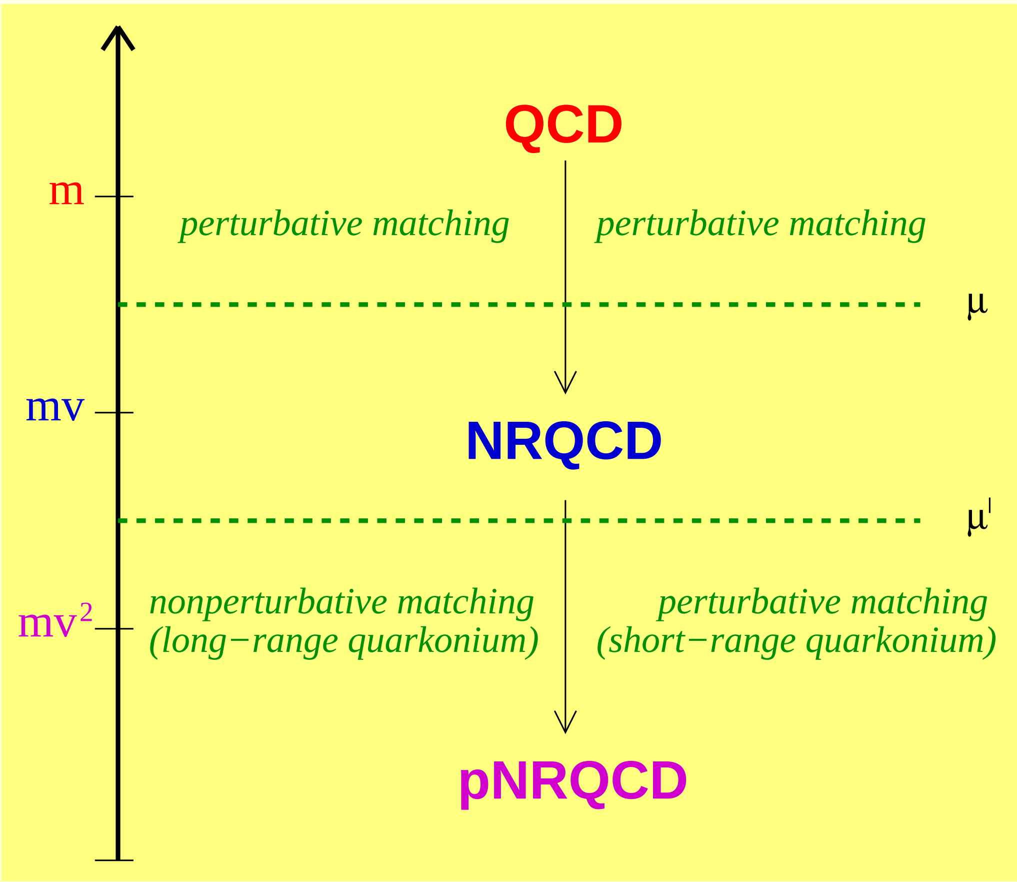
EFTs for systems made of two heavy quarks

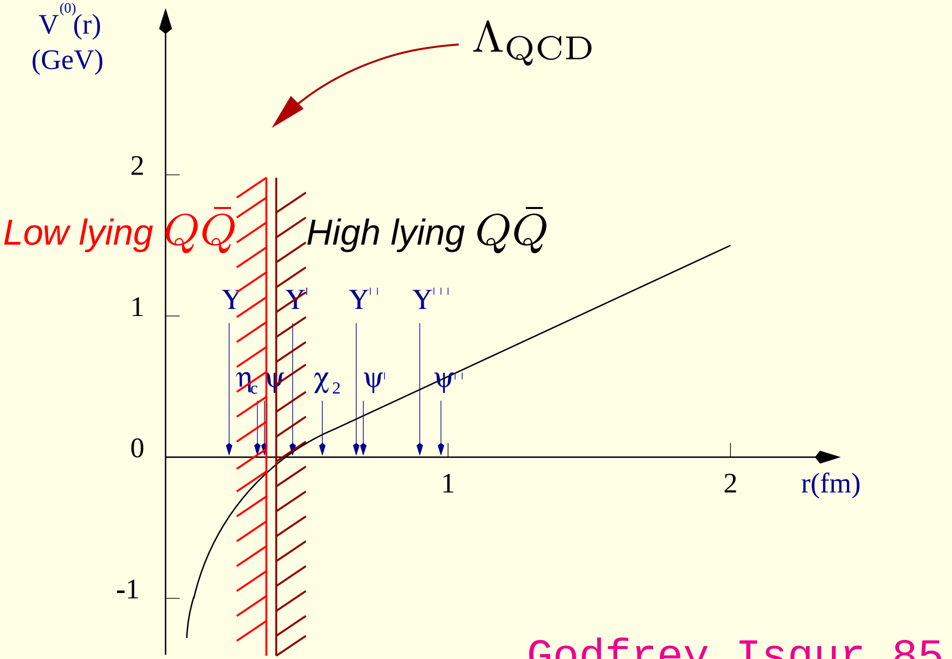


Caswell Lepage 86, Lepage Thacker 88
Bodwin Braaten Lepage 95, ...

Pineda Soto 97, Brambilla et al 99
Kniehl et al 99, ...
Luke Manohar 97, Luke Savage 98
Labelle 98, Grinstein Rothstein 98
Griesshammer 98, Luke et al 00
Hoang 02, ... $\longrightarrow$ vNRQCD

EFTs for systems made of two heavy quarks





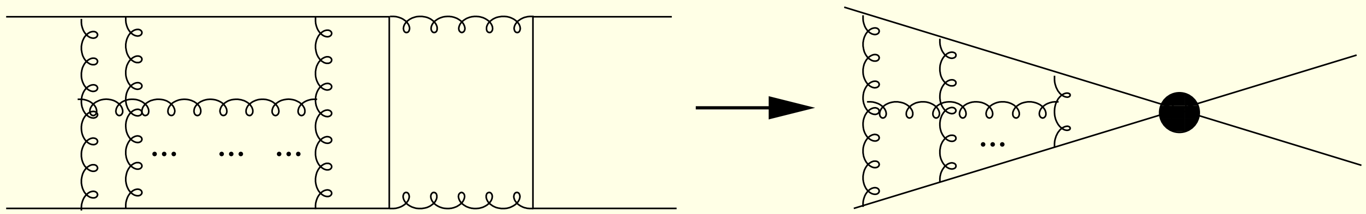
Godfrey Isgur 85

A potential picture arises at the level of pNRQCD:

- the potential is perturbative if $mv \gg \Lambda_{\rm QCD}$
- the potential is non-perturbative if $mv \sim \Lambda_{\rm QCD}$

NRQCD

NRQCD is obtained by integrating out modes associated with the scale m



$\times c(\mu/m)$

QCD NRQCD

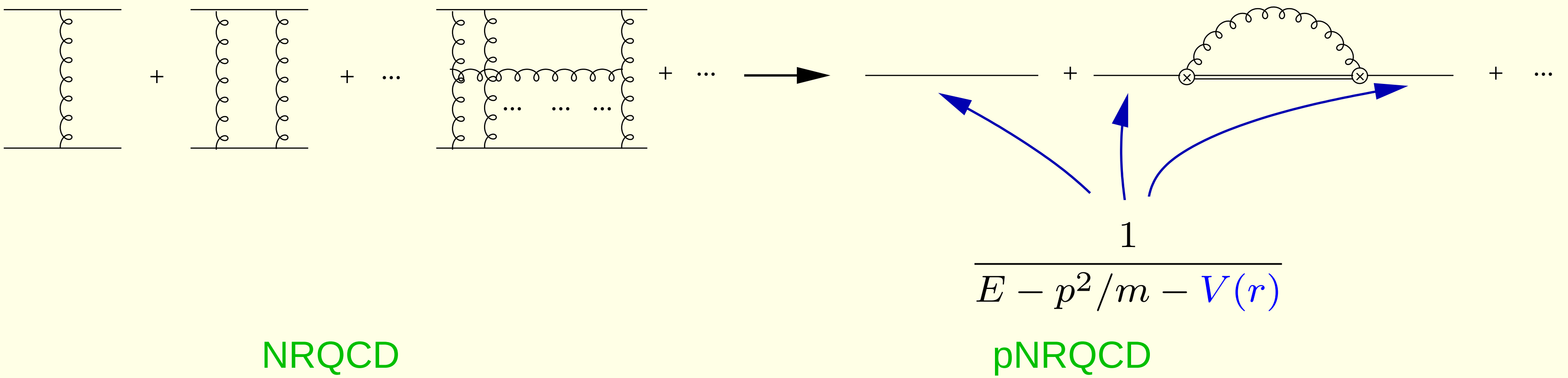
- The matching is perturbative.
- The Lagrangian is organized as an expansion in $1/m$ and $\alpha_s(m)$:

$$\mathcal{L}_{\rm NRQCD} = \sum_n c(\alpha_s(m/\mu)) \times O_n(\mu, \lambda)/m^n$$

Suitable to describe annihilation and production of quarkonium.

pNRQCD

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim mv$



- The Lagrangian is organized as an expansion in $1/m$, r, and $\alpha_s(m)$:

$$\mathcal{L}_{\rm pNRQCD} = \sum_k \sum_n \frac{1}{m^k} \times c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda)\, r^n$$

pNRQCD for $mv \gg \Lambda_{\rm QCD}$

- Degrees of freedom (ultrasoft) at scales lower than mv:

  Q-$\bar{Q}$ states, with energy $\sim \Lambda_{\rm QCD}, mv^2$ and momentum $\lesssim mv$
  $\Rightarrow$ (i) singlet $\rm S$ (ii) octet $\rm O$

  Gluons with energy and momentum $\sim \Lambda_{\rm QCD}, mv^2$

- Power counting:

  $p \sim \frac{1}{r} \sim mv$;
  all gauge fields are multipole expanded: $A(R, r, t) = A(R, t) + \mathbf{r} \cdot \boldsymbol{\nabla} A(R, t) + \dots$
  and scale like $(\Lambda_{\rm QCD}$ or $mv^2)^{\rm dimension}$.

  Non-analytic behaviour in $r \to$ matching coefficients V

pNRQCD for $mv \gg \Lambda_{\rm QCD}$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + {\rm Tr} \left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} + {\rm O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) {\rm O} \right\}$$

LO in r

$$\theta(T)\, e^{-iTH_s}$$

$$\theta(T)\, e^{-iTH_o} \left(e^{-i\int dt\, A^{\rm adj}} \right)$$

pNRQCD for $mv \gg \Lambda_{\rm QCD}$

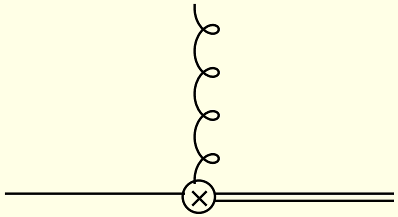
$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} + \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} + \mathrm{O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - V_o \right) \mathrm{O} \right\}$$

LO in r

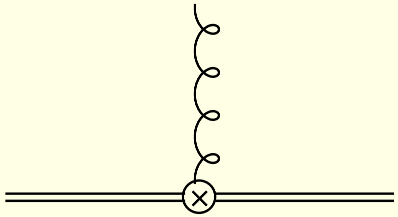
$$+ V_A \mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{S} + \mathrm{S}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} \right\} + \frac{V_B}{2} \mathrm{Tr}\left\{ \mathrm{O}^\dagger \mathbf{r} \cdot g\mathbf{E}\, \mathrm{O} + \mathrm{O}^\dagger \mathrm{O} \mathbf{r} \cdot g\mathbf{E} \right\} + \cdots$$

NLO in r

pNRQCD for $mv \gg \Lambda_{\rm QCD}$



$O^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{S}$



$O^\dagger \{\mathbf{r}\cdot g\mathbf{E},\, \mathrm{O}\}$

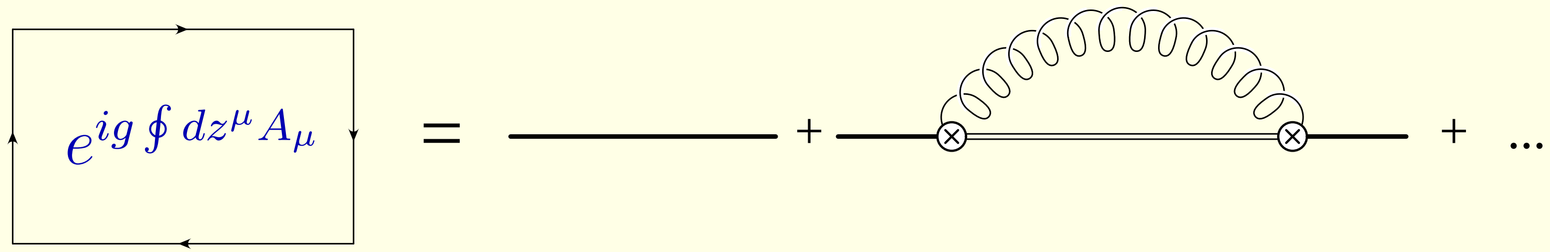
$$+V_A \mathrm{Tr}\left\{\mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{S} + \mathrm{S}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{O}\right\}$$

$$+\frac{V_B}{2}\mathrm{Tr}\left\{\mathrm{O}^\dagger \mathbf{r}\cdot g\mathbf{E}\,\mathrm{O} + \mathrm{O}^\dagger \mathrm{O}\mathbf{r}\cdot g\mathbf{E}\right\}$$

NLO in r

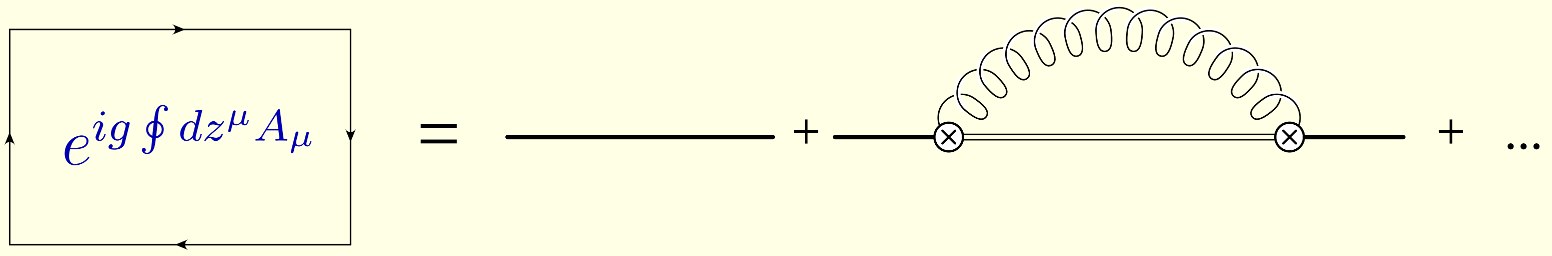
2. Calculation

The Static Potential



NRQCD pNRQCD

The Static Potential

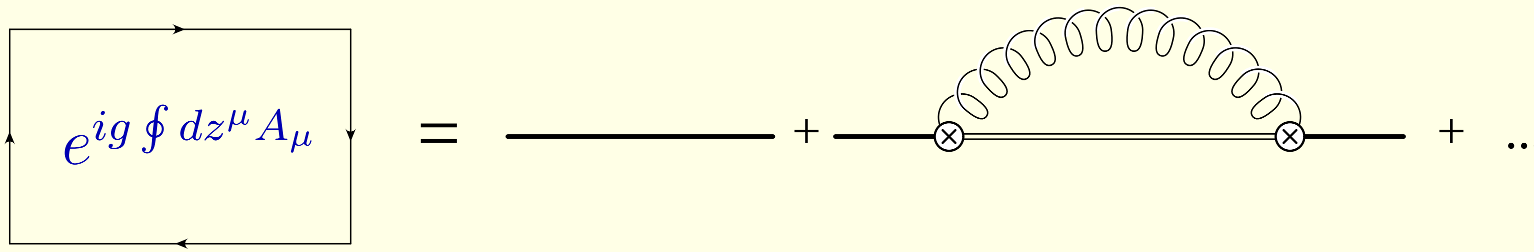


NRQCD pNRQCD

$$\lim_{T\to\infty} \frac{i}{T} \ln \left\langle \Box \right\rangle = V_s(r,\mu) - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt\, e^{-it(V_o - V_s)} \langle \mathrm{Tr}(\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0))\rangle(\mu) + \ldots$$

ultrasoft contribution

The Static Potential



NRQCD pNRQCD

$$\lim_{T\to\infty}\frac{i}{T}\ln\langle \Box \rangle = V_s(r,\mu) - i\frac{g^2}{N_c}V_A^2\int_0^\infty dt\, e^{-it(V_o-V_s)}\,\langle \mathrm{Tr}(\mathbf{r}\cdot\mathbf{E}(t)\,\mathbf{r}\cdot\mathbf{E}(0))\rangle(\mu)+\dots$$

ultrasoft contribution

* The μ dependence cancels between the two terms in the right-hand side:

$V_s \sim \ln r\mu, \ln^2 r\mu,$ ...

ultrasoft contribution $\sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \dots \ln r\mu, \ln^2 r\mu, \dots$

Static Wilson loop

$$\lim_{T\to\infty} \frac{i}{T} \ln \left\langle \Box \right\rangle = -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 + \ldots \right]$$

is known at two loops:

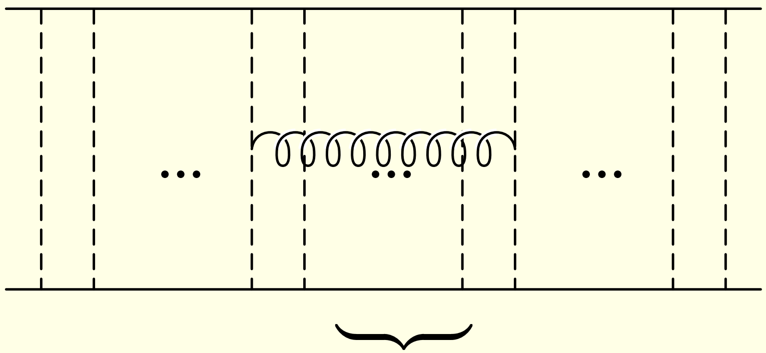
$$a_1 = \frac{31}{9} C_A - \frac{10}{9} n_f + 2\gamma_E \beta_0,$$

Billoire 80

$$\begin{aligned} a_2 \quad = \quad & \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3}\zeta(3) \right) C_A^2 - \left(\frac{899}{81} + \frac{28}{3}\zeta(3) \right) C_A n_f \\ & - \left(\frac{55}{6} - 8\zeta(3) \right) C_F n_f + \frac{100}{81} n_f^2 + 4\gamma_E \beta_0 a_1 + \left(\frac{\pi^2}{3} - 4\gamma_E^2 \right) \beta_0^2 + 2\gamma_E \beta_1 \end{aligned}$$

Schröder 99, Peter 97

Appelquist–Dine–Muzinich diagrams



$$= -\frac{C_F C_A^3}{12}\,\frac{\alpha_s}{r}\,\frac{\alpha_s^3}{\pi}\,\ln\left[\frac{C_A\alpha_s}{2r}\times r\right]$$

$$\sim \exp(-i(V_o - V_s)\,T)$$

Appelquist Dine Muzinich 78, Brambilla Pineda Soto Vairo 99

Static octet potential

$$\lim_{T\to\infty} \frac{i}{T} \ln \frac{\langle \square \rangle}{\langle \phi_{ab}^{\mathrm{adj}} \rangle} = \frac{1}{2N_c} \frac{\alpha_{\mathrm{s}}(1/r)}{r} \left[1 + b_1 \frac{\alpha_{\mathrm{s}}(1/r)}{4\pi} + b_2 \left(\frac{\alpha_{\mathrm{s}}(1/r)}{4\pi} \right)^2 + \dots \right]$$

Is known at two loops.

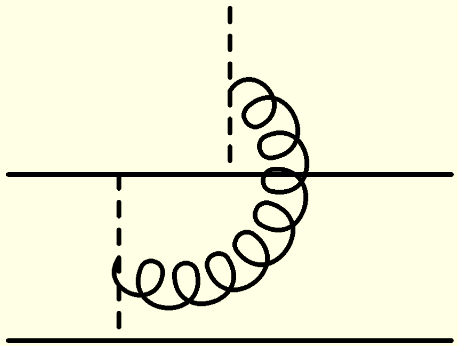
$$b_1 = a_1$$

$$b_2 = a_2 + C_A^2(\pi^4 - 12\pi^2)$$

Kniehl et al 04

V_A

The first contributing diagrams are of the type:

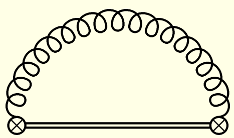


Therefore

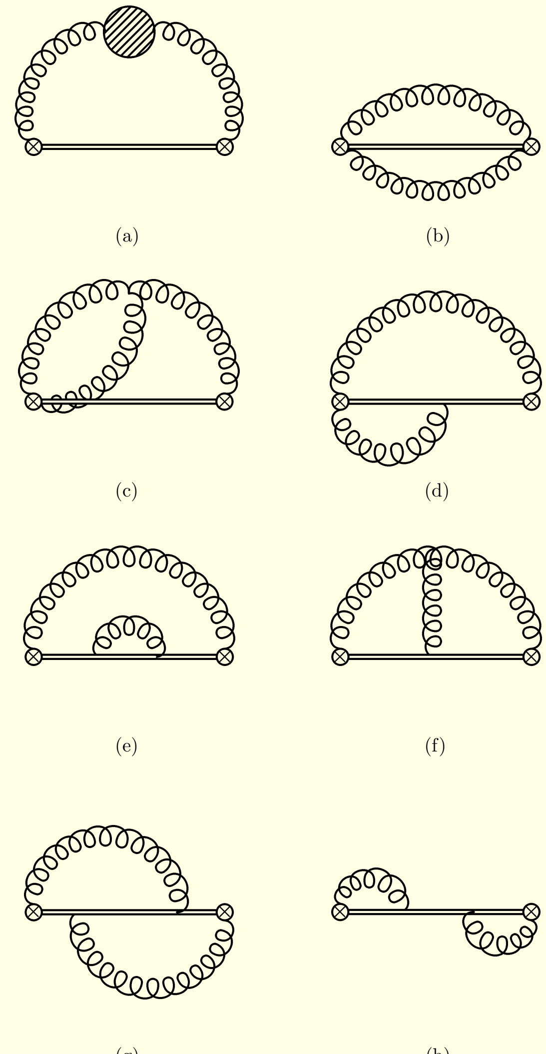
$$V_A(r, \mu) = 1 + \mathcal{O}(\alpha_s^2)$$

Chromoelectric field correlator: $\langle E(t)E(0)\rangle$

Is known at NLO.



LO



NLO

Eidemüller Jamin 97

Static singlet potential

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\
& + \left(\frac{16\,\pi^2}{3} C_A^3 \ \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\
& \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\, C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right]
\end{aligned}
$$

$$
\begin{aligned}
a_4^{L2} \;&=\; -\frac{16\pi^2}{3} C_A^3\, \beta_0 \\
a_4^{L} \;&=\; 16\pi^2 C_A^3 \left[a_1 + 2\gamma_E \beta_0 + n_f \left(-\frac{20}{27} + \frac{4}{9} \ln 2 \right) \right. \\
& \qquad \left. + C_A \left(\frac{149}{27} - \frac{22}{9} \ln 2 + \frac{4}{9} \pi^2 \right) \right]
\end{aligned}
$$

Brambilla et al 99, 06

Static singlet potential

$$
\begin{aligned}
V_s(r,\mu) \;=\; -C_F\frac{\alpha_{\rm s}(1/r)}{r}\Bigg[1 &+ a_1\frac{\alpha_{\rm s}(1/r)}{4\pi} + a_2\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^2 \\
&+ \left(\frac{16\,\pi^2}{3}C_A^3\,\ln r\mu + a_3\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^3 \\
&+ \left(a_4^{L2}\ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\,C_A^3\beta_0(-5+6\ln 2)\right)\ln r\mu + a_4\right)\left(\frac{\alpha_{\rm s}(1/r)}{4\pi}\right)^4\Bigg]
\end{aligned}
$$

- The logarithmic contribution at N^3LO may be extracted from the one-loop calculation of the ultrasoft contribution;
- the single logarithmic contribution at N^4LO may be extracted from the two-loop calculation of the ultrasoft contribution.

Static singlet potential

$$
\begin{aligned}
V_s(r,\mu) \;=\; & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi}\right)^2 \right. \\
& + \left(\frac{16\,\pi^2}{3} C_A^3 \;\ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi}\right)^3 \\
& \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2\, C_A^3 \beta_0 (-5 + 6\ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi}\right)^4 \right]
\end{aligned}
$$

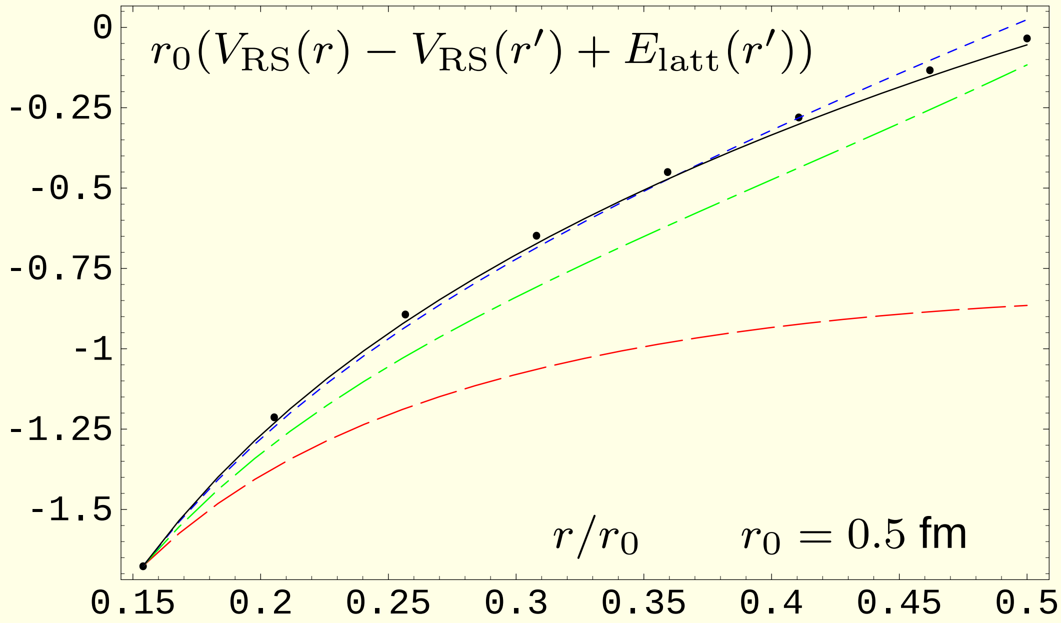
The potential is a Wilson coefficient of an EFT. In general, it undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which allow to resum large logarithms.
It carries also large $(\alpha_s \beta_0)^n$ *contributions of the renormalon type.*

3. Applications

Static energy

$$
\begin{aligned}
E_0(r) \quad = \quad & -\frac{C_F\alpha_s(1/r)}{r}\Bigg\{1+\frac{\alpha_s(1/r)}{4\pi}\left[a_1+2\gamma_E\beta_0\right] \\
& +\left(\frac{\alpha_s(1/r)}{4\pi}\right)^2\left[a_2+\left(\frac{\pi^2}{3}+4\gamma_E^2\right)\beta_0^2+\gamma_E\left(4a_1\beta_0+2\beta_1\right)\right] \\
& +\left(\frac{\alpha_s(1/r)}{4\pi}\right)^3\left[\frac{16\pi^2}{3}C_A^3\ln\frac{C_A\alpha_s(1/r)}{2}+\tilde{a}_3\right] \\
& +\left(\frac{\alpha_s(1/r)}{4\pi}\right)^4\left[a_4^{L2}\ln^2\frac{C_A\alpha_s(1/r)}{2}+a_4^L\ln\frac{C_A\alpha_s(1/r)}{2}+\tilde{a}_4\right] \\
& + \qquad \cdots \qquad \Bigg\}
\end{aligned}
$$

Static energy vs lattice QCD



NNLL + 3 loop est.

NNLO

NLO

LO

$\alpha_{\rm s} = \alpha_{\rm s}(1/r)$

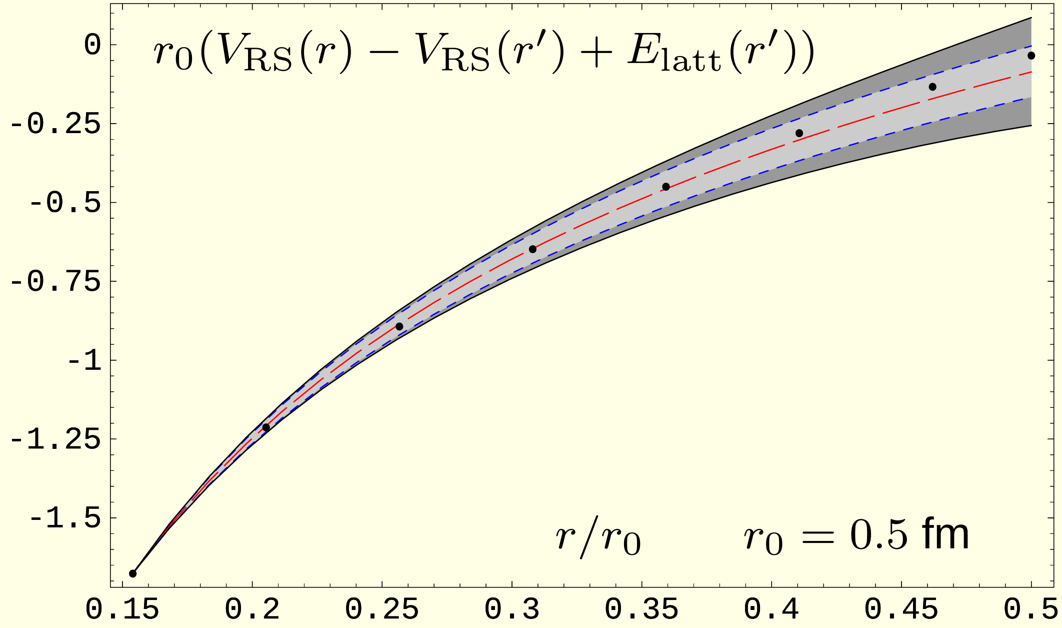
$\nu_f = \nu_{us} = 2.5\, r_0^{-1}$

$r' = 0.15399\, r_0$

Necco Sommer 01, Pineda 02

Sumino 02, Lee 02 03

Static energy vs lattice QCD



Pineda 02

No signal of short-range linear non-perturbative effects.

Quarkonium Spectrum at $\mathcal{O}(m\alpha_s^5)$

Low lying $Q\bar{Q}$ states are assumed to realize the hierarchy: $m \gg 1/r \sim mv \gg \Lambda_{QCD}$

$$E_n = \langle n| \frac{\mathbf{p}^2}{m^2} + V_s(\mu)|n\rangle - i\frac{g^2}{3N_c}\int_0^\infty dt\, \langle n|\mathbf{r} e^{it(E_n^{(0)}-H_o)}\mathbf{r}|n\rangle\, \langle \mathbf{E}(t)\,\mathbf{E}(0)\rangle(\mu)$$

$m\alpha_s^5 \ln \alpha_s$ Brambilla Pineda Soto Vairo 99, Kniehl Penin 99
$m\alpha_s^5$ Kniehl Penin Smirnov Steinhauser 02 NNLL Pineda 02

c and b masses

reference	order	$\overline{m}_b(\overline{m}_b)$ (GeV)
Brambilla et al 01	NNLO +charm ($\Upsilon(1S)$)	$4.190 \pm 0.020 \pm 0.025$
Penin Steinhauser 02	NNNLO* ($\Upsilon(1S)$)	4.346 ± 0.070
Lee 03	NNNLO* ($\Upsilon(1S)$)	4.20 ± 0.04
Contreras et al 03	NNNLO* ($\Upsilon(1S)$)	4.241 ± 0.070
Pineda Signer 06	NNLL* high moments SR	4.19 ± 0.06

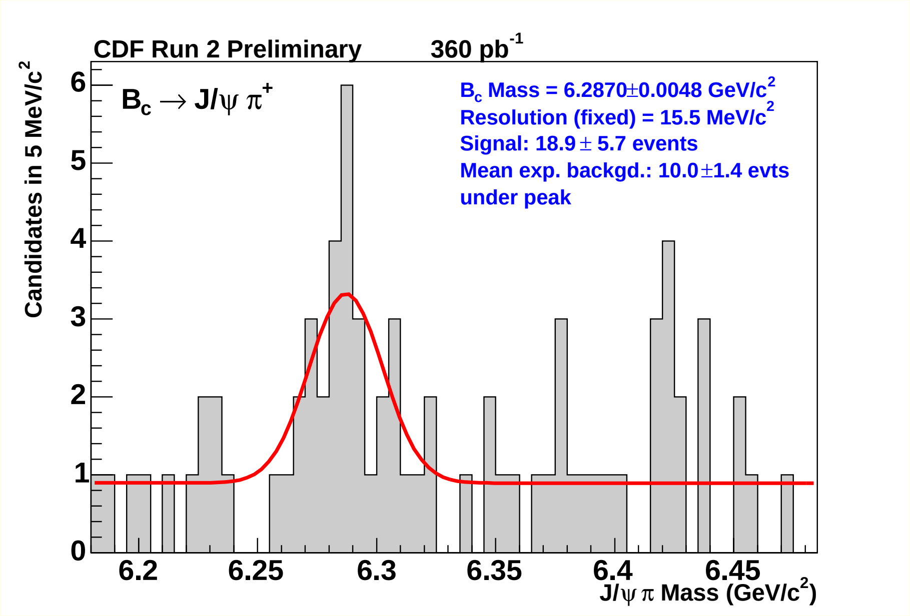
reference	order	$\overline{m}_c(\overline{m}_c)$ (GeV)
Brambilla et al 01	NNLO (J/ψ)	1.24 ± 0.020
Eidemüller 02	NNLO high moments SR	1.19 ± 0.11

B_c mass

State	expt	lattice04	BV00	BSV01	BSV02
B_c mass (MeV)					
1^1S_0	$6400(400)$	6304(16)	6326(29)	6324(22)	6307(17)

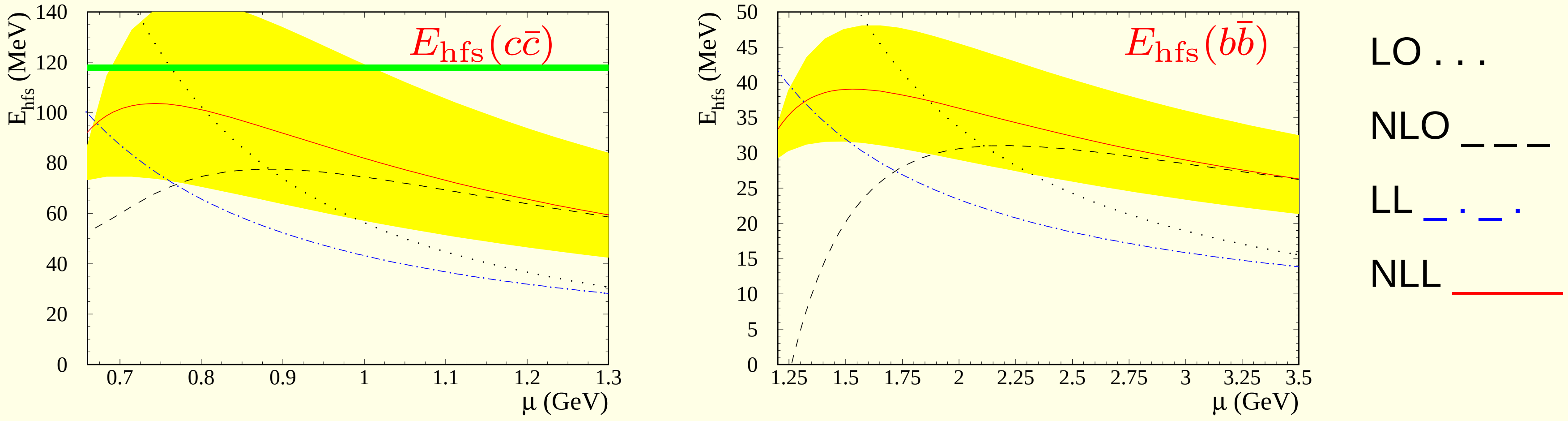
Brambilla et al 01 02, Brambilla Vairo 00, HPQCD-FNAL-UKQCD 04

In CDF 05 B_c is found in $B_c \to J/\psi\ \pi$.



$$M_{B_c} = 6287 \pm 4.8 \pm 1.1 \text{ MeV}$$

Hfs and the η_b mass



$$M(\eta_b) = 9421 \pm 10\,(\mathrm{th})\,{}^{+9}_{-8}\,(\delta\alpha_s)\ \mathrm{MeV}$$

Kniehl et al 03

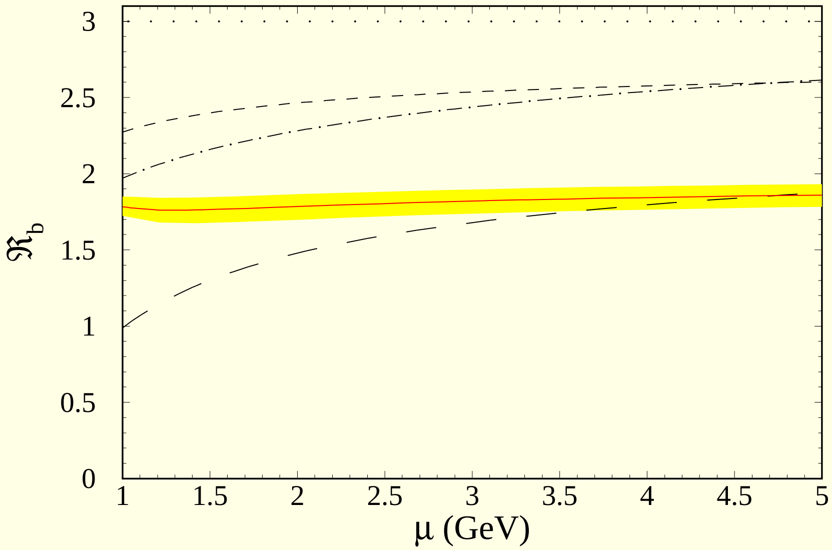
- A similar analysis in the B_c case gives:

$$M(B_c^*) - M(B_c) = 65 \pm 24\,(\mathrm{th})\,{}^{+19}_{-16}\,(\delta\alpha_s)\ \mathrm{MeV}$$

Penin et al 04

Em decays of $\Upsilon(1S)$ and η_b

$$\mathcal{R}_b = \frac{\Gamma(\Upsilon(1S)\rightarrow e^+e^-)}{\Gamma(\eta_b\rightarrow\gamma\gamma)}$$



LO . . .

NLO _ _ _

NNLO __ __ __

NLL _ . _ .

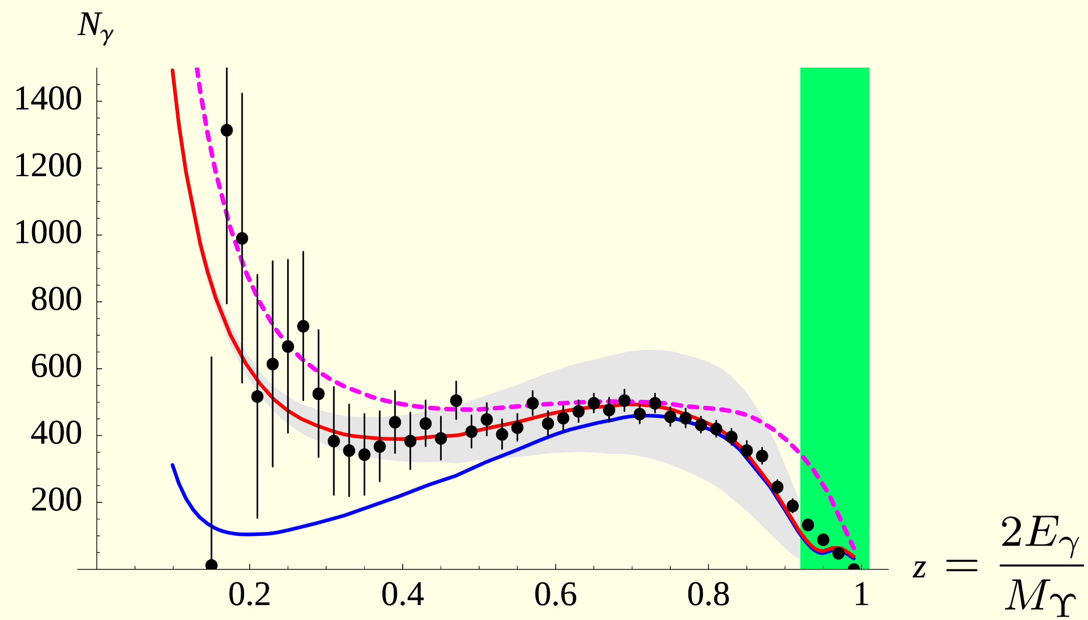
NNLL ____

$$\Gamma(\eta_b(1S) \rightarrow \gamma\gamma) = 0.659 \pm 0.089(\mathrm{th.})^{+0.019}_{-0.018}(\delta\alpha_s) \pm 0.015(\mathrm{exp.})\ \mathrm{keV}$$

Penin Pineda Smirnov Steinhauser 04

Pineda Signer 06

$\Upsilon(1S) \to \gamma\, X$



Photon spectrum at NLO (continuous lines, pNRQCD + SCET) vs CLEO data

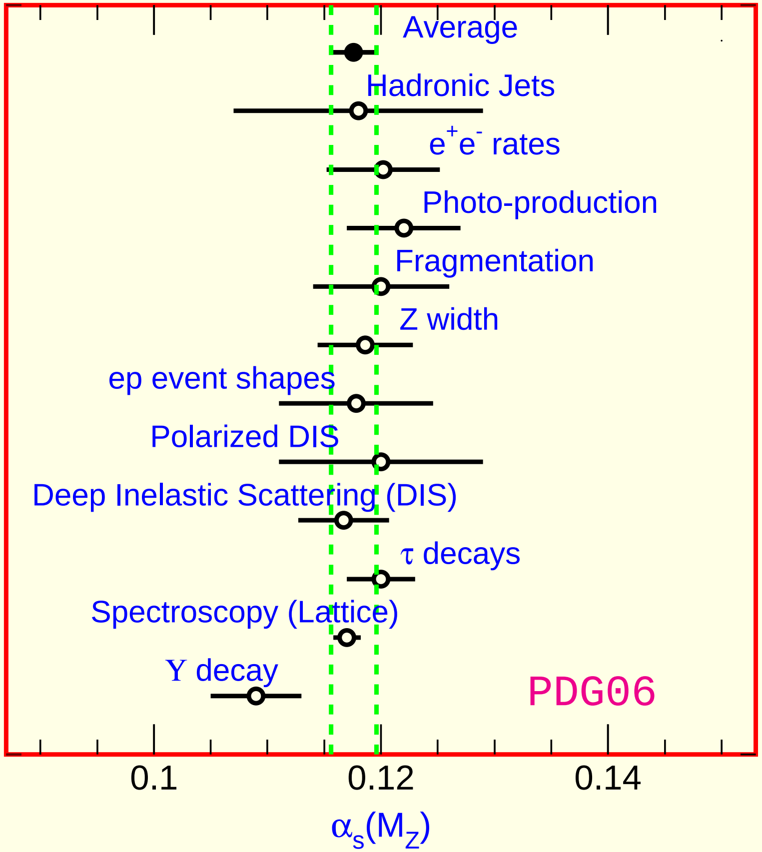
Garcia Soto 04 05, Fleming Leibovich 03

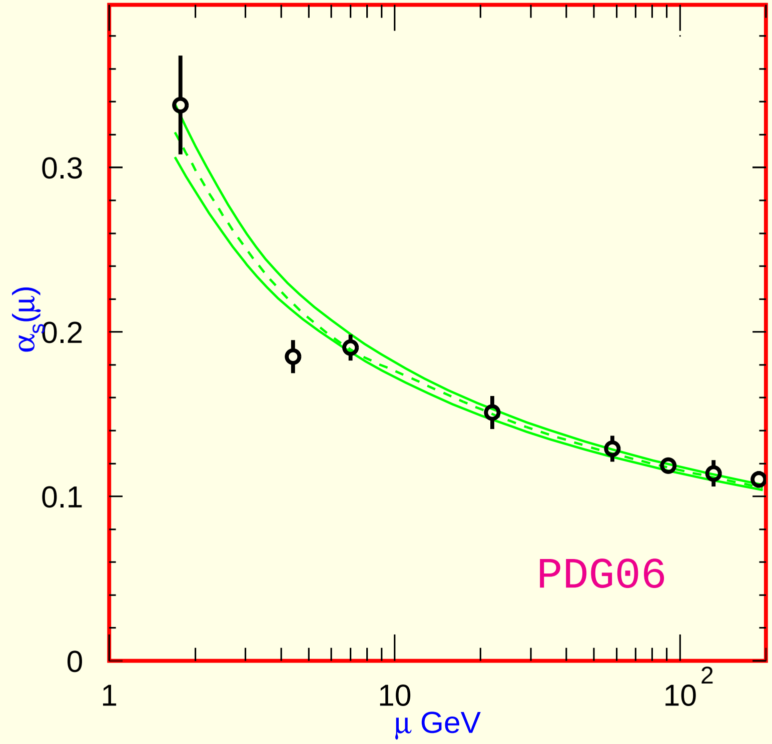
From $\Gamma(\Upsilon(1S) \to \gamma\, X)/\Gamma(\Upsilon(1S) \to X)$:

$$\alpha_s(M_{\Upsilon(1S)}) = 0.184^{+0.014}_{-0.013}, \qquad \alpha_s(M_Z) = 0.119^{+0.006}_{-0.005}$$

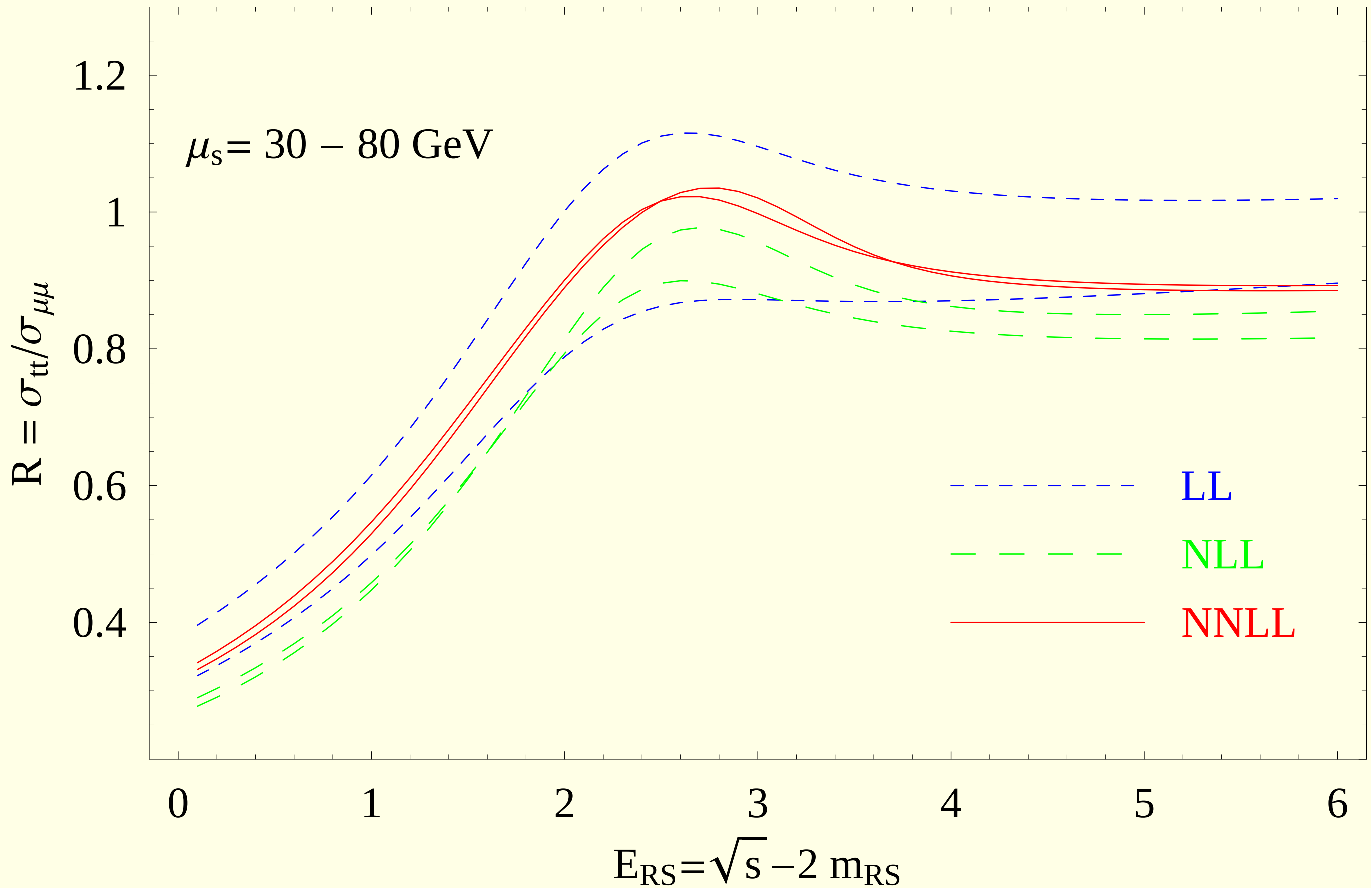
Brambilla Garcia Soto Vairo 07

α_s from the Υ system





$t\bar{t}$ production near threshold



Pineda Signer 06, Hoang Teubner 01

4. Non-perturbative potential

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out.

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

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- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

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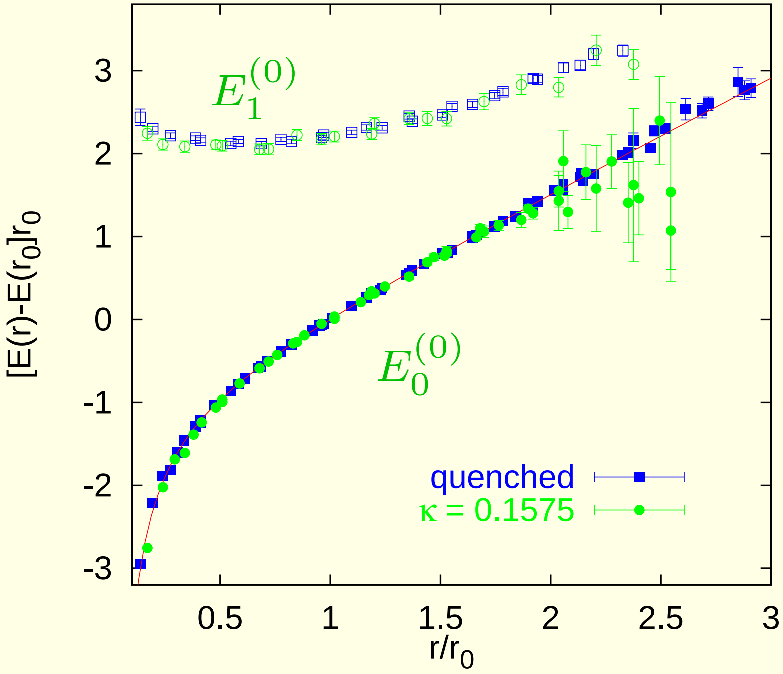


Bali et al. 98

($r_0 \simeq 0.5$ fm)

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

- All scales above mv^2 are integrated out.
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order $\Lambda_{\rm QCD}$ with the static $Q\bar{Q}$ energy.

$\Rightarrow$ The singlet quarkonium field S of energy mv^2 is the only the degree of freedom of pNRQCD(up to ultrasoft light quarks, e.g. pions).

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = \mathrm{Tr}\left\{ \mathrm{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) \mathrm{S} \right\}$$

Brambilla Pineda Soto Vairo 00

pNRQCD for $mv \sim \Lambda_{\rm QCD}$

$$\mathcal{L} = {\rm Tr}\left\{ {\rm S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - V_s \right) {\rm S} \right\}$$

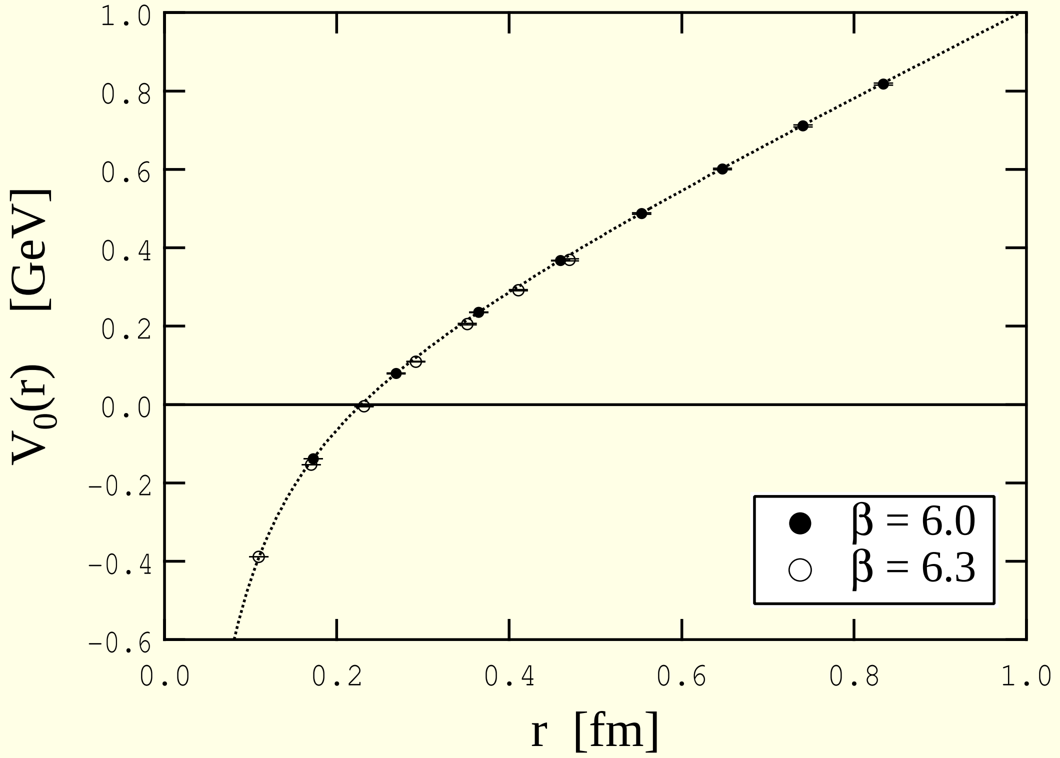
Brambilla Pineda Soto Vairo 00

- The potential V_s (${\rm Re}\, V_s + i\, {\rm Im}\, V_s$) is non-perturbative:
  (a) to be determined from the lattice;
  (b) to be determined from QCD vacuum models.

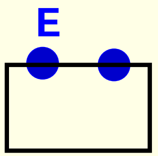
Creutz et al. 82, Campostrini 85, Michael 85, Born et al. 94,
Bali et al 97, Bali 00, Koma et al 06, Brambilla et al. 93, 95, 97, 98

Static potential

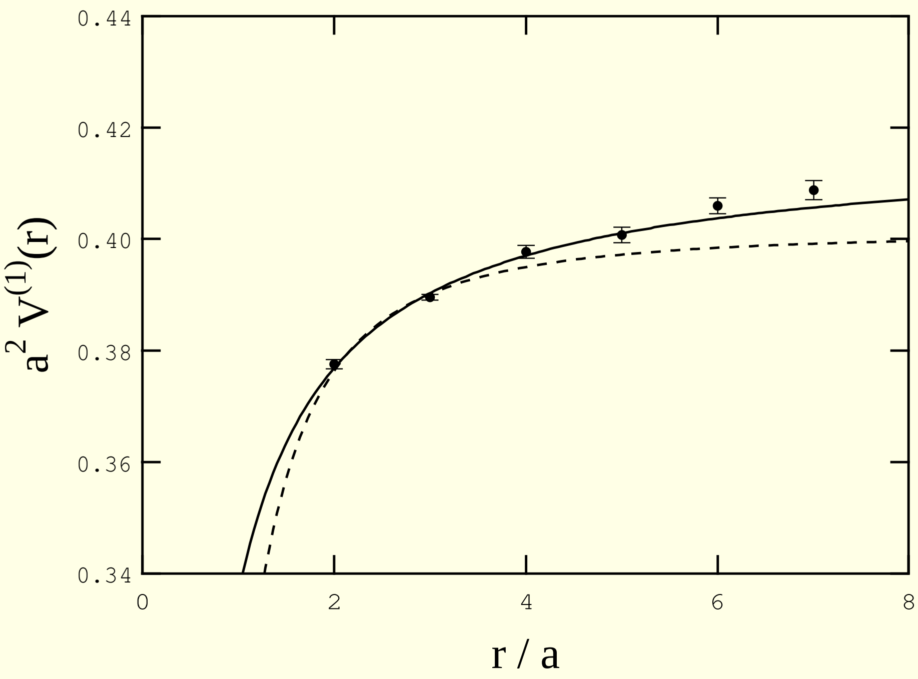
$$V_s^{(0)} = \lim_{T\to\infty} \frac{i}{T} \ln \langle \square \rangle$$



$1/m$ potential

$$\frac{V_s^{(1)}}{m} = -\frac{1}{2m}\int_0^{\infty} dt\, t \left\langle \right.$$  $$\left. \right\rangle$$

Brambilla et al 00



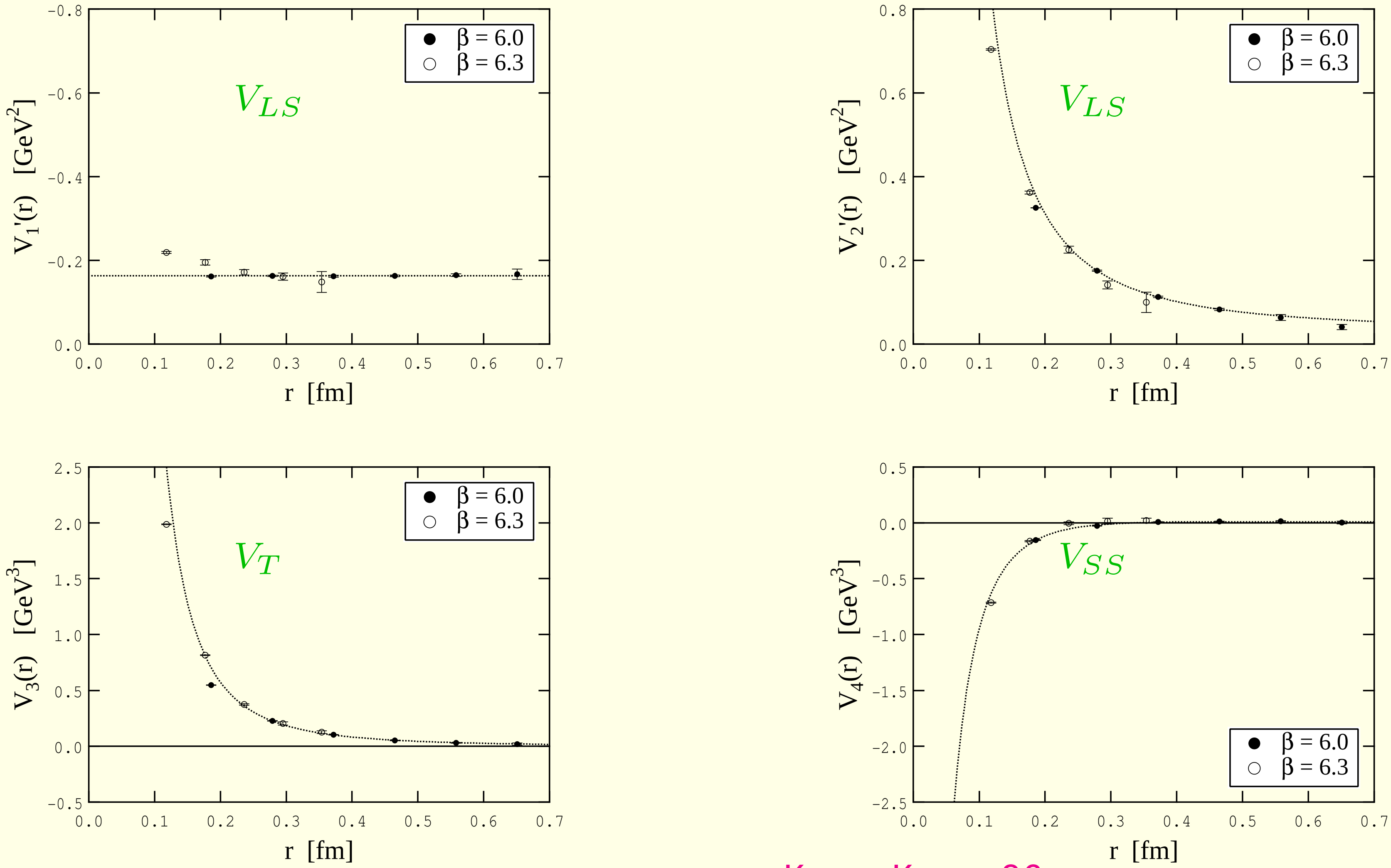
$$V^{(1)} = -\frac{c}{r} + d$$

$$\frac{2c}{m_c}\,\frac{1}{r} \approx \frac{1}{r}$$ part of the static potential

$$\frac{2c}{m_b}\,\frac{1}{r} \approx$$ 26% of the $\frac{1}{r}$ part of the static potential

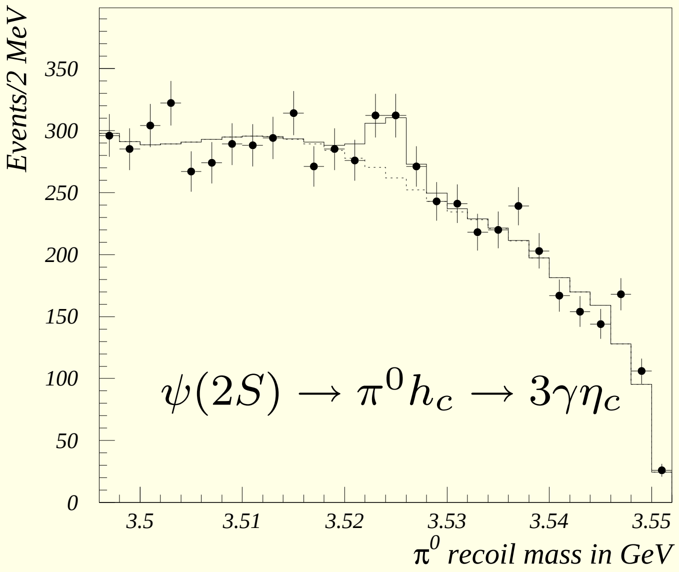
Koma Koma Wittig 06

Spin-dependent potentials



Koma Koma 06

h_c



$$M = 3524.4 \pm 0.6 \pm 0.4 \text{ MeV}$$

CLEO 05

Also

$$M = 3525.8 \pm 0.2 \pm 0.2 \text{ MeV}, \qquad \Gamma < 1 \text{ MeV}$$

E835 05

- To be compared with $M_{\text{c.o.g.}}(1P) = 3525.36 \pm 0.2 \pm 0.2$ MeV.

Conclusions

Non-relativistic EFTs provide a rigorous definition of the potential between two heavy quarks.

- In the perturbative regime, it is a key ingredient for precision calculations of several threshold observables.

- In the non-perturbative regime, it may be calculated on the lattice. Embedded in an EFT it provides an alternative to more traditional lattice calculations with heavy quarks (e.g. NRQCD).

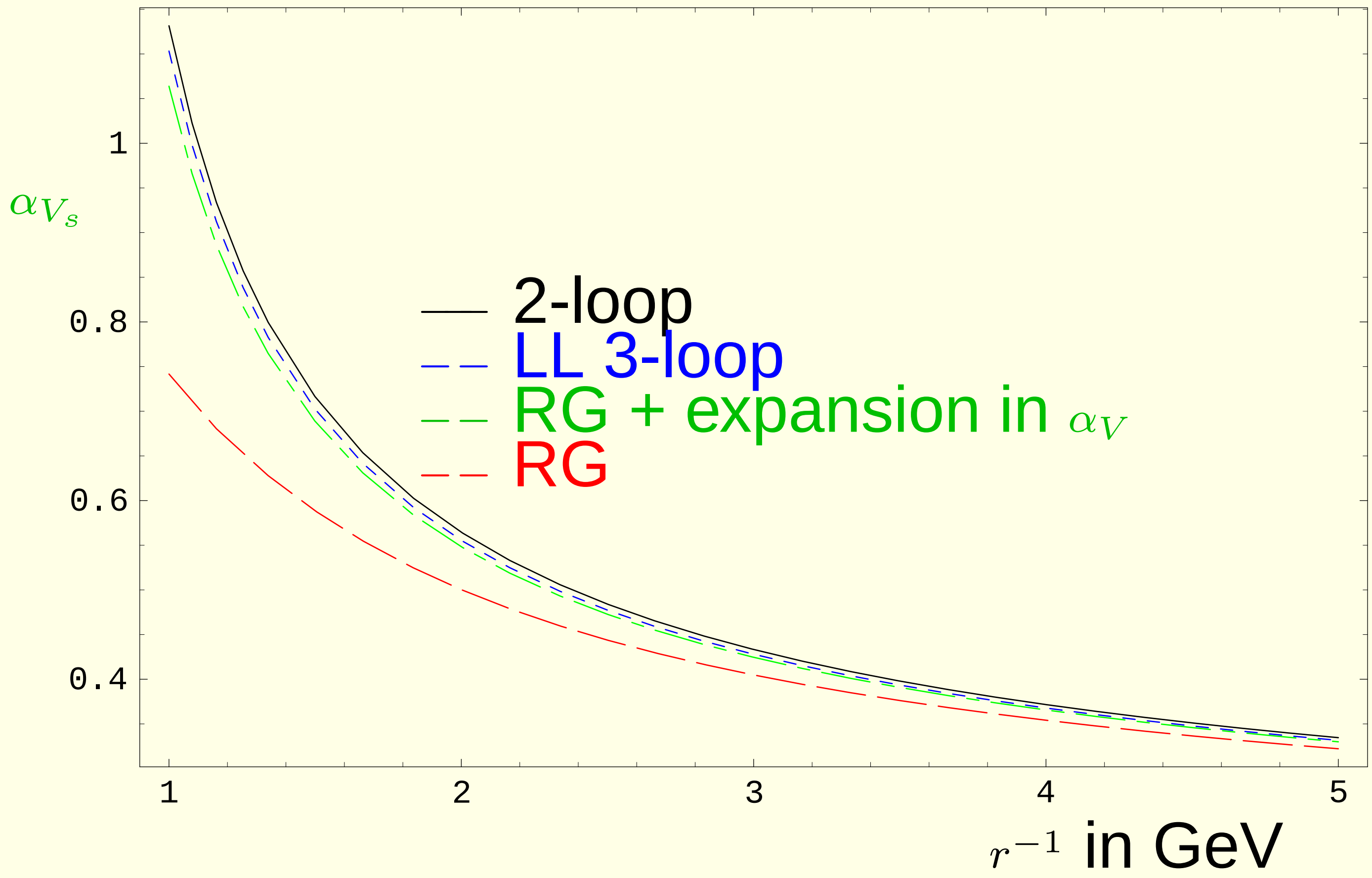
Backup slides

Summing Logs

$$\begin{cases} \mu \dfrac{d}{d\mu} \alpha_{V_s} = \dfrac{2}{3} \dfrac{\alpha_s}{\pi} V_A^2 \left[\dfrac{\alpha_{V_o}}{6} + \dfrac{4}{3} \alpha_{V_s} \right]^3 \\ \mu \dfrac{d}{d\mu} \alpha_{V_o} = \dfrac{2}{3} \dfrac{\alpha_s}{\pi} V_A^2 \left[\dfrac{\alpha_{V_o}}{6} + \dfrac{4}{3} \alpha_{V_s} \right]^3 \\ \mu \dfrac{d}{d\mu} \alpha_s = \alpha_s \beta(\alpha_s) \\ \mu \dfrac{d}{d\mu} V_A = 0 \\ \mu \dfrac{d}{d\mu} V_B = 0 \end{cases}$$

Pineda Soto 00

Summing Logs



$(\mu = \alpha_s(r)/r;\ N_f = 4)$

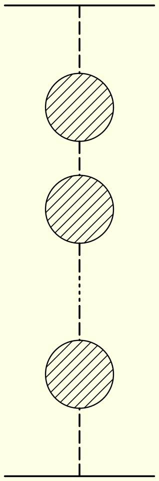
Pineda Soto 00

Summing $(\alpha_s \beta_0)^n$

V_s is affected by renormalons: $V_s(renormalon) = C_0 + C_2 r^2 + \ldots$

$$C_0 \simeq -2 \frac{C_F \alpha_s(\mu)}{\pi} \mu \sum_{n=0}^{\infty} n! \left(\frac{\beta_0 \alpha_s(\mu)}{2\pi} \right)^n$$

$$\Rightarrow \delta C_0 \sim \Lambda_{\mathrm{QCD}}$$



$$C_2 \simeq \frac{1}{9} \frac{C_F \alpha_s(\mu)}{\pi} \mu^3 \sum_{n=0}^{\infty} n! \left(\frac{\beta_0 \alpha_s(\mu)}{6\pi} \right)^n$$

$$\Rightarrow \delta C_2 \sim \Lambda_{\mathrm{QCD}}^3$$

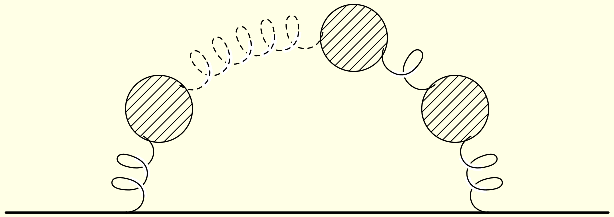
$1/r \gg \mu \gg \Lambda_{\mathrm{QCD}}$

Aglietti Ligeti 95

Summing $(\alpha_s \beta_0)^n$

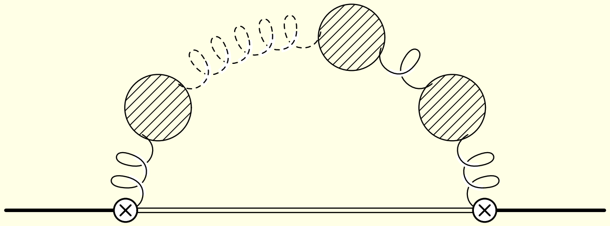
V_s is affected by renormalons: $V_s(renormalon) = C_0 + C_2 r^2 + \ldots$

The $\mathcal{O}(\Lambda_{QCD})$ renormalon cancels against the pole mass.

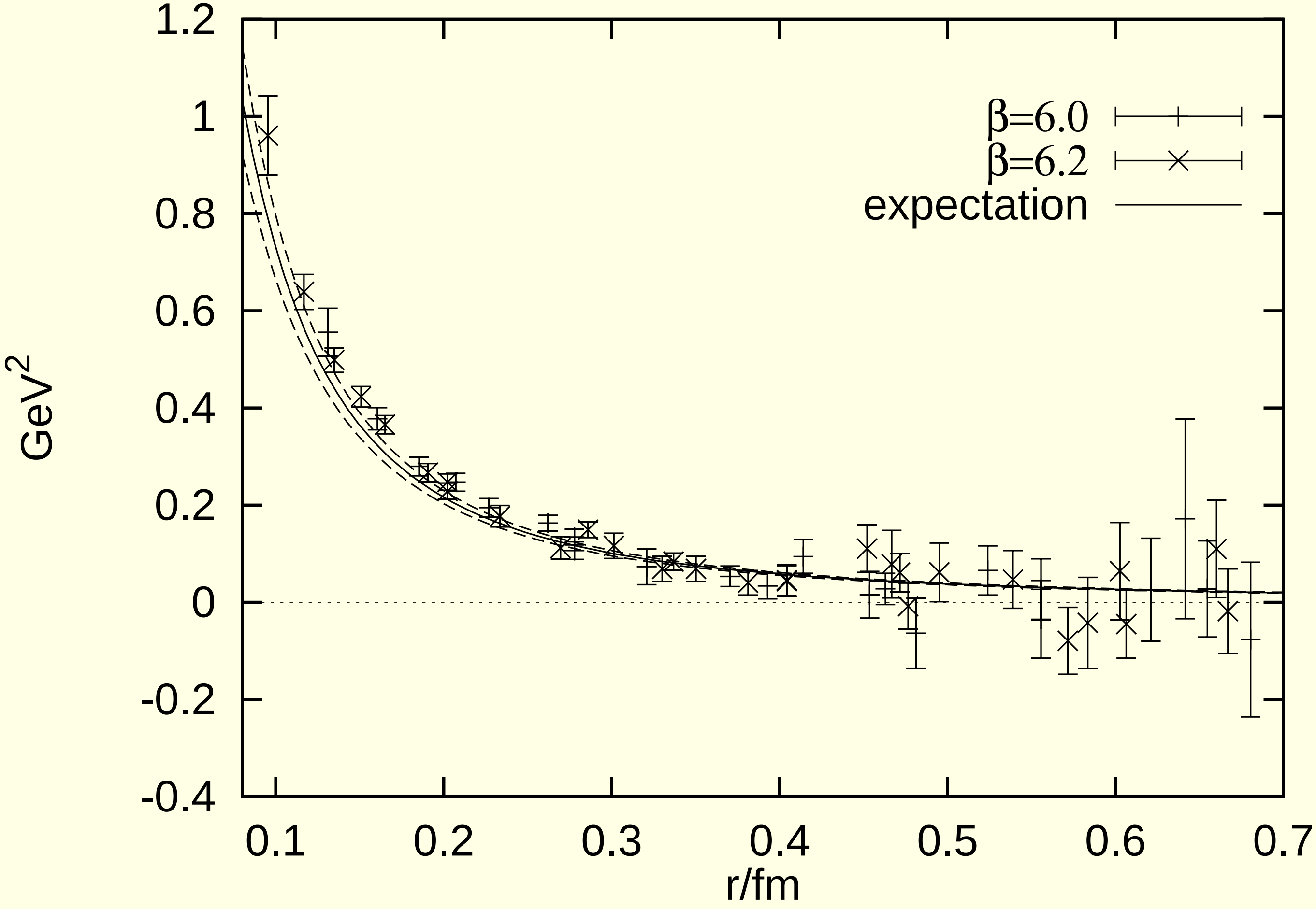
$2\times$  $= -C_0$

Beneke 98, Pineda 98, Hoang Smith Stelzer Willenbrock 99

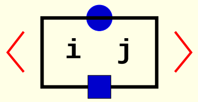
The $\mathcal{O}(\Lambda^3_{QCD})$ renormalon cancels in pNRQCD.

 $= -C_2 r^2$

Brambilla Pineda Soto Vairo 99



$$\epsilon^{kij}\,\hat{r}^k \int_0^\infty dt\, t$$



Bali Schilling Wachter 97

Potential Models

State	PDG[16]	BGS[12]	GI[12]	EFG[13]	Cornell[3]
$J/\psi(1^3S_1)$	3096.87 ± 0.04	3090	3098	3096	3095 [3095]
$\eta_c(1^1S_0)$	2979.2 ± 1.3	2982	2975	2979	3095
$\psi'(2^3S_1)$	3685.96 ± 0.09	3672	3676	3686	3684 [3684]
$\eta_c'(2^1S_0)$	3637.7 ± 4.4	3630	3623	3588	3684
$\psi(3^3S_1)$	4040 ± 10	4072	4100	4088	4110 [4225]
$\eta_c(3^1S_0)$		4043	4064	3991	4110
$\psi(4^3S_1)$	4415 ± 6	4406	4450	–	4460 [4625]
$\eta_c(4^1S_0)$		4384	4425	–	4460
$\chi_2(1^3P_2)$	3556.18 ± 0.13	3556	3550	3566	3522 [3523]
$\chi_1(1^3P_1)$	3510.51 ± 0.12	3505	3510	3510	3522 [3517]
$\chi_0(1^3P_0)$	3415.3 ± 0.4	3424	3445	3424	3522
$h_c(1^1P_1)$	3524 ± 1^a	3516	3517	3526	3522 [3519]
$\chi_2(2^3P_2)$	3931 ± 5^a	3972	3979	3972	–
$\chi_1(2^3P_1)$		3925	3953	3929	–
$\chi_0(2^3P_0)$		3852	3916	3854	–
$h_c(2^1P_1)$		3934	3956	3945	–
$\chi_2(3^3P_2)$		4317	4337	–	–
$\chi_1(3^3P_1)$		4271	4317	–	–
$\chi_0(3^3P_0)$		4202	4292	–	–
$h_c(3^1P_1)$		4279	4318	–	–
$\psi_3(1^3D_3)$		3806	3849	3815	3810
$\psi_2(1^3D_2)$		3800	3838	3811	3810
$\psi(1^3D_1)$	3769.9 ± 2.5	3785	3819	3798	3810 [3755]
$\eta_{c2}(1^1D_2)$		3799	3837	3811	3810
$\psi_3(2^3D_3)$		4167	4217	–	4190
$\psi_2(2^3D_2)$		4158	4208	–	4190
$\psi(2^3D_1)$	4159 ± 20	4142	4194	–	4190 [4230]
$\eta_{c2}(2^1D_2)$		4158	4208	–	4190

- Potential models grasp the main features of the quarkonium spectrum above and below threshold.
- No error bars.
- Hardly improvable. Non renormalizable.

Swanson 06