Form Factors from Lattice QCD

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– QCDSF Collaboration –

Special mention:

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Outline

Lattice Simulation

Basics

Nucleon

Pion

Conclusions & Outlook

Lattice Simulation

Action

$N_f = 2$

$$S = S_G + S_F$$

$$S_G = \beta \sum_{x,\mu<\nu} \left(1 - \frac{1}{3}\mathrm{Re\,Tr}\, U_{\mu\nu}(x)\right)$$

$$S_F = \sum_x \Big\{ \bar\psi(x)\psi(x) - \kappa\, \bar\psi(x) U_\mu^\dagger(x-\hat\mu)[1+\gamma_\mu]\psi(x-\hat\mu) - \kappa\, \bar\psi(x) U_\mu(x)[1-\gamma_\mu]\psi(x+\hat\mu) - \frac{1}{2}\kappa\, c_{SW}\, g\, \bar\psi(x)\sigma_{\mu\nu}F_{\mu\nu}(x)\psi(x) \Big\}$$

$\Updownarrow$

$\partial_\mu A_\mu^{\mathrm{imp}} = 2m_q P$

Clover Fermions

Advantages

- Local
- Transfer matrix
- $O(a)$ improved
- Flavor symmetry

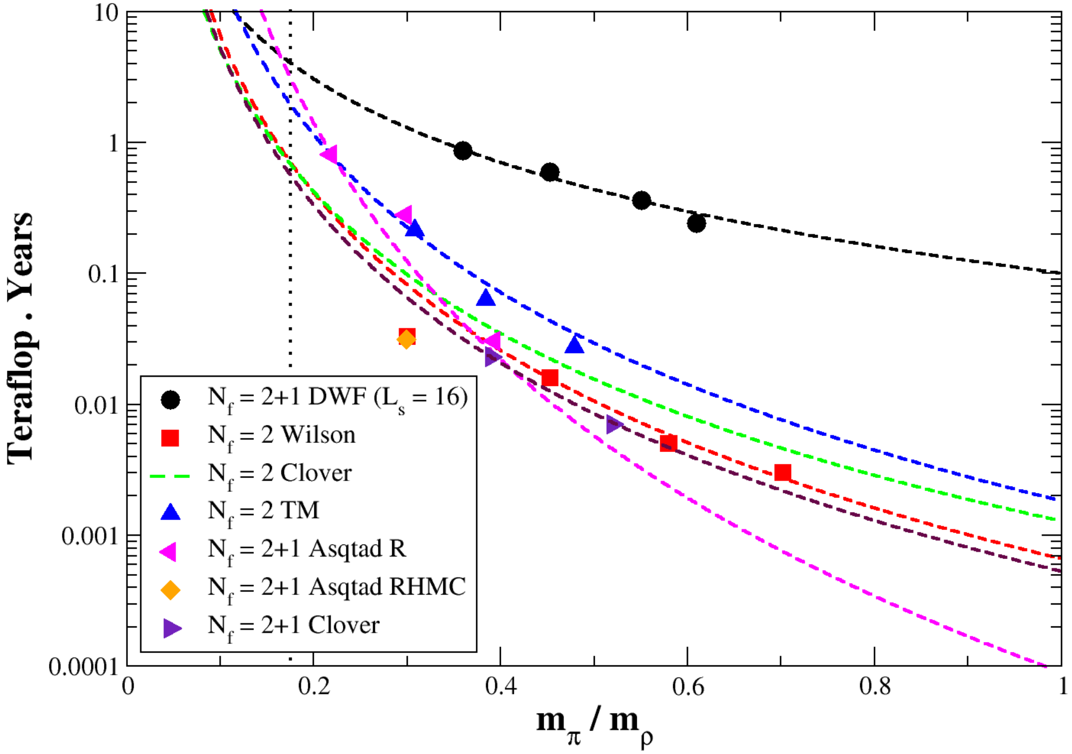
  Prerequisite to making contact with $SU(2)$ ChPT

  – Finite size corrections

  – Chiral extrapolation

  – Determination of low-energy constants
- Fast to simulate

Cost of Simulation

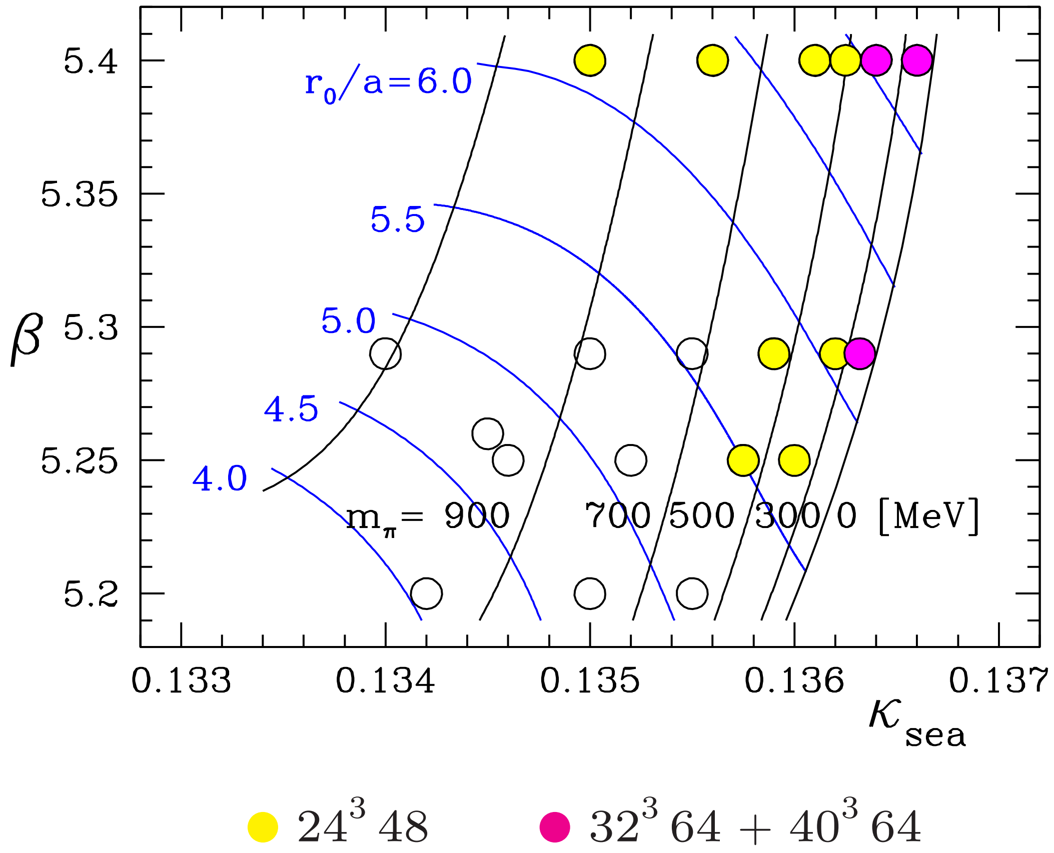


Clark

Lattices

β	$\kappa_{\rm sea}$	Volume	a [fm]	m_{PS} [MeV]
5.20	0.13420	$16^3 \times 32$	0.11	1010
5.20	0.13500	$16^3 \times 32$	0.10	830
5.20	0.13550	$16^3 \times 32$	0.09	620
5.25	0.13460	$16^3 \times 32$	0.10	990
5.25	0.13520	$16^3 \times 32$	0.09	830
5.25	0.13575	$24^3 \times 48$	0.08	600
5.25	0.13600	$24^3 \times 48$	0.08	450
5.26	0.13450	$16^3 \times 32$	0.10	1010
5.29	0.13400	$16^3 \times 32$	0.10	1170
5.29	0.13500	$16^3 \times 32$	0.09	930
5.29	0.13550	$24^3 \times 48$	0.08	770
5.29	0.13550	$16^3 \times 32$		780
5.29	0.13550	$12^3 \times 32$		880
5.29	0.13590	$24^3 \times 48$	0.08	590
5.29	0.13590	$16^3 \times 32$		630
5.29	0.13590	$12^3 \times 32$		870
5.29	0.13620	$24^3 \times 48$	0.08	400
5.29	0.13632	$32^3 \times 64$	0.08	340
Not yet analyzed $\longrightarrow$ 5.29	0.13632	$40^3 \times 64$	0.08	290
5.40	0.13500	$24^3 \times 48$	0.08	1040
5.40	0.13560	$24^3 \times 48$	0.07	840
5.40	0.13610	$24^3 \times 48$	0.07	630
5.40	0.13625	$24^3 \times 48$	0.07	530
5.40	0.13640	$24^3 \times 48$	0.07	440
5.40	0.13640	$32^3 \times 64$	0.07	440
Not yet analyzed $\longrightarrow$ 5.40	0.13660	$32^3 \times 64$	0.07	280

Coverage



← $a = 0.07$ fm

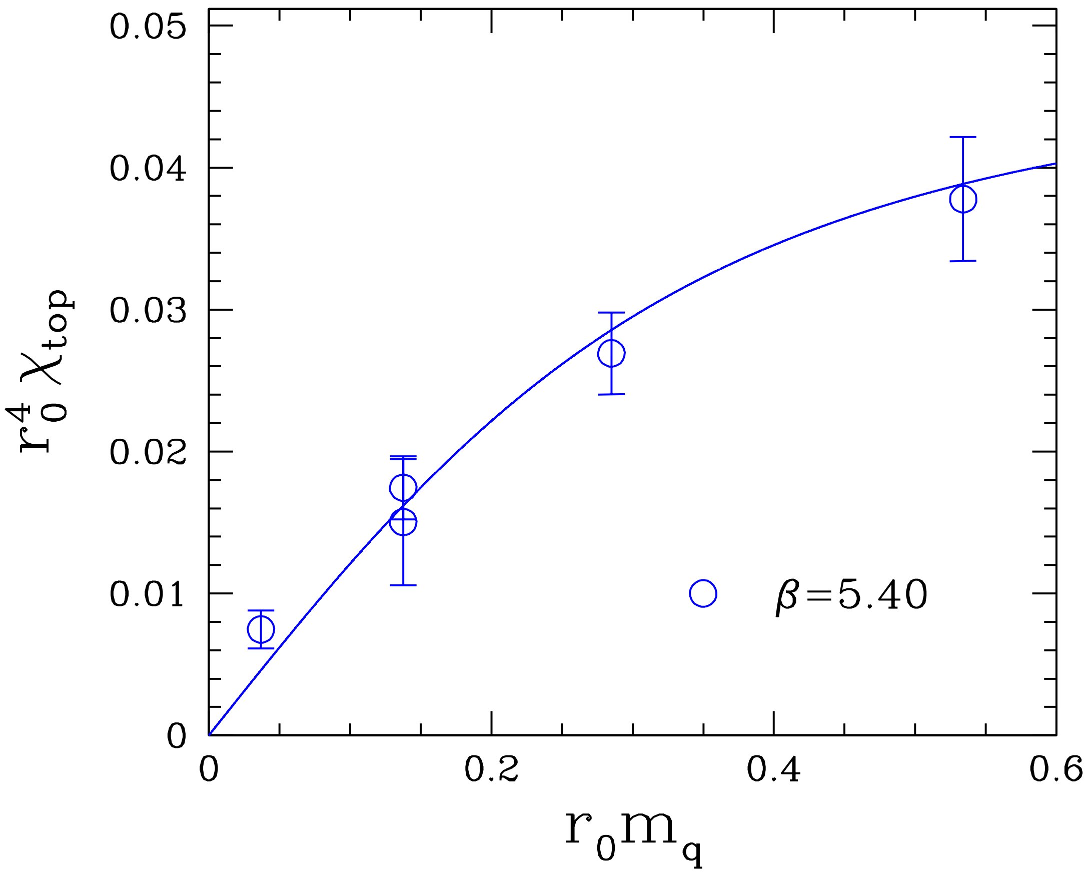
← $a = 0.10$ fm

For gauge field sampling we use 'ordinary' HMC algorithm with Hasenbusch integration + 3 time scales

Effect of Unquenching ?

$$\chi_{\rm top} \equiv \frac{\langle Q^2 \rangle}{V} = \frac{\Sigma\, m_q}{2}$$

Vector Ward Identity √

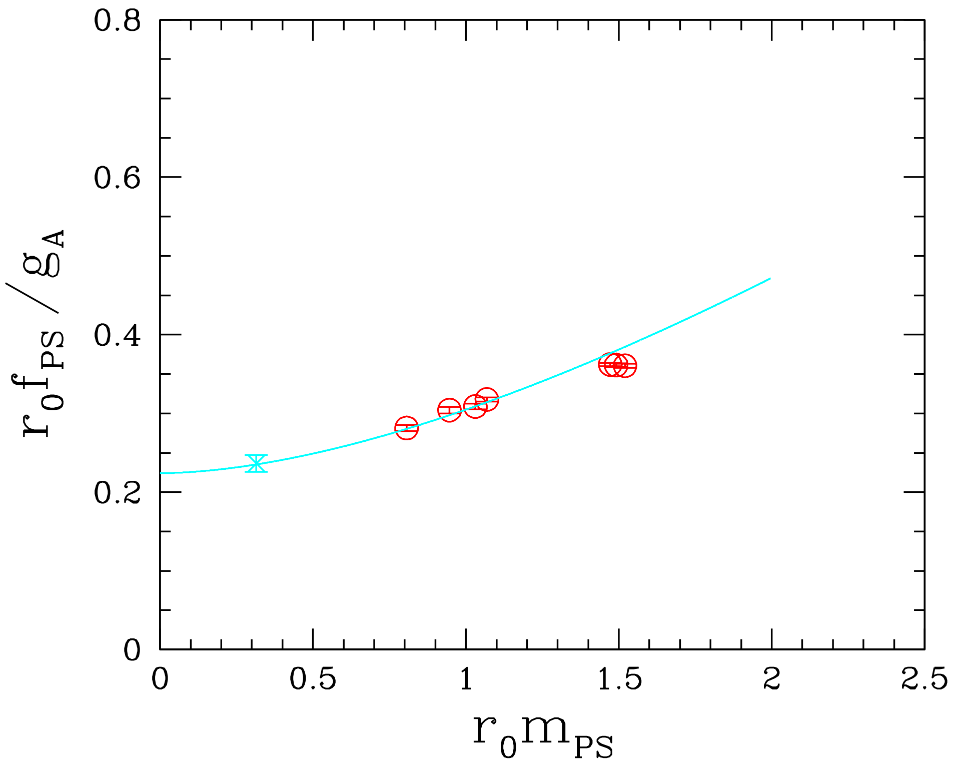


$$\left(\frac{1}{\chi_{\rm top}}\right)^2 = \left(\frac{2}{\Sigma\, m_q}\right)^2 + \left(\frac{1}{\chi^\infty_{\rm top}}\right)^2$$

Dürr

$\Sigma^{\overline{MS}}(2\ \text{GeV}) = [276(12)\ \text{MeV}]^3$

Determination of Scale

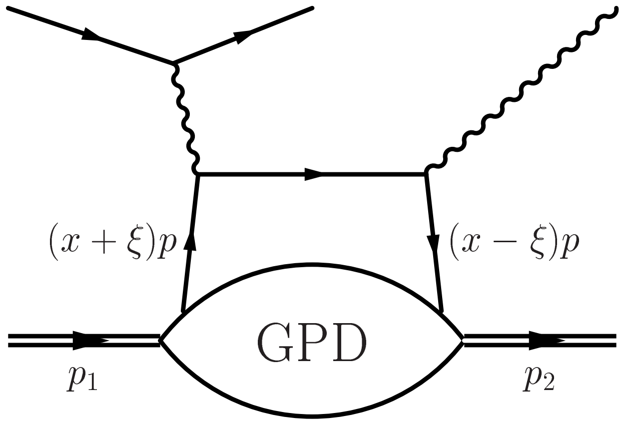


Z_A cancels, FS effects & leading log's largely cancel

$r_0 = 0.45(1)$ fm

Basics

OPE

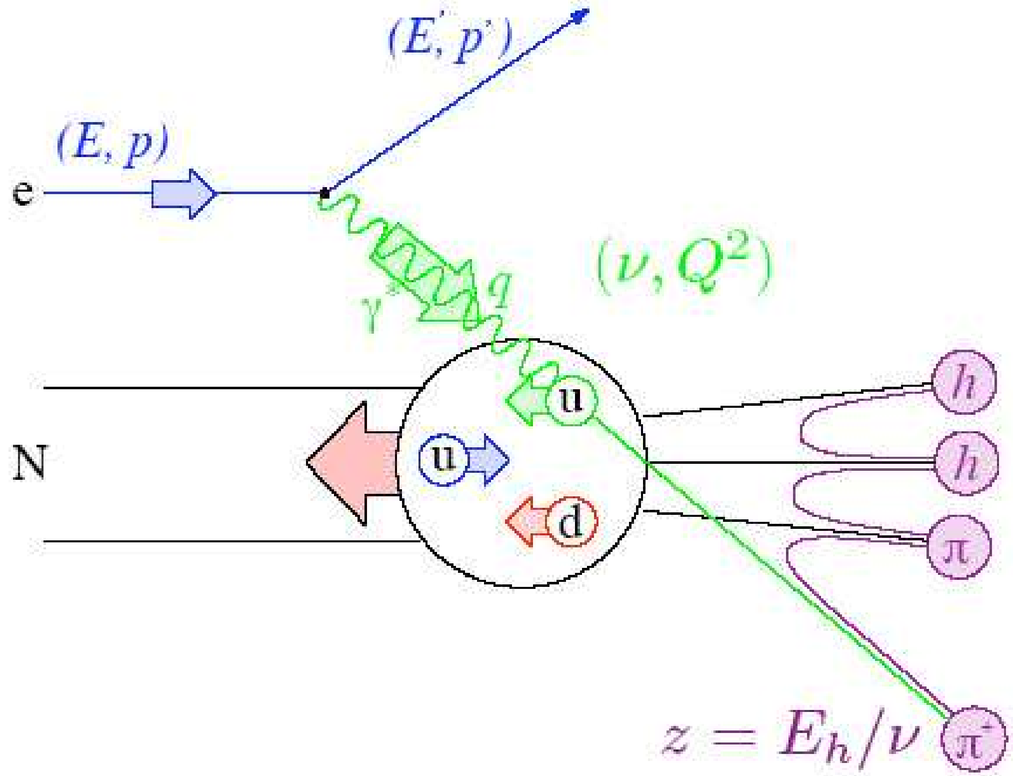


$p = \frac{1}{2}(p_1 + p_2)$, $\Delta = p_2 - p_1$, $q = \frac{1}{2}(q_1 + q_2)$

$\xi = 0$: Momentum transfer of the struck parton purely transverse, i.e. $\Delta = \mathbf{\Delta}_\perp$

$$J(q)\, J(-q) \;=\; \sum_n c_n \;\times\; \left\{ \begin{array}{l} \mathcal{O}^q_{\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n-1} \bar{q}\gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} q \\ \mathcal{O}^{5q}_{\sigma\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n} \bar{q}\gamma_\sigma\gamma_5 \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q \\ \mathcal{O}^{Tq}_{\mu\nu\mu_1\cdots\mu_n} = \left(\frac{i}{2}\right)^{n} \bar{q}\sigma_{\mu\nu}\gamma_5 \overleftrightarrow{D}_{\mu_1} \cdots \overleftrightarrow{D}_{\mu_n} q \end{array} \right.$$

Experiment



Nucleon

$$\langle p_1, s|\, \mathcal{O}^q_{\{\mu_1\cdots\mu_n\}}\, |p_2, s\rangle = \bar{u}(p_1, s)\Big[A^q_n(\Delta^2)\; \gamma_{\{\mu_1} + B^q_n(\Delta^2)\; \frac{\mathrm{i}\Delta^\alpha}{2m_N}\sigma_{\alpha\{\mu_1}\Big] p_{\mu_2}\cdots p_{\mu_n\}}\, u(p_2, s) \;+\; \cdots$$

$$\langle p_1, s|\mathcal{O}^{5q}_{\{\mu\mu_1\cdots\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\Big[\tilde{A}^q_{n+1}(\Delta^2)\; \gamma_{\{\mu}\gamma_5 p_{\mu_1}\cdots p_{\mu_n\}}\Big] u(p_2, s) \;+\; \cdots$$

$$\langle p_1, s|\mathcal{O}^{Tq}_{\mu\{\nu\mu_1\cdots\mu_n\}}|p_2, s\rangle = \bar{u}(p_1, s)\Big[A^{Tq}_{n+1}(\Delta^2)\; \sigma_{\mu\{\nu}\gamma_5 - \tilde{A}^{Tq}_{n+1}(\Delta^2)\big(\frac{\Delta^2}{2m_N^2}\sigma_{\mu\{\nu} - \frac{\Delta_\mu\Delta_\alpha}{2m_N^2}\sigma_{\alpha\{\nu}\big)\gamma_5 + \bar{B}^{Tq}_{n+1}(\Delta^2)\; \epsilon_{\alpha\beta\mu\{\nu}\frac{\Delta_\alpha\gamma_\beta}{2m_N}\Big]\; p_{\mu_1}\cdot\cdot p_{\mu_n\}} u(p_2, s) \;+\; \cdots$$

$$A_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} H^q(x, \Delta^2)$$

$$H^q(x,0) = q(x)$$

$$B_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} E^q(x, \Delta^2)$$

$$\tilde{A}_n^q(\Delta^2) = \int_0^1 dx\, x^{n-1} \tilde{H}^q(x, \Delta^2)$$

$$\tilde{H}^q(x,0) = \Delta q(x)$$

$$A_n^{Tq}(\Delta^2) = \int_0^1 dx\, x^{n-1} H^{Tq}(x, \Delta^2)$$

$$H^{Tq}(x,0) = \delta q(x)$$

↑ GFFs

↑ GPDs

$$\boxed{\frac{1}{2}\left(A_2^q(0) + B_2^q(0)\right) = J^q}$$

Ji

$$A_1^q(\Delta^2) = F_1^q(\Delta^2)$$

$$B_1^q(\Delta^2) = F_2^q(\Delta^2)$$

$$\tilde{A}_1^q(\Delta^2) = g_A^q(\Delta^2)$$

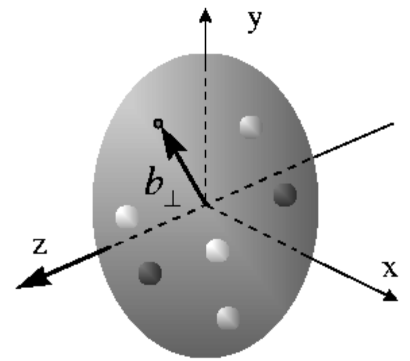
$$A_1^{Tq}(\Delta^2) = g_T^q(\Delta^2)$$

$$\Delta^2 = t = -Q^2$$

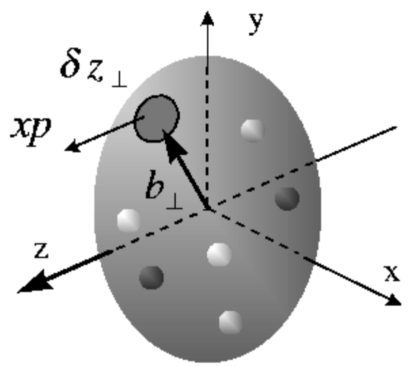
Impact Parameter Space

Generically

$$A_n^q(\mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} A_n^q(\mathbf{\Delta}_\perp^2)$$



$$H^q(x, \mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} \mathrm{e}^{\,\mathrm{i}\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} H^q(x, \mathbf{\Delta}_\perp^2)$$



Probability interpretation

Burkardt

$$H^q(x,\Delta^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\Delta^2\right) q(y)$$

Similarly for $\tilde{H}^q$ and $H^{T\,q}$

$$\int_0^1 dx\, x^n C(x,\Delta^2) = \frac{A_{n+1}(\Delta^2)}{A_{n+1}(0)} \overset{e.g.}{=} \frac{1}{(1-\Delta^2/M_n^2)^p}$$

By inverse Mellin transform

$$H^q(x,\mathbf{b}_\perp^2) = \int_x^1 \frac{dy}{y} C\left(\frac{x}{y},\mathbf{b}_\perp^2\right) q(y)$$

$$C(x,\mathbf{b}_\perp^2) = \int \frac{d^2\mathbf{\Delta}_\perp}{(2\pi)^2} e^{i\,\mathbf{b}_\perp \mathbf{\Delta}_\perp} C(x,\mathbf{\Delta}_\perp^2)$$

Nucleon

$N_f = 2$ 'Valence' quark distributions

Form Factors

Sachs form factors

$$F_1(\Delta^2) = A_1(\Delta^2)$$

$$F_2(\Delta^2) = B_1(\Delta^2)$$

$$G_e(\Delta^2) = F_1(\Delta^2) + \frac{\Delta^2}{4m_N^2} F_2(\Delta^2)$$

$$G_m(\Delta^2) = F_1(\Delta^2) + F_2(\Delta^2)$$

$$F_1(0) = e^N$$

$$F_2(0) = \mu^N - e^N = \kappa^N$$

$$G_e(0) = e^N$$

$$G_m(0) = \mu^N = 1 + \kappa^N$$

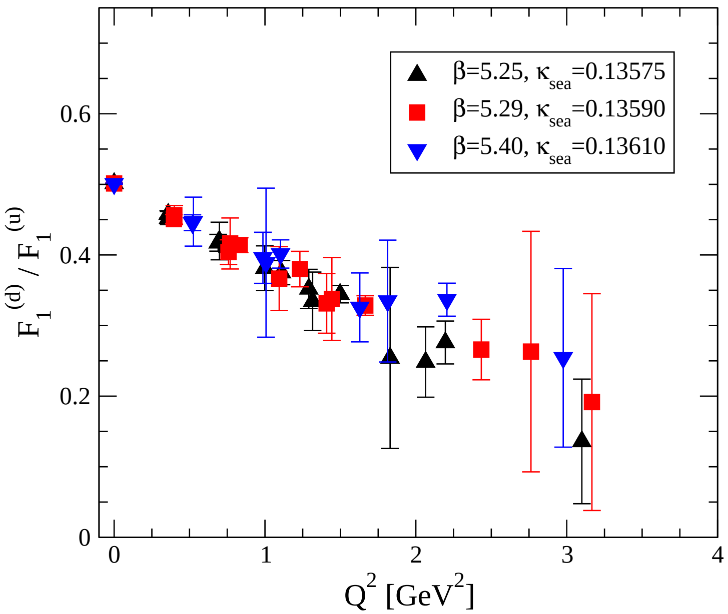
Benchmark calculation

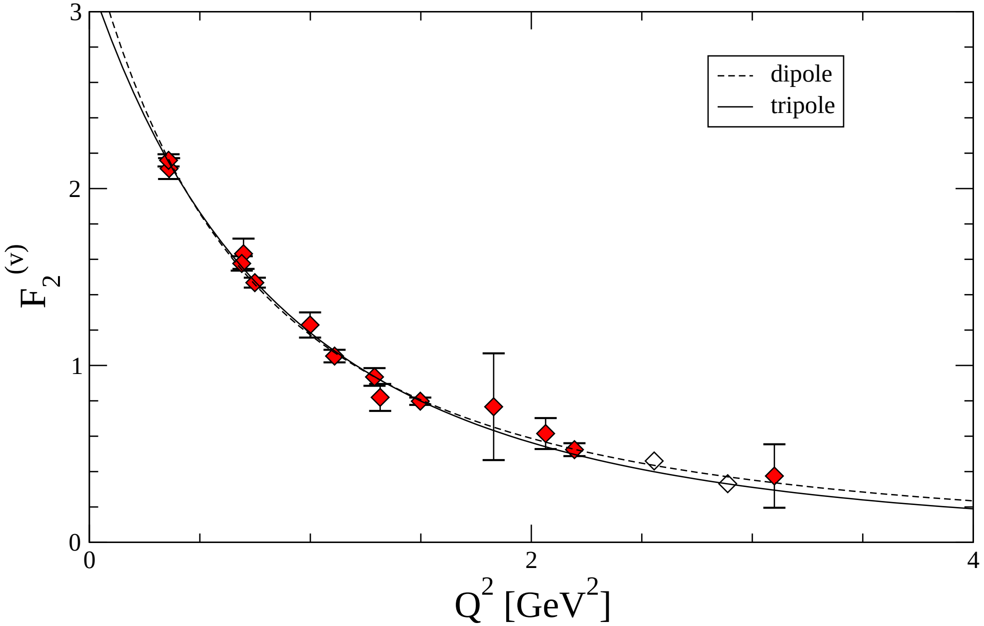
Expect (dimensional counting)

$$F_1(Q^2) \propto \frac{1}{(Q^2)^2}$$

$$F_2(Q^2) \propto \frac{1}{(Q^2)^3}$$

$$Q^2 = -\Delta^2$$





$$F^p = \frac{2}{3}F^u - \frac{1}{3}F^d$$

$$F^n = -\frac{1}{3}F^u + \frac{2}{3}F^d$$

$$F^v = F^u - F^d$$

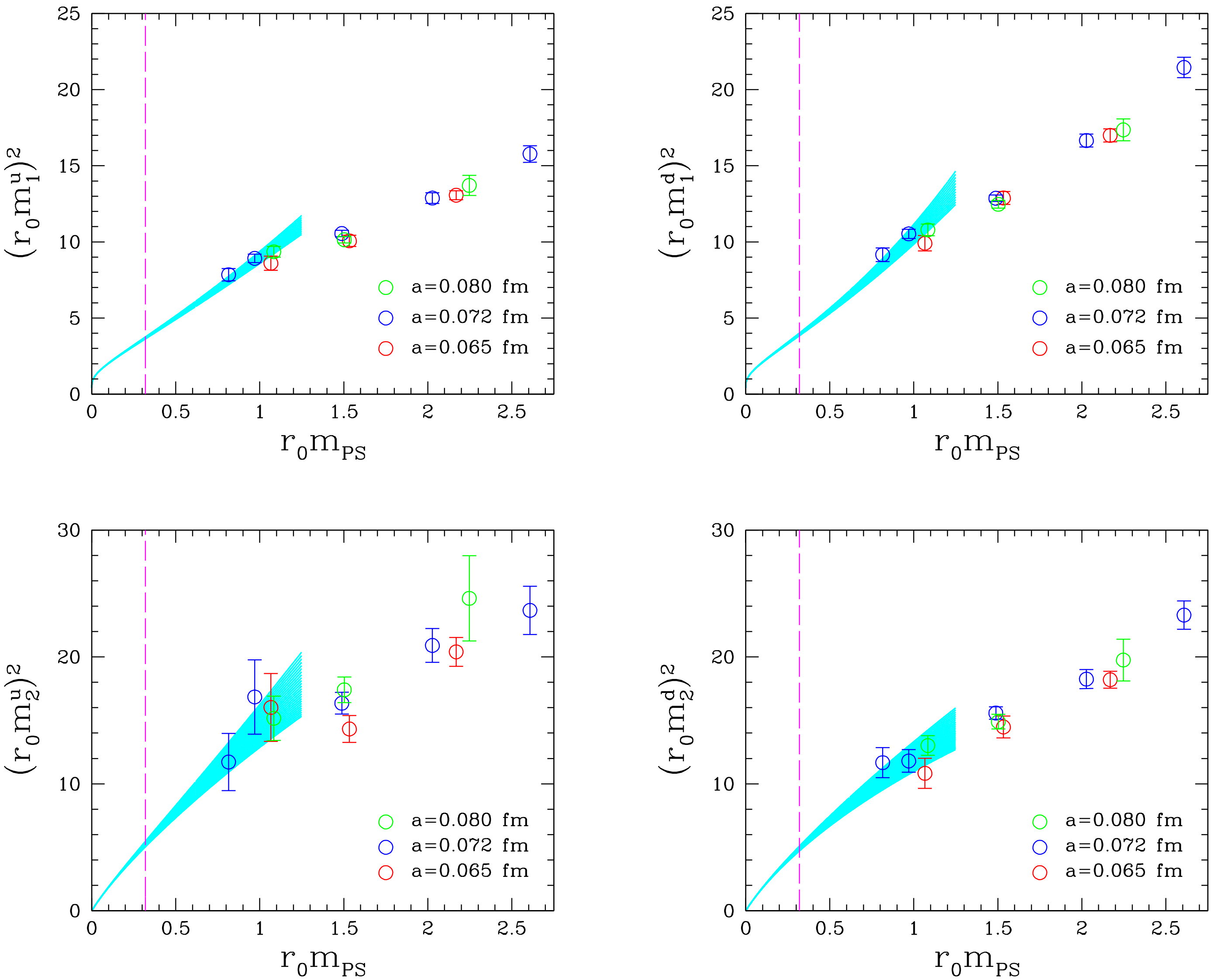
Pole ansatz

$$\boxed{F(Q^2) = F(0)\big(1 + Q^2/m^2\big)^{-n}}$$

$F_1^u \qquad n = 2$

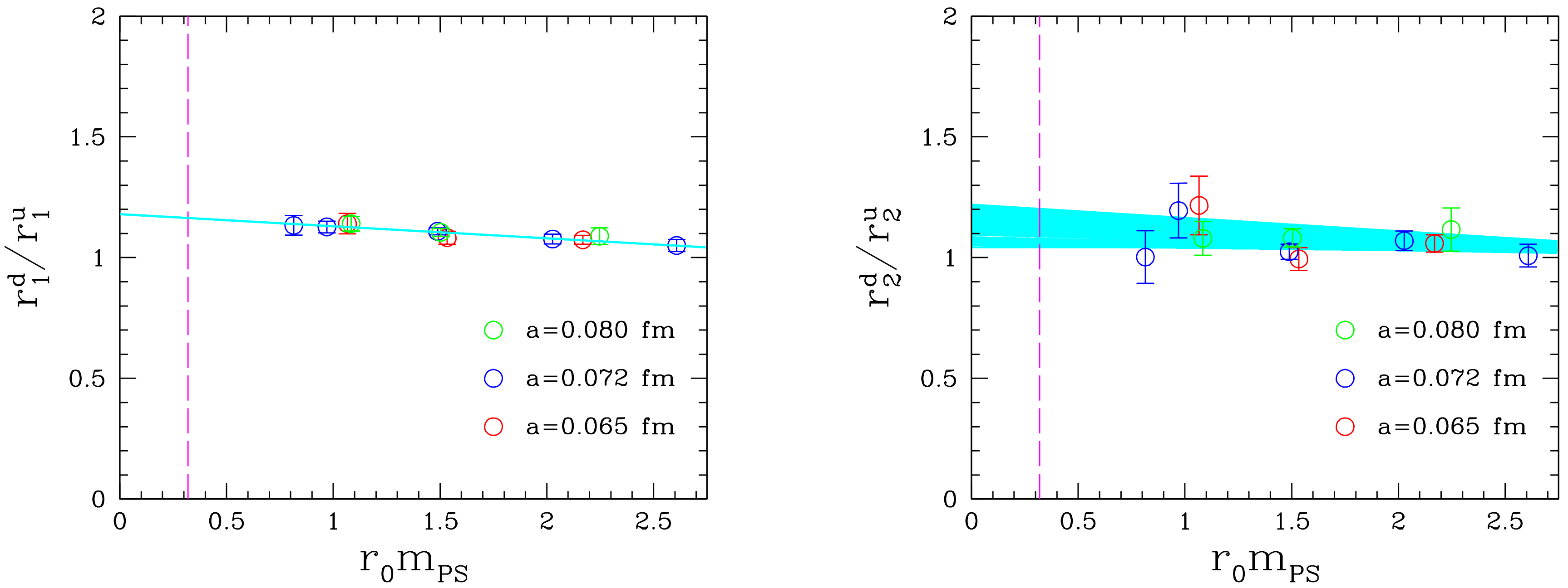
$$\left.\begin{matrix} F_1^d \\ F_2^{u,d} \end{matrix}\right\} \quad n = 3$$

Chiral extrapolation



$$F_i(Q^2) = F_i(0)\big(1 - \frac{1}{6} r_i^2 Q^2 + O(Q^4)\big)$$

$$r_i^2 = 6\,n/m_i^2$$



$$r_{1,2}^d > r_{1,2}^u$$

ChPT

$$r_1^2 = -\frac{1}{(4\pi F_\pi)^2}\left\{1 + 7g_A^2 + (10g_A^2 + 2)\log\left[\frac{m_\pi}{\lambda}\right]\right\} - \frac{12B_{10}^{(r)}(\lambda)}{(4\pi F_\pi)^2}$$
$$+ \frac{c_A^2}{54\pi^2 F_\pi^2}\left\{26 + 30\log\left[\frac{m_\pi}{\lambda}\right] + 30\frac{\Delta}{\sqrt{\Delta^2 - m_\pi^2}}\log R(m_\pi)\right\}$$

$$r_2^2 = \frac{g_A^2 m_N}{8F_\pi^2 \kappa \pi m_\pi} + \frac{c_A^2 m_N}{9F_\pi^2 \kappa \pi^2 \sqrt{\Delta^2 - m_\pi^2}}\log R(m_\pi) + \frac{24 m_N}{\kappa} B_{c2}$$

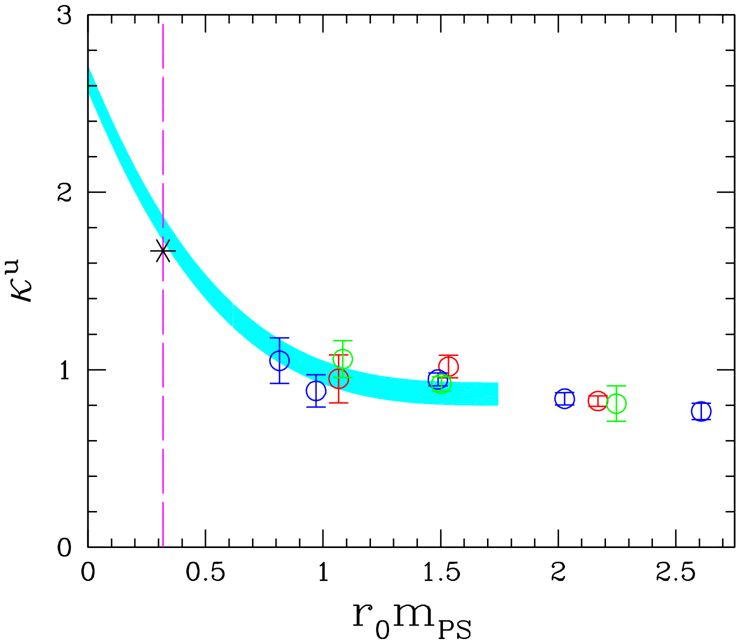
$$R(m) = \frac{\Delta}{m} + \sqrt{\frac{\Delta^2}{m^2} - 1}$$

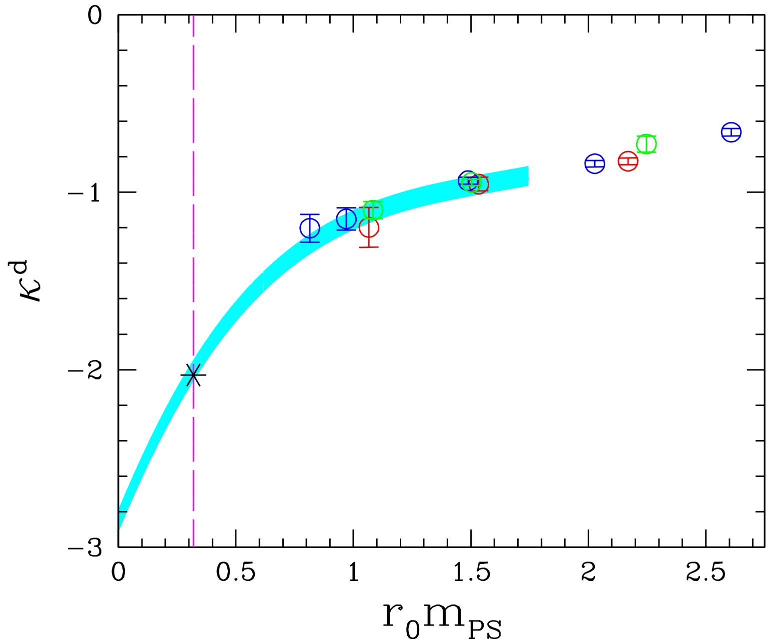
PRD 71, 034508 (2005)

Radii

[fm^2]	Lattice	Experiment	ChPT
$(r_1^u)^2$	0.67(3)		
$(r_1^d)^2$	0.93(4)		
$(r_1^v)^2$	0.41(5)	0.58	0.71
$(r_2^u)^2$	0.69(3)		
$(r_2^d)^2$	0.74(5)		
$(r_2^v)^2$	0.72(6)	0.80	0.60

Chiral extrapolation (ctd.)





$$\kappa^p = \frac{2}{3}\kappa^u - \frac{1}{3}\kappa^d$$

$$\kappa^n = -\frac{1}{3}\kappa^u + \frac{2}{3}\kappa^d$$

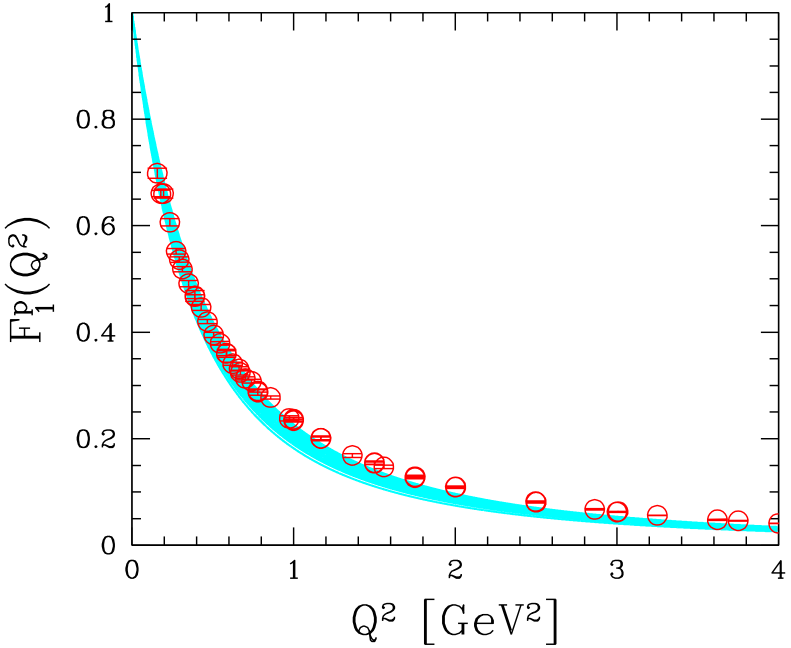
ChPT

$$\kappa(m_\pi) = \kappa_v^0 - \frac{g_A^2\, m_\pi M_N}{4\pi F_\pi^2} + \frac{2c_A^2 \Delta M_N}{9\pi^2 F_\pi^2} \left\{ \sqrt{1 - \frac{m_\pi^2}{\Delta^2}} \log R(m_\pi) + \log \left[\frac{m_\pi}{2\Delta}\right] \right\}$$

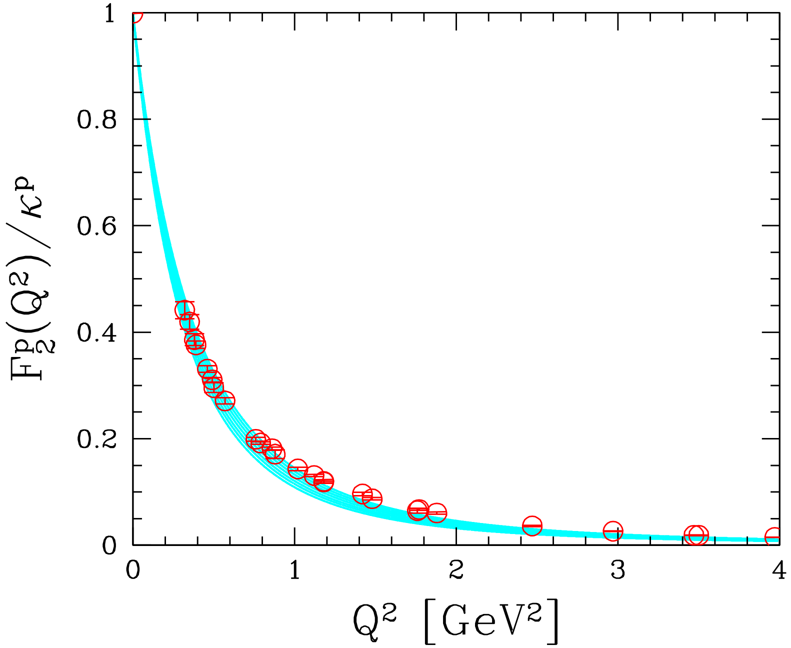
$$- 8 E_1^{(r)}(\lambda) M_N m_\pi^2 + \frac{4 c_A c_V g_A M_N m_\pi^2}{9\pi^2 F_\pi^2} \log \left[\frac{2\Delta}{\lambda}\right] + \frac{4 c_A c_V g_A M_N m_\pi^3}{27\pi F_\pi^2 \Delta}$$

$$- \frac{8 c_A c_V g_A \Delta^2 M_N}{27\pi^2 F_\pi^2} \left\{ \left(1 - \frac{m_\pi^2}{\Delta^2}\right)^{3/2} \log R(m_\pi) + \left(1 - \frac{3m_\pi^2}{2\Delta^2}\right) \log \left[\frac{m_\pi}{2\Delta}\right] \right\}$$

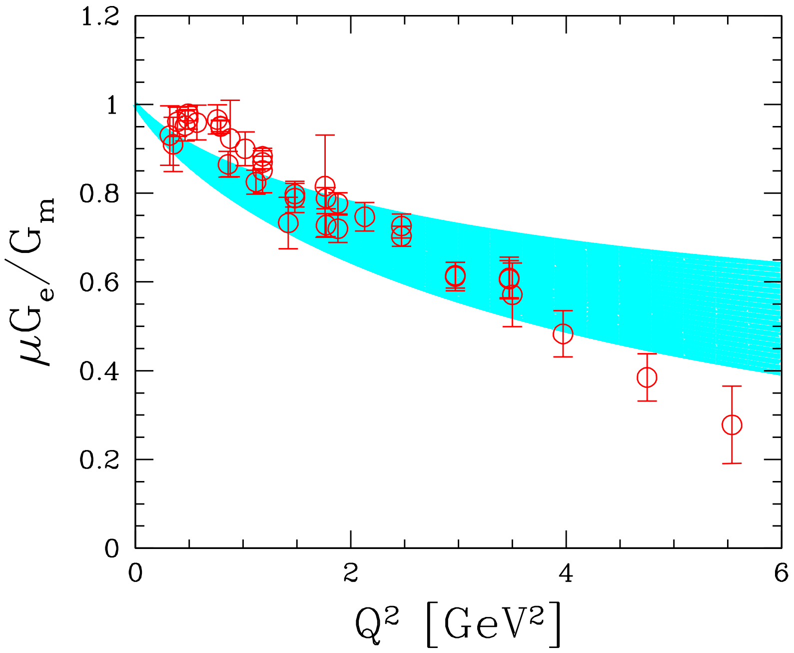
Finally

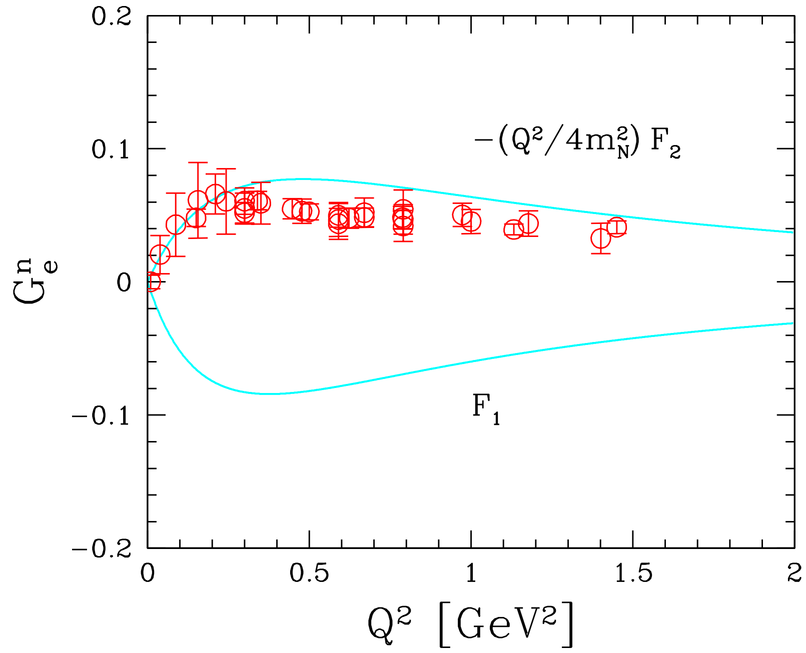


$F_1^p(Q^2)$
Q^2 [GeV^2]
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4



$F_2^p(Q^2)/\kappa^p$
Q^2 [GeV^2]
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4





Very preliminary

Multi-pole ansatz

$$F(Q^2) = F(0)\big(1 + c_1 Q^2 + c_n Q^{2n}\big)^{-1}$$

$F_1^{u,d}$ $\quad n = 2$

$F_2^{u,d}$ $\quad n = 3$

- Fulfills superconvergence relations

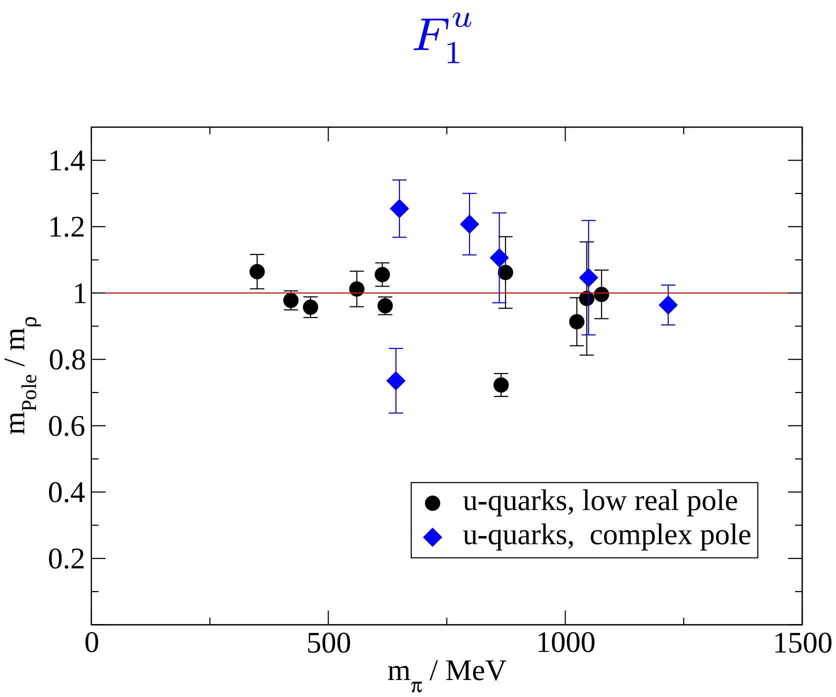
$$\int d\,Q^2\,\mathrm{Im}\,F_{1,2}(Q^2) = 0\,, \quad \int d\,Q^2\,Q^2\,\mathrm{Im}\,F_2(Q^2) = 0$$

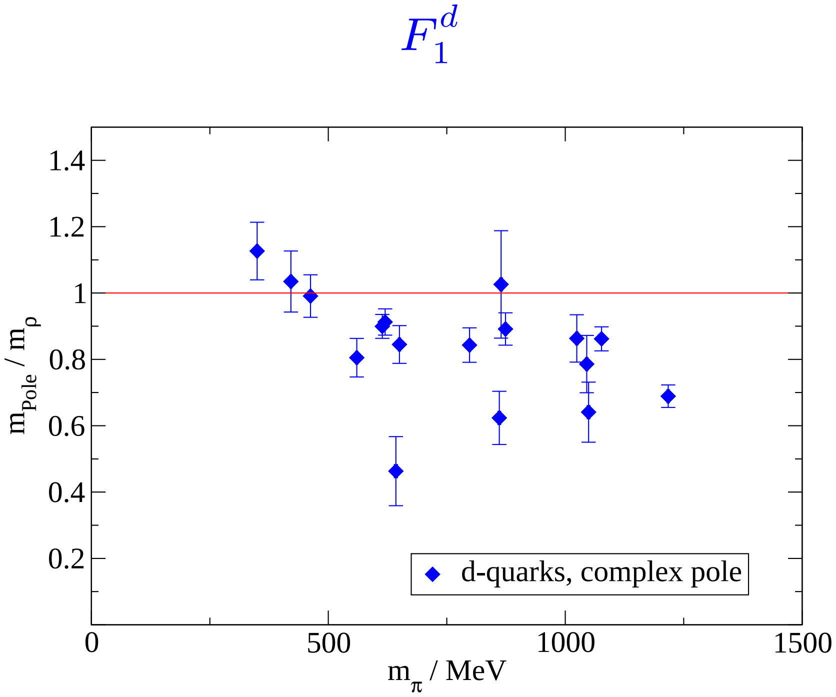
- Enforces pQCD behavior

- Exhibits effective resonance pole at small Q^2

Belushkin, Hammer & Meißner

Lowest pole mass



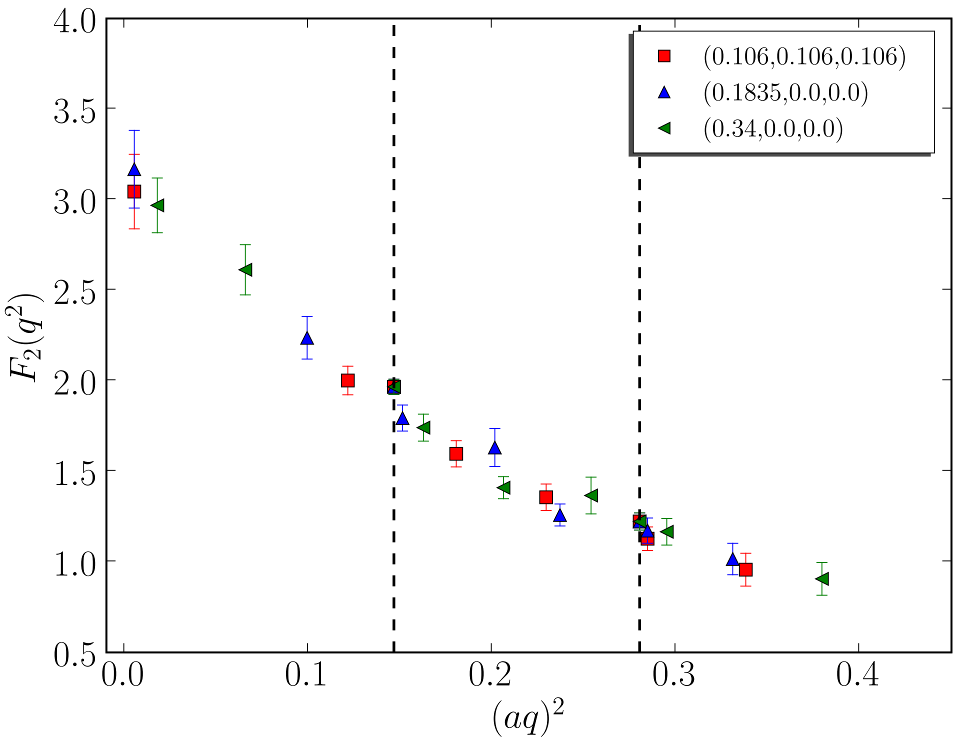


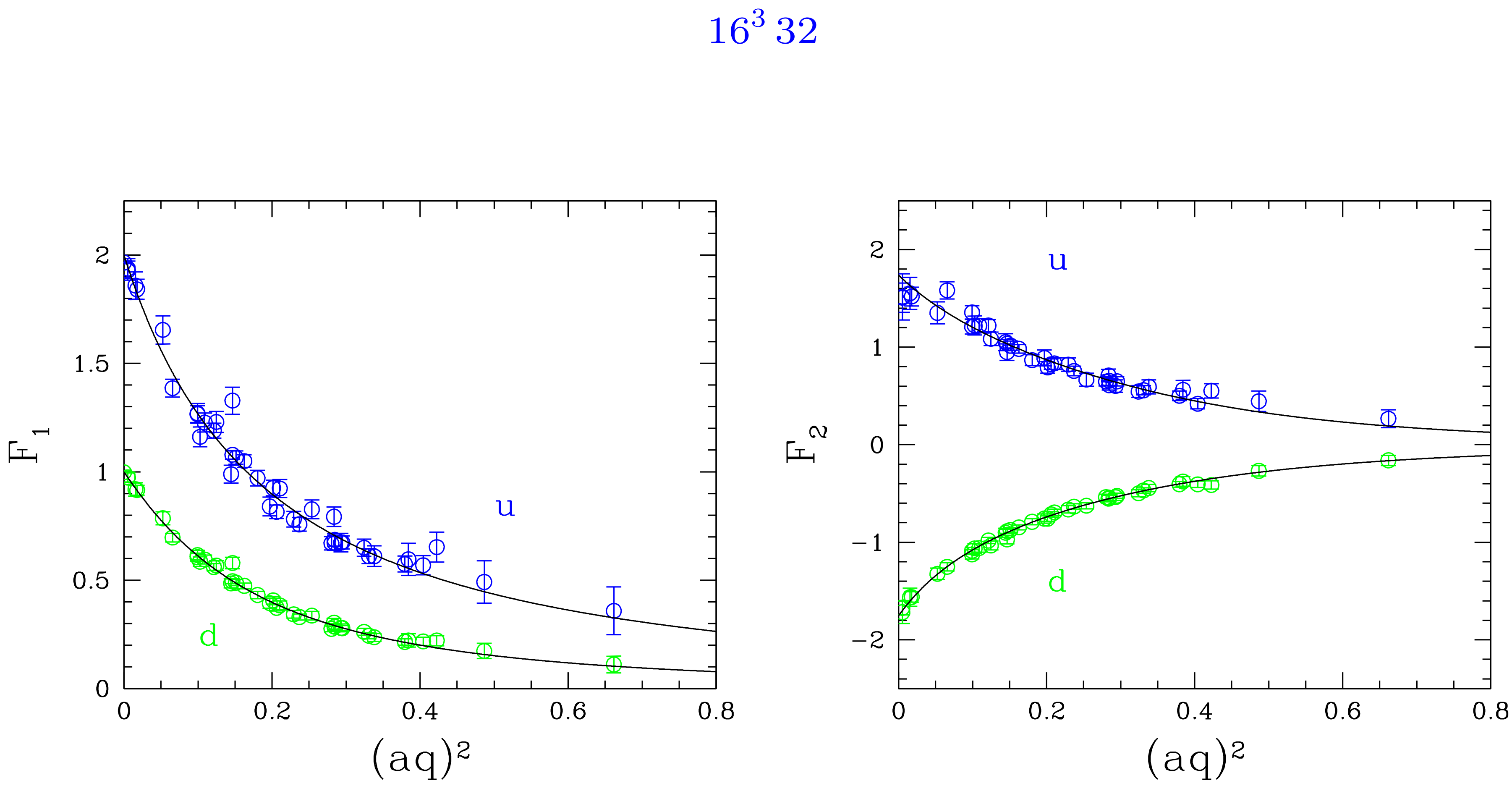
Twisted boundary conditions

With ordinary (periodic) boundary conditions momenta are quantized in units of $2\pi/L$. By imposing partially twisted boundary conditions

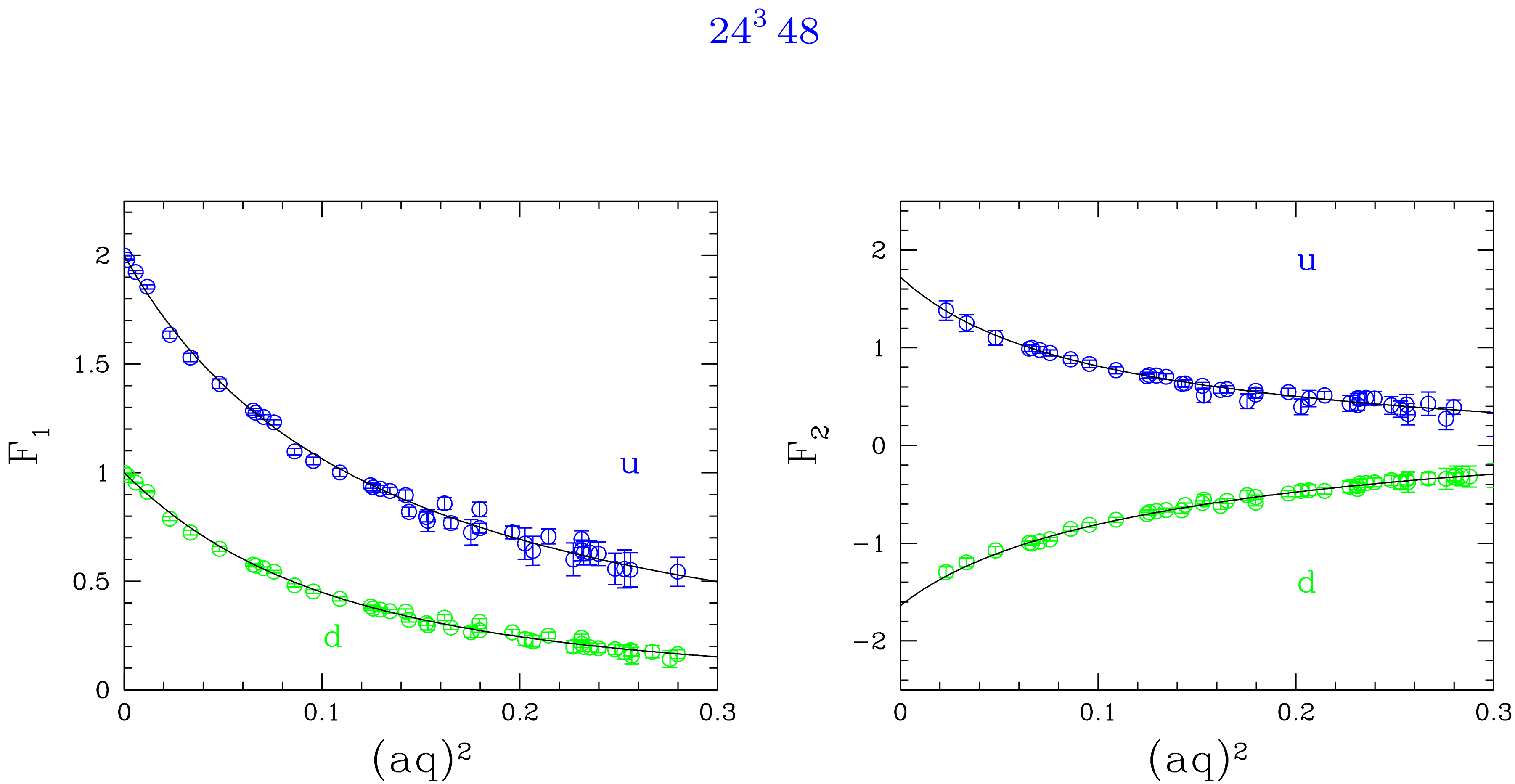
$$\psi(x_i + L) = e^{i\,\theta_i}\,\psi(x_i)\,,\ i = 1, 2, 3$$

on the valence quarks, one can tune the momenta of the nucleon continuously according to $\vec{p} \longrightarrow \vec{p} + \vec{\theta}$

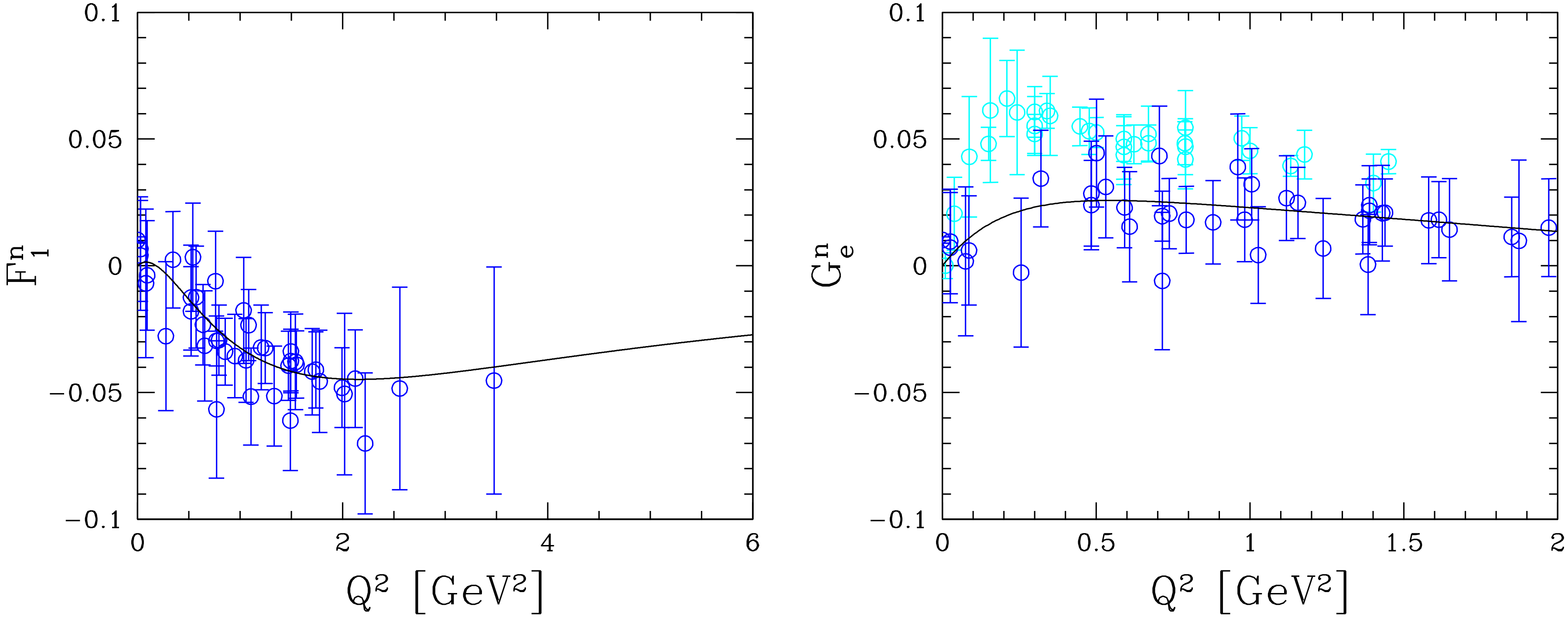




16³ 32
F₁
F₂
u
d
(aq)²
0
0.2
0.4
0.6
0.8
0.5
1
1.5
2
−2
−1



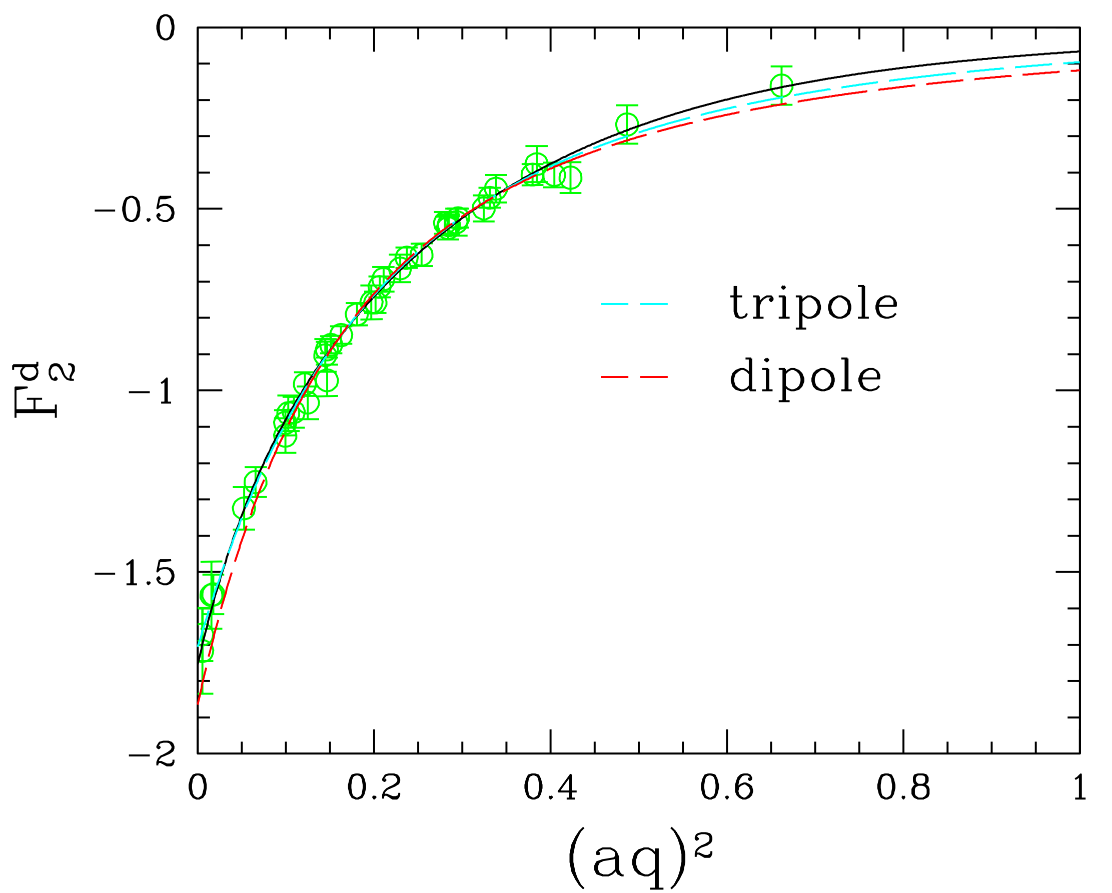
24³ 48
F 1
2
1.5
1
0.5
0
0.1
0.2
0.3
(aq)²
u
d
F 2
−1
−2



Preliminary

Proton	$\Longleftrightarrow$	Neutron
u	$\Longleftrightarrow$	d
$r_d > r_u$	$\Longleftrightarrow$	$\boxed{r_u > r_d}$

Significance of tbc data?



Spin Asymmetries

$\lambda_\perp$ quark spin
$s_\perp$ nucleon spin

Transverse spin density

$\Downarrow$

$$\langle p_+, s_\perp | \bar{q}(\mathbf{b}_\perp) \left[\gamma_+ - \lambda_{\perp i}\, \sigma_{+j}\, \gamma_5\right] q(\mathbf{b}_\perp) | p_+, s_\perp \rangle = \Big\{ A_1^q(\mathbf{b}_\perp^2) + \lambda_{\perp i}\, s_{\perp i} \Big[A_1^{Tq}(\mathbf{b}_\perp^2)$$

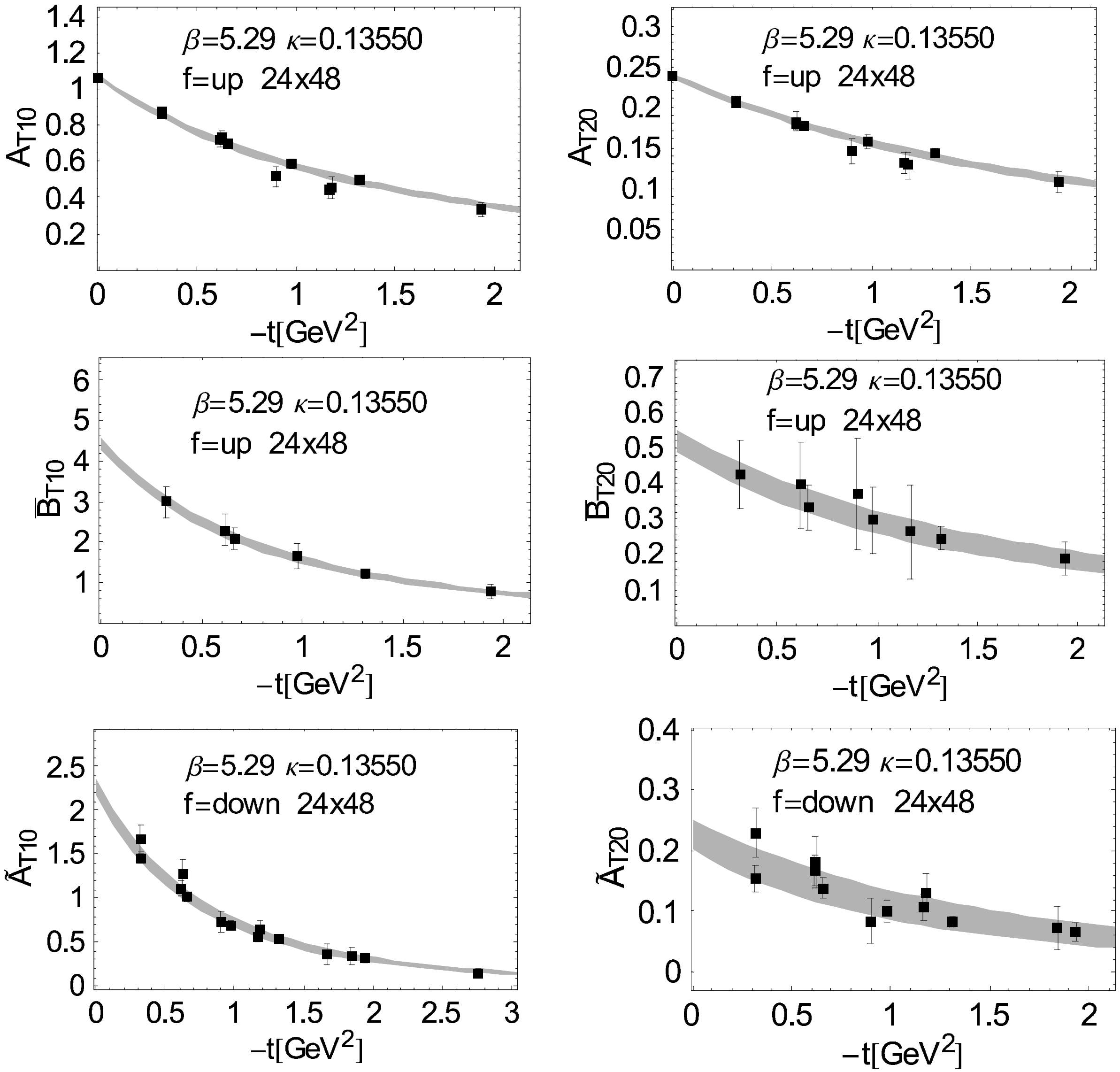
$$- \frac{1}{4m_N^2} \Delta_{b_\perp} \tilde{A}_1^{Tq}(\mathbf{b}_\perp^2) \Big] - \frac{1}{m_N} \epsilon_{ij} b_{\perp j} \Big[s_{\perp i} B_1^q(\mathbf{b}_\perp^2)' + \lambda_{\perp i}\, \bar{B}_1^{Tq}(\mathbf{b}_\perp^2)' \Big]$$

$$+ \frac{1}{m_N^2} \lambda_{\perp i} \left(2 b_{\perp i}\, b_{\perp j} - \mathbf{b}_\perp^2 \delta_{ij}\right) s_{\perp j}\, \tilde{A}_1^{Tq}(\mathbf{b}_\perp^2)'' \Big\}$$

$\Uparrow$

Quadrupole

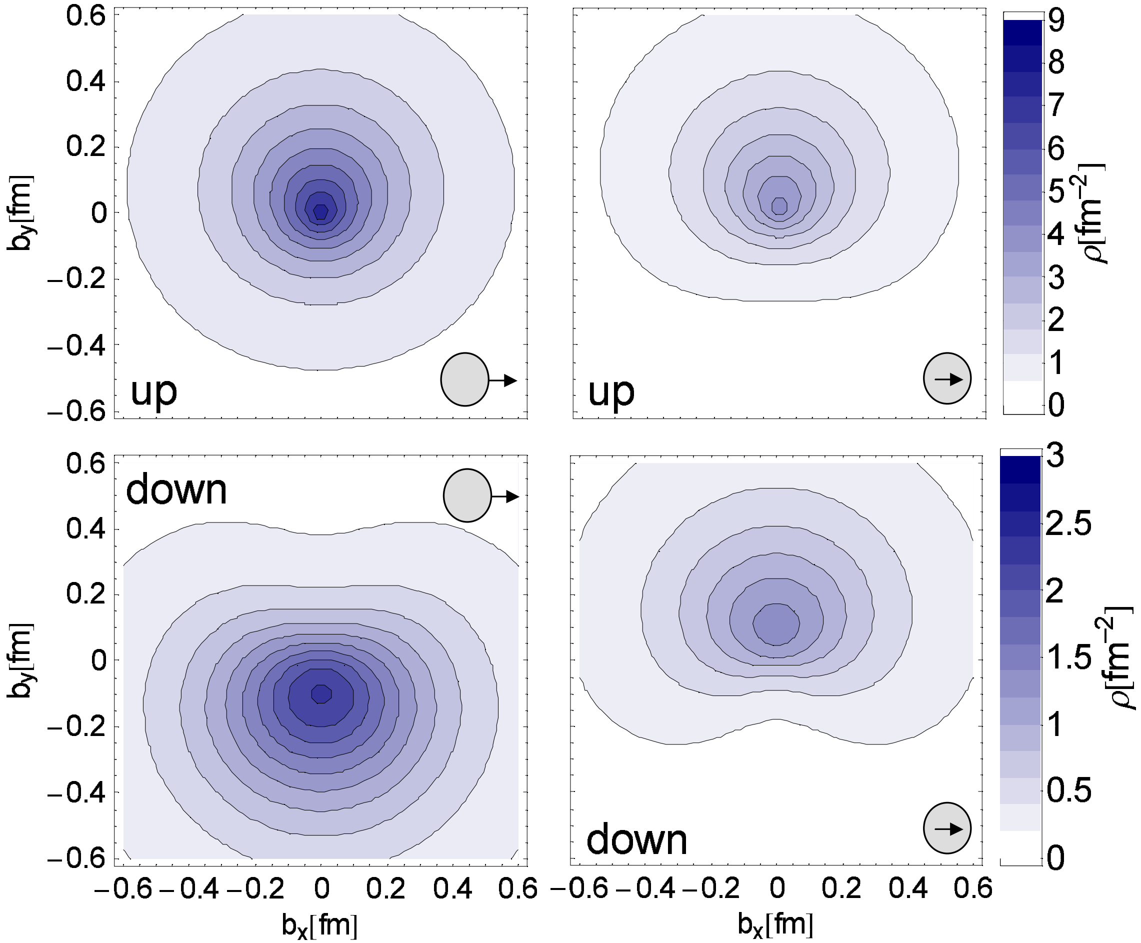
Diehl & Hägler



Dipole fit

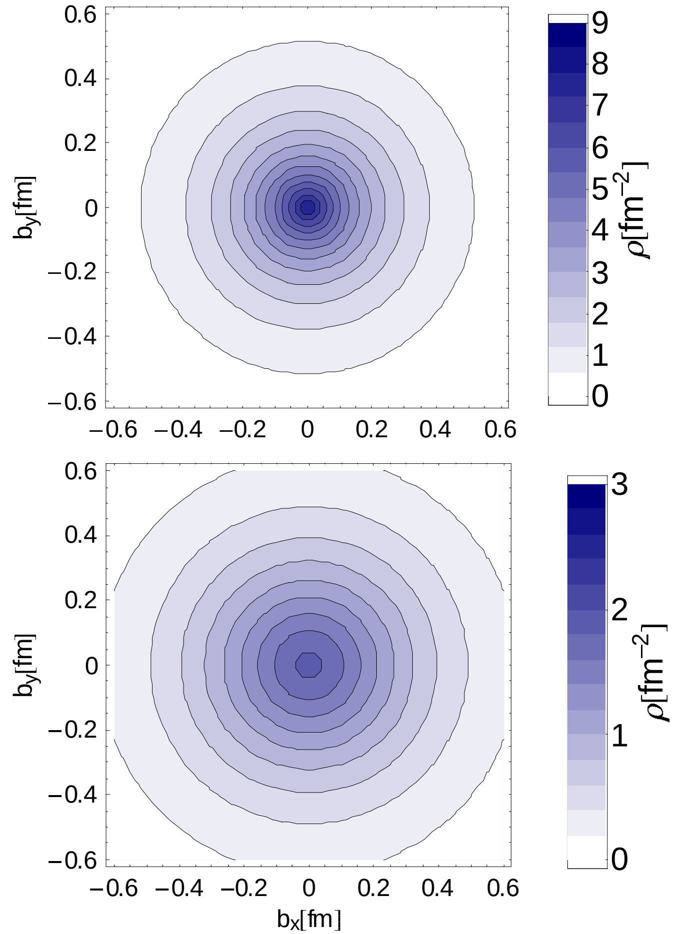
1^{st} moment

To be extrapolated to chiral limit



Sivers effect

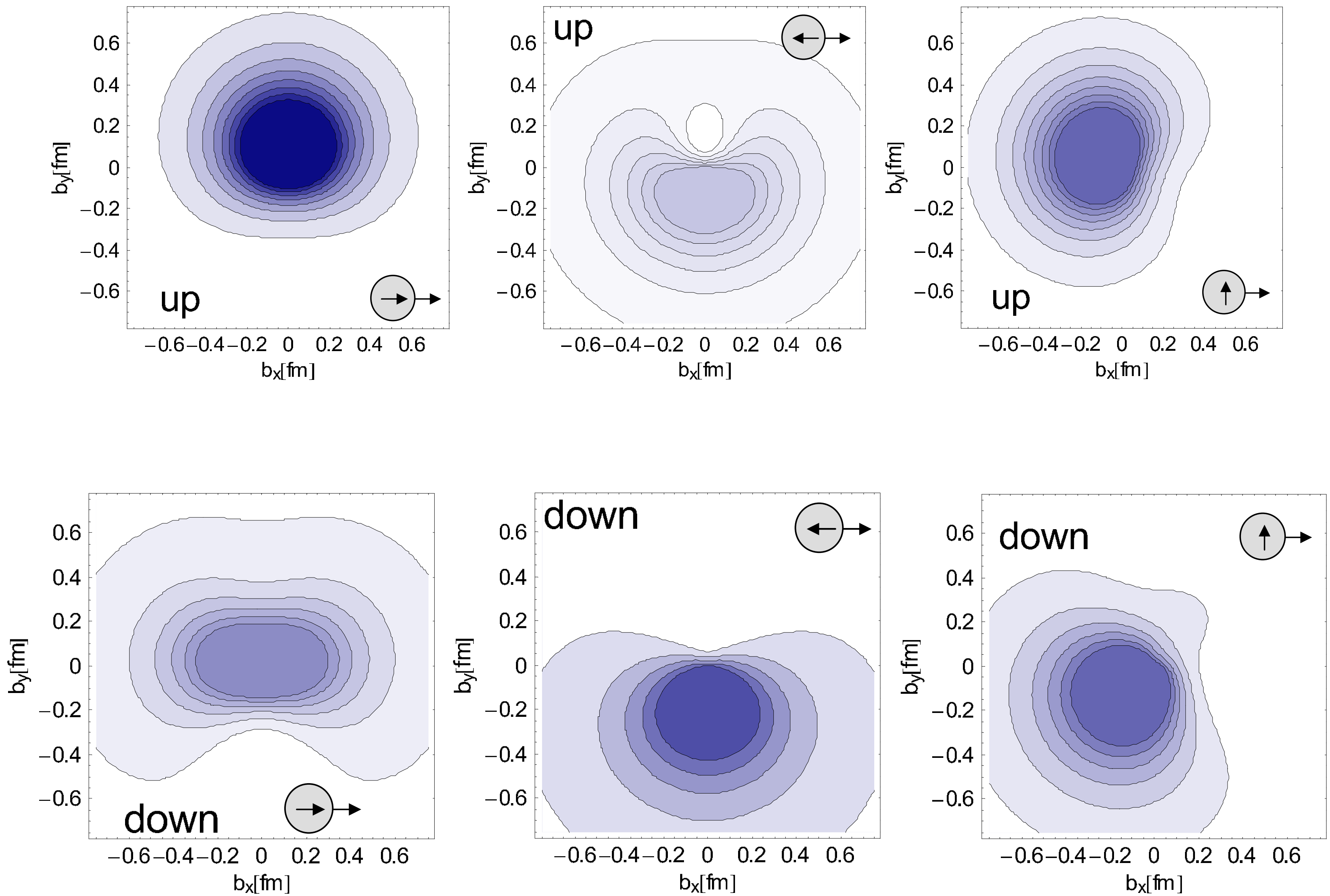
Boer-Mulders effect



Unpolarized

$$r_T^d > r_T^u$$

Nucleon and quarks both polarized



Pion

$N_f = 2$ ‘Valence’ quark distributions

Form Factor

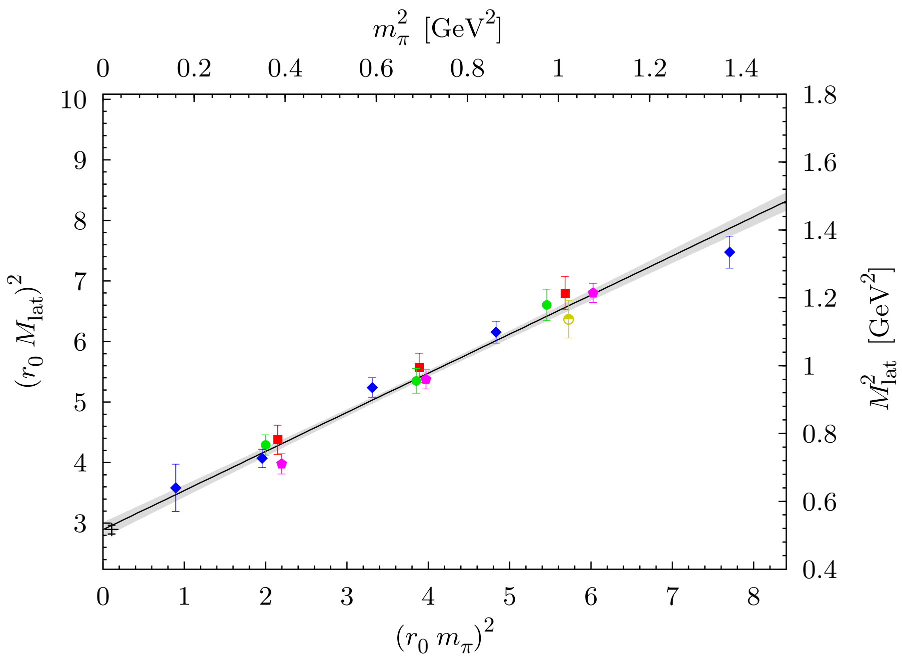
$$F^{\pi}(\Delta^2) = A_1(\Delta^2)$$

Expect (dimensional counting)

$$F^{\pi}(Q^2) \propto \frac{1}{Q^2} \qquad Q^2 = -\Delta^2$$

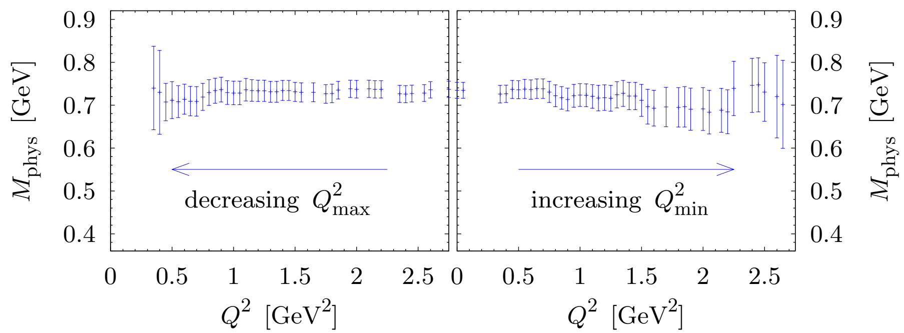
Ansatz

$$\boxed{F^{\pi}(Q^2) = 1/(1 + Q^2/m^2)^{-1}}$$



$$M_{\rm lat} = m$$

$$M_{\rm phys} = M_{\rm lat}\,(m_{\pi,\rm phys})$$

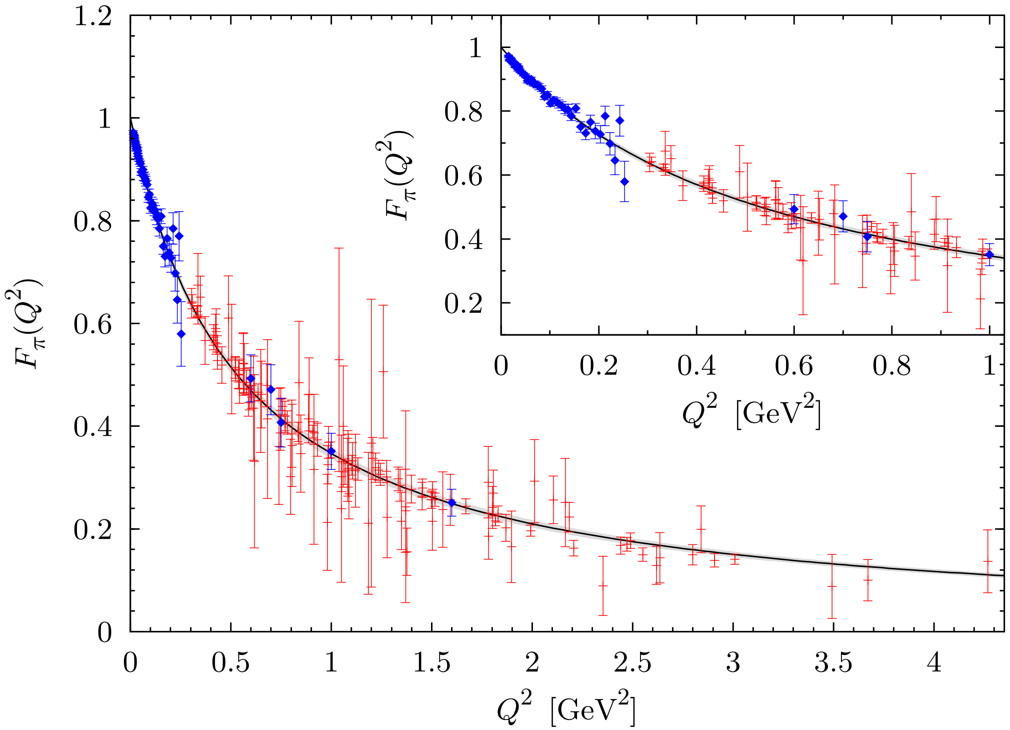


$\Leftarrow$ Stability of monopole fit

$M_{\rm phys} = 773(17)$ MeV $\quad Q^2 < 1$ GeV^2

VDM

$M_{\rm phys} = 729(16)$ MeV $\quad$ all Q^2



Experiment Lattice

⇑

Shifted by $(1 + Q^2/M^2_{\text{phys}})^{-1} - (1 + Q^2/M^2_{\text{lat}})^{-1}$

Spin Asymmetries

Transverse spin density

$\Downarrow$

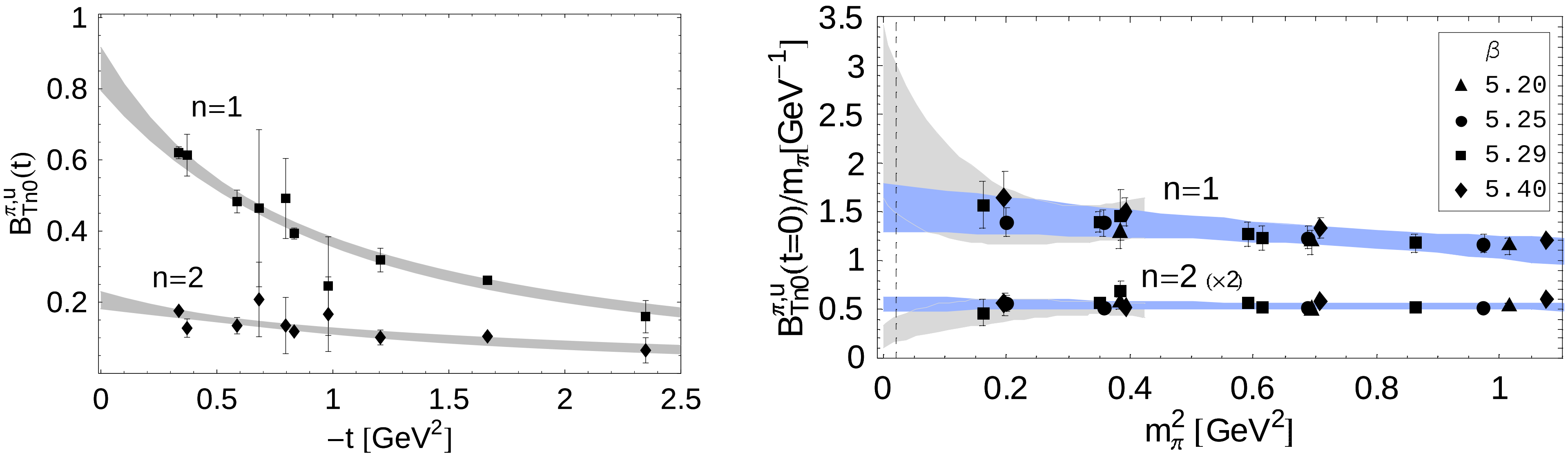
$\lambda_\perp$ quark spin

$$\langle p_+ | \bar{q}(\mathbf{b}_\perp) \left[\gamma_+ - \boldsymbol{\lambda}_{\perp i}\, \boldsymbol{\sigma}_{+j}\, \gamma_5 \right] q(\mathbf{b}_\perp) | p_+ \rangle = \left\{ A_1^q(\mathbf{b}_\perp^2) - \frac{1}{m_\pi} \epsilon_{ij} b_{\perp j} \boldsymbol{\lambda}_{\perp i}\, \bar{B}_1^{Tq}(\mathbf{b}_\perp^2)' \right\}$$

$\Uparrow$ Monopole

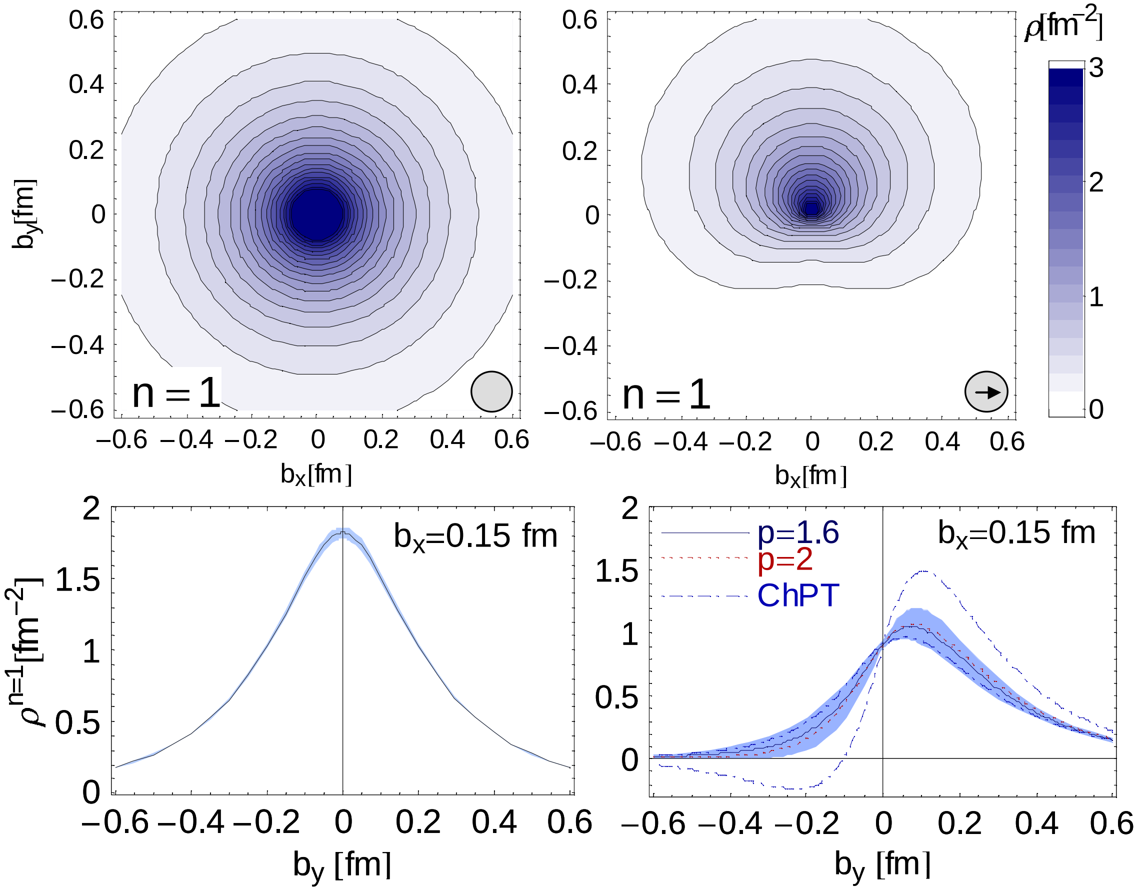
$\Uparrow$ Dipole

Hägler et al.



n=1
n=2
$B^{\pi,u}_{Tn0}(t)$
$-t$ [GeV^2]
β
5.20
5.25
5.29
5.40
n=1
n=2 (×2)
$B^{\pi,u}_{Tn0}(t=0)/m_\pi$[GeV^{-1}]
m_π^2 [GeV^2]

$p-$pole fit



Boer-Mulders effect

ChPT: Diehl, Manashov & Schäfer

Conclusions & Outlook

- Simulations at small pion masses m_π with Wilson-type fermions feasible

- Improvement of algorithms
- Increase of computing power

- Extrapolation to chiral limit and infinite volume greatly improved

FS corrections surprisingly well described by ChPT

Begin to resolve pion cloud (chiral logs)

- Twisted boundary conditions allow to tune hadron momenta continuously

- Challenge: Evaluation of disconnected diagrams

- Simulations with $N_f = 2 + 1$ SLiNC fermions down to $m_\pi \approx 200$ MeV under way

- 2008/9 : $a = 0.08$ fm
- 2009/10 : $a = 0.05$ fm

On spatial volumes $\gtrsim (4\,\text{fm})^3$