Third order relativistic dissipative hydrodynamics

arXiv:0907.4500

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P.Huovinen, D.Rischke, A.Muronga

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Outline

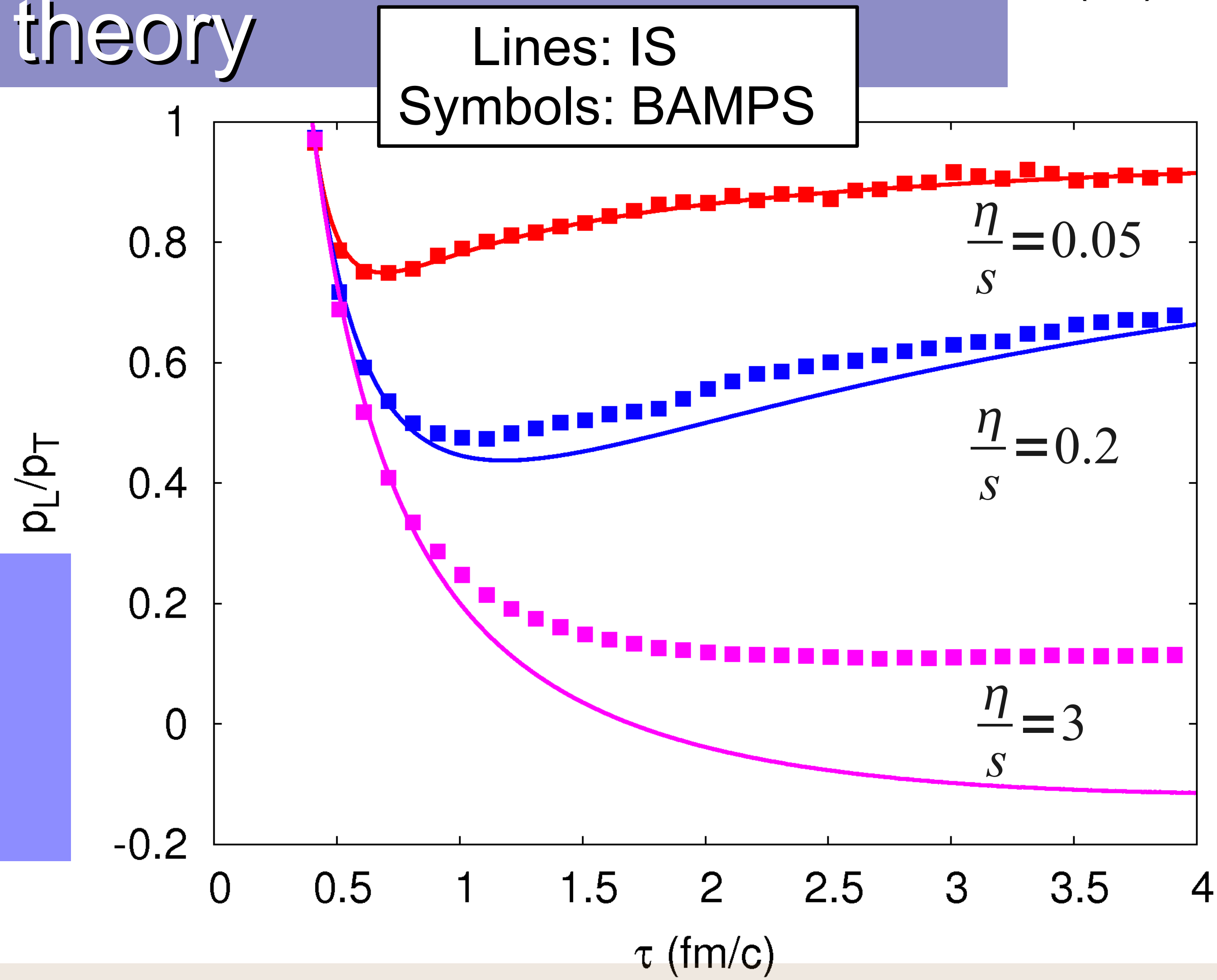
- Israel-Stewart equations for 1-Dim boost-invariant systems
- Third-order theory
- Higher-orders (1-Dim only)
- Comparison with kinetic transport

Israel-Stewart theory

$$p_L = p - \pi$$
$$p_T = p + \pi/2$$

P.Huovinen, D.Molnar, PRC 79 (2009) 014906.

M.Martinez, M.Strickland, PRC 79 (2009) 044903.



Troubles:

$$p_L = \frac{e}{3} - \pi < 0 \quad \Rightarrow \quad \pi > e/3$$

...and since $\dot{T} = -\frac{T}{3\tau} + \frac{\pi}{e}\frac{T}{\tau}$, reheating as soon as $\pi > e/3$

Third-order theory: entropy current

$$s^{\mu} = s_0 u^{\mu} - \beta_2 \pi_{\mu\nu} \pi^{\mu\nu} \frac{u^{\mu}}{2T} + \alpha \beta_2^2 \pi_{\alpha\beta} \pi^{\alpha}_{\sigma} \pi^{\beta\sigma} \frac{u^{\mu}}{T}$$

Israel-Stewart approach

Higher-order correction to Israel-Stewart.
Negative! (i.e. less entropy)

Third-order theory: entropy production

$$T\partial_\mu s^\mu = \pi_{\mu\nu}\left(\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\mu\nu} - \frac{1}{2}T\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu}\right) +$$

$$+\alpha T\pi_{\alpha\beta}\pi^\alpha_\sigma\pi^{\beta\sigma}\partial_\mu\left(\beta_2^2\frac{u^\mu}{T}\right) + \underbrace{3\alpha\beta_2^2\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma}}$$

$$3\alpha\beta_2^2\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma} = \underbrace{3(1-\theta\tau_\pi)\alpha\beta_2^2\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma}} + 3\theta\tau_\pi\alpha\beta_2^2\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma}$$

$\theta\tau_\pi > 1 \Rightarrow$ strong dissipation , π increasing

$\theta\tau_\pi < 1 \Rightarrow$ π decreases, relaxation

This term is thus always positive!

Third-order theory: second law

$$T\partial_\mu s^\mu = \pi_{\mu\nu}\left(\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\mu\nu} - \frac{1}{2}T\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu}\right) + \alpha T\pi_{\alpha\beta}\pi^\alpha_\sigma\pi^{\beta\sigma}\partial_\mu\left(\beta_2^2\frac{u^\mu}{T}\right) +$$

$$+3\alpha\beta_2^2\theta\tau_\pi\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma} + \underbrace{3\alpha\beta_2^2\left(1-\theta\tau_\pi\right)\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma}}_{\text{always} > 0} \geqslant 0$$

$$\pi_{\mu\nu}\left(\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\mu\nu} - \frac{1}{2}T\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu} + \alpha T\pi^{<\mu}_\sigma\pi^{\nu>\sigma}\partial_\mu\left(\beta_2^2\frac{u^\mu}{T}\right) + 3\alpha\beta_2^2\theta\tau_\pi\pi^{<\mu}_\sigma\dot{\pi}^{\nu>\sigma}\right) \geqslant 0$$

$$\pi^{\mu\nu} = 2\eta\left(\underbrace{\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\mu\nu} - \frac{1}{2}T\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu}}_{\text{Israel-Stewart eq.}} + \alpha T\pi^{<\mu}_\sigma\pi^{\nu>\sigma}\partial_\mu\left(\beta_2^2\frac{u^\mu}{T}\right) + 3\alpha\beta_2^2\theta\tau_\pi\pi^{<\mu}_\sigma\dot{\pi}^{\nu>\sigma}\right)$$

Since then $$T\partial_\mu s^\mu = \frac{1}{2\eta}\pi_{\mu\nu}\pi^{\mu\nu} + 3\alpha\beta_2^2\left(1-\theta\tau_\pi\right)\pi_{\alpha\beta}\pi^\alpha_\sigma\dot{\pi}^{\beta\sigma} \geqslant 0$$

Third-order theory: what are α and β_2 ?

Third order ansatz:

(1) $$s = s_0 - \frac{\beta_2}{2T}\pi_{\mu\nu}\pi^{\mu\nu} + \alpha\frac{\beta_2^2}{T}\pi_{\alpha\beta}\pi^{\alpha}_{\sigma}\pi^{\beta\sigma}$$

Kinetic theory:

(2) $$s = \int u_\mu p^\mu f\left(\ln f - 1\right)\frac{d^3p}{p_0}$$

Taking Grad's approximation for *f*

$$f(x,p) = f_0\left(1+\phi\right)$$ with $$\phi(x,p) = C_0\pi_{\mu\nu}p^\mu p^\nu = C_0\pi\left(\frac{1}{2}p_T^2 - p_z^2\right)$$

and expanding the log in (2) up to third order in Φ we obtain:

$$s = s_0 - \frac{27}{16}\frac{\pi^2}{eT} - \frac{27}{8}\frac{\pi^3}{e^2T}$$

compare to (1) →

$$\beta_2 = \frac{9}{4e} \qquad \alpha = -\frac{8}{9}$$

Third-order theory: 1-Dim expansion

$$\pi_{\mu\nu} = 2\eta\left(\sigma^{\mu\nu} - \beta_2\dot{\pi}^{\mu\nu} - \frac{1}{2}T\,\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu} + \alpha T\,\pi_\sigma^{<\mu}\pi^{\nu>\sigma}\partial_\mu\left(\beta_2^2\frac{u^\mu}{T}\right) + 3\alpha\beta_2^2\theta\tau_\pi\pi_\sigma^{<\mu}\dot{\pi}^{\nu>\sigma}\right)$$

$$v = \frac{z}{t} \longrightarrow u^\mu = (\cosh\eta, 0, 0, \sinh\eta) \longrightarrow \partial_\mu u^\mu = 1/\tau$$

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - 3\frac{\pi^2}{e\tau} - 3\frac{\tau_\pi}{\tau}\frac{\pi}{e}\dot{\pi}$$

Multiplying both sides with τ_π/e:

$$\tau_\pi\frac{\dot{\pi}}{e} = -\frac{\pi}{e} - \frac{4}{3}\frac{\pi}{e}\frac{\tau_\pi}{\tau} + \frac{8}{27}\frac{\tau_\pi}{\tau} - 3\frac{\pi^2}{e^2}\frac{\tau_\pi}{\tau} - 3\frac{\tau_\pi^2}{\tau}\dot{\pi}\frac{\pi}{e^2}$$

$$O(2) \quad O(1) \quad O(2) \quad O(1) \quad O(3) \quad O(4) \longrightarrow \text{neglect!}$$

All orders of corrections in 1-D

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - 3\frac{\pi^2}{e\tau}$$

Third order equation for 1-Dim system

Way to estimate all higher-order corrections:

We make the *ansatz*

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - x\frac{\pi^2}{e\tau}$$

x contains all corrections (third order and higher)

If all corrections are included, it should be possible to solve the $\sigma = 0$, $\eta \to \infty$ limit !

I.e., free streaming: $\dot{e} = -\frac{e}{\tau}$ and $\pi = p = \frac{e}{3}$ (i.e. $p_L = 0$) $\Rightarrow \dot{\pi} = -\frac{\dot{e}}{3}$

$\Rightarrow$ $x = \frac{5}{3}$

Third-order theory: 1-Dim expansion

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - 3\frac{\pi^2}{e\tau}$$

Third order equation for 1-Dim system

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - \frac{5}{3}\frac{\pi^2}{e\tau}$$

"*All orders*" equation for 1-Dim system

$$\dot{e} = -\frac{4}{3}\frac{e}{\tau} + \frac{\pi}{\tau}$$

$$\dot{n} = -\frac{n}{\tau}$$

Hydro vs BAMPS

BAMPS: Z.Xu, C.Greiner, PRC 71 (2005) 064901; Talks by I.Bouras, Z.Xu

BAMPS :

$$\eta = \frac{6}{5}\frac{T}{\sigma_{22}} \longrightarrow \sigma_{22} = \frac{6}{5}\frac{T}{\frac{\eta}{s}\cdot(4n - n\ln\lambda)}$$

Isotropic elastic cross section, thermal initial conditions, ideal EoS

Solutions

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - x\frac{\pi^2}{e\tau}$$

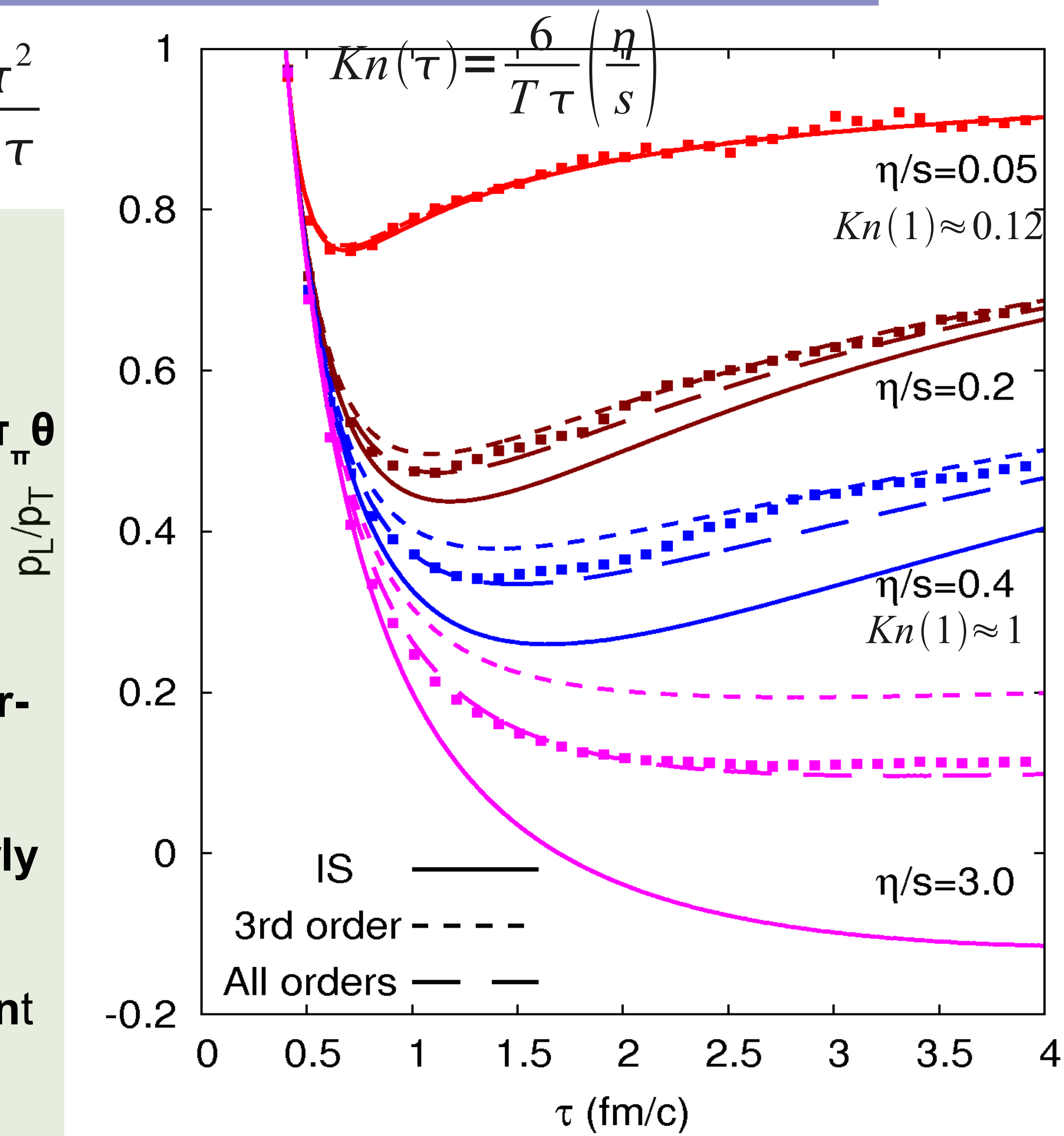
"All-oders" *ansatz* fits very good at early times

x may be dependend on Kn=$\tau_\pi\theta$

Third order is much closer to kinetic transport than IS

Oscillating behavior of higher-oder corrections?

Third order: deviations at early times
(dissipation at largest there)
→ higher orders still important



Initial time

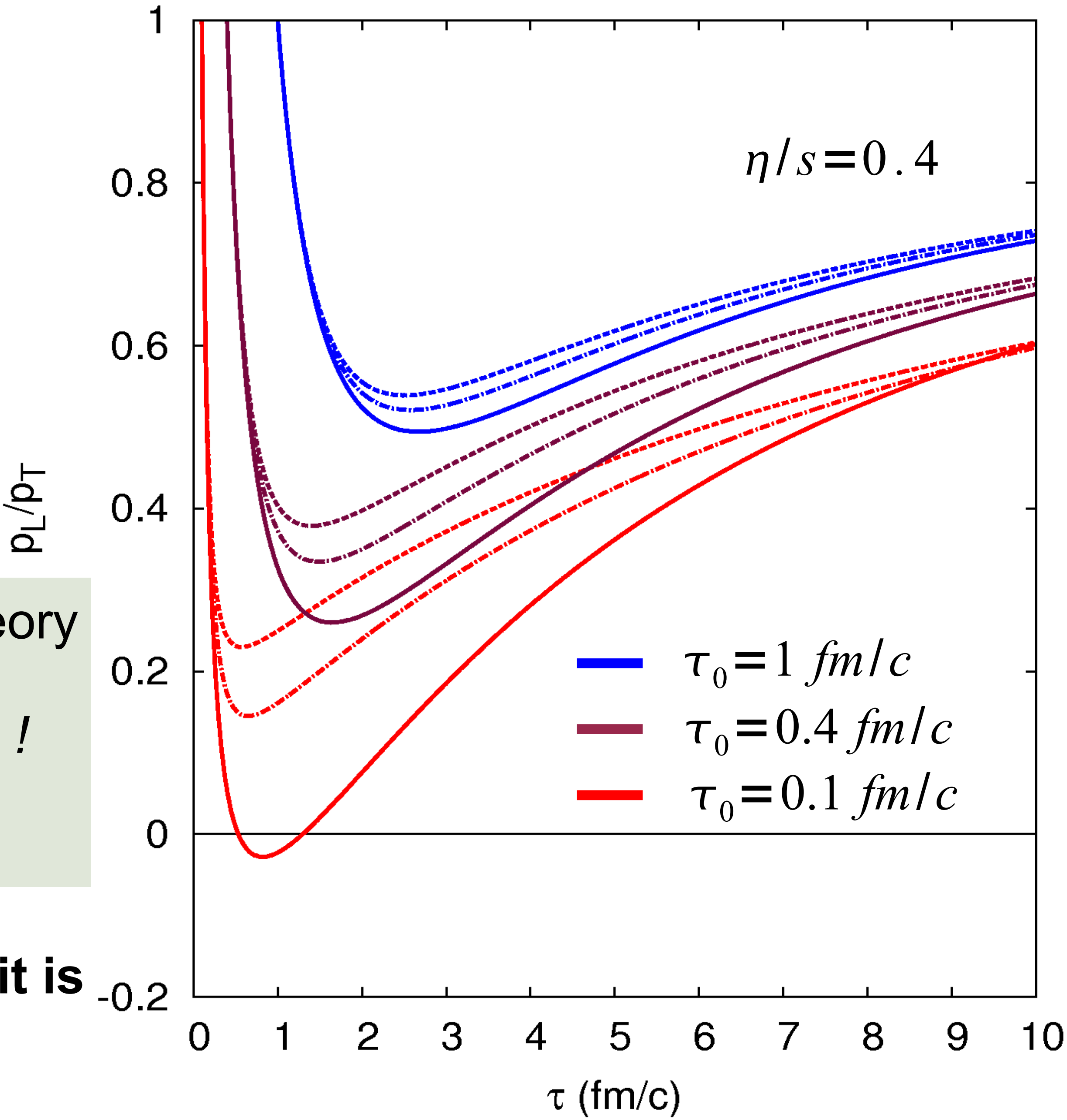
$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - x\frac{\pi^2}{e\tau}$$

Study

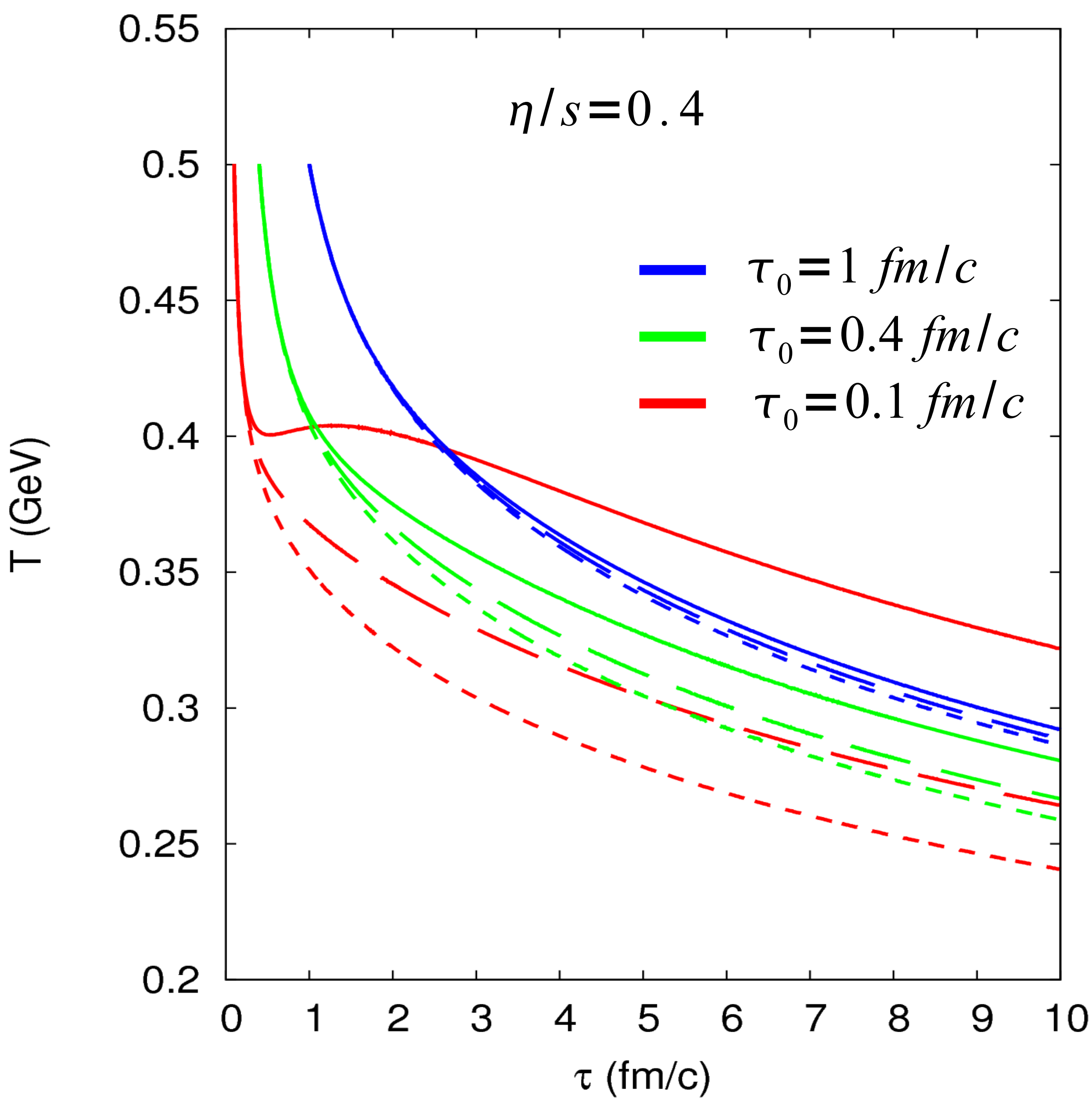
$\frac{\partial}{\partial\tau}\left(\frac{\pi}{e}\right)$ at $\pi = \frac{e}{3}$

→In third (and higher) order theory
π cannot grow larger than p,
no matter what η/s or τ_0 chosen !

In second-order theory (x=0) it is possible



Third-order theory: Temperature



Summary

- We extended Israel and Stewarts approach to derive a *third-order* dissipative hydrodynamic equation
- Comparisons with kinetic transport (BAMPS) in 1D demonstrate the importance of third and higher order corrections
- *10%* corrections for $\eta/s=0.2$, *30%* for *0.4*. Almost *200%* for $\eta/s=3$
- Remarkably good agreement with kin. transport even for large η/s (Kn)
- Longitudinal pressure cannot become negative in third-order equation! No reheating.
- Corrections of all orders can be estimated from the $\eta \to \infty$ limit
- Heat and bulk have been neglected...
- Third-order equation formulated for a general case. “All-orders” is difficult to generalize.

Hydro is a matter of belief

Church of the Third Order in Bahia, Brazil

Third-order theory: entropy production

BACKUP

$$T\partial_\mu s^\mu = \pi_{\mu\nu}\left(\sigma^{\mu\nu} - \beta_2 \dot{\pi}^{\mu\nu} - \frac{1}{2} T\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu}\right) + \alpha T \pi_{\alpha\beta}\pi^\alpha_\sigma \pi^{\beta\sigma}\partial_\mu\left(\beta_2^2 \frac{u^\mu}{T}\right) + 3\alpha\beta_2^2 \pi_{\alpha\beta}\pi^\alpha_\sigma \dot{\pi}^{\beta\sigma}$$

$$3\alpha\beta_2^2 \pi_{\alpha\beta}\pi^\alpha_\sigma \dot{\pi}^{\beta\sigma} = 3(1-\theta\tau_\pi)\alpha\beta_2^2 \pi_{\alpha\beta}\pi^\alpha_\sigma \dot{\pi}^{\beta\sigma} + 3\theta\tau_\pi \alpha\beta_2^2 \pi_{\alpha\beta}\pi^\alpha_\sigma \dot{\pi}^{\beta\sigma}$$

$\theta\tau_\pi > 1$

strong dissipation

$\pi^{\beta\sigma}$ increasing

$(1-\theta\tau_\pi) < 0$

$\alpha\pi_{\alpha\sigma}\pi^\sigma_\beta \dot{\pi}^{\alpha\beta} < 0$

(same sign as $\alpha\pi_{\alpha\sigma}\pi^\sigma_\beta \pi^{\alpha\beta}$)

$\theta\tau_\pi < 1$

$\pi^{\beta\sigma}$ decreases, relaxation

$(1-\theta\tau_\pi) > 0$

$\alpha\pi_{\alpha\sigma}\pi^\sigma_\beta \dot{\pi}^{\alpha\beta} > 0$

(opposite sign as $\alpha\pi_{\alpha\sigma}\pi^\sigma_\beta \pi^{\alpha\beta}$)

Third-order theory: 1-Dim expansion

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - x\frac{\pi^2}{e\tau}$$

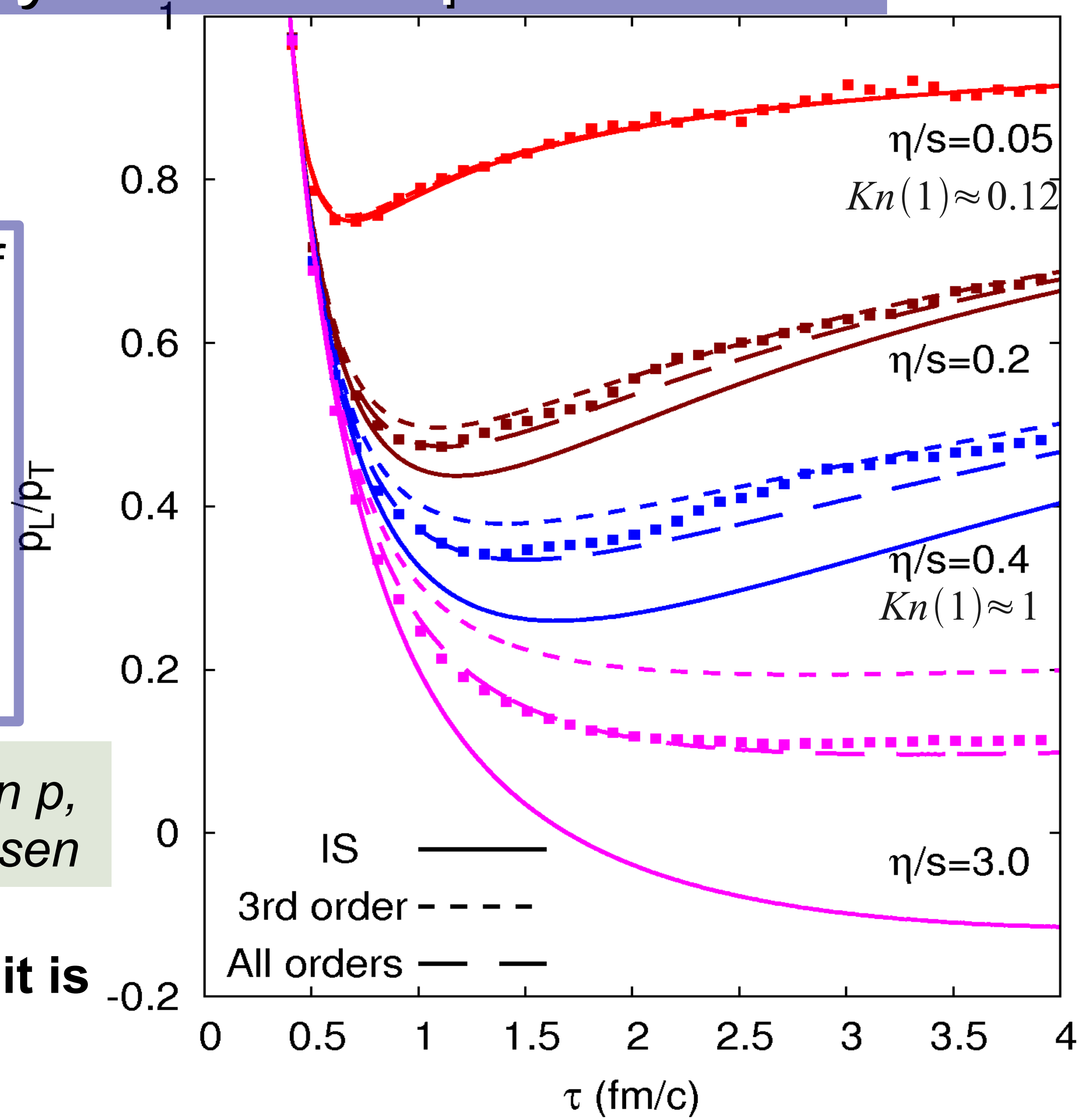
Investigate the behavior of $\frac{\partial}{\partial\tau}\left(\frac{\pi}{e}\right)$ at $\pi = \frac{e}{3}$:

For *x=5/3* and *x=3*,

$$\frac{\partial}{\partial\tau}\left(\frac{\pi}{e}\right)_{\pi=e/3} < 0$$

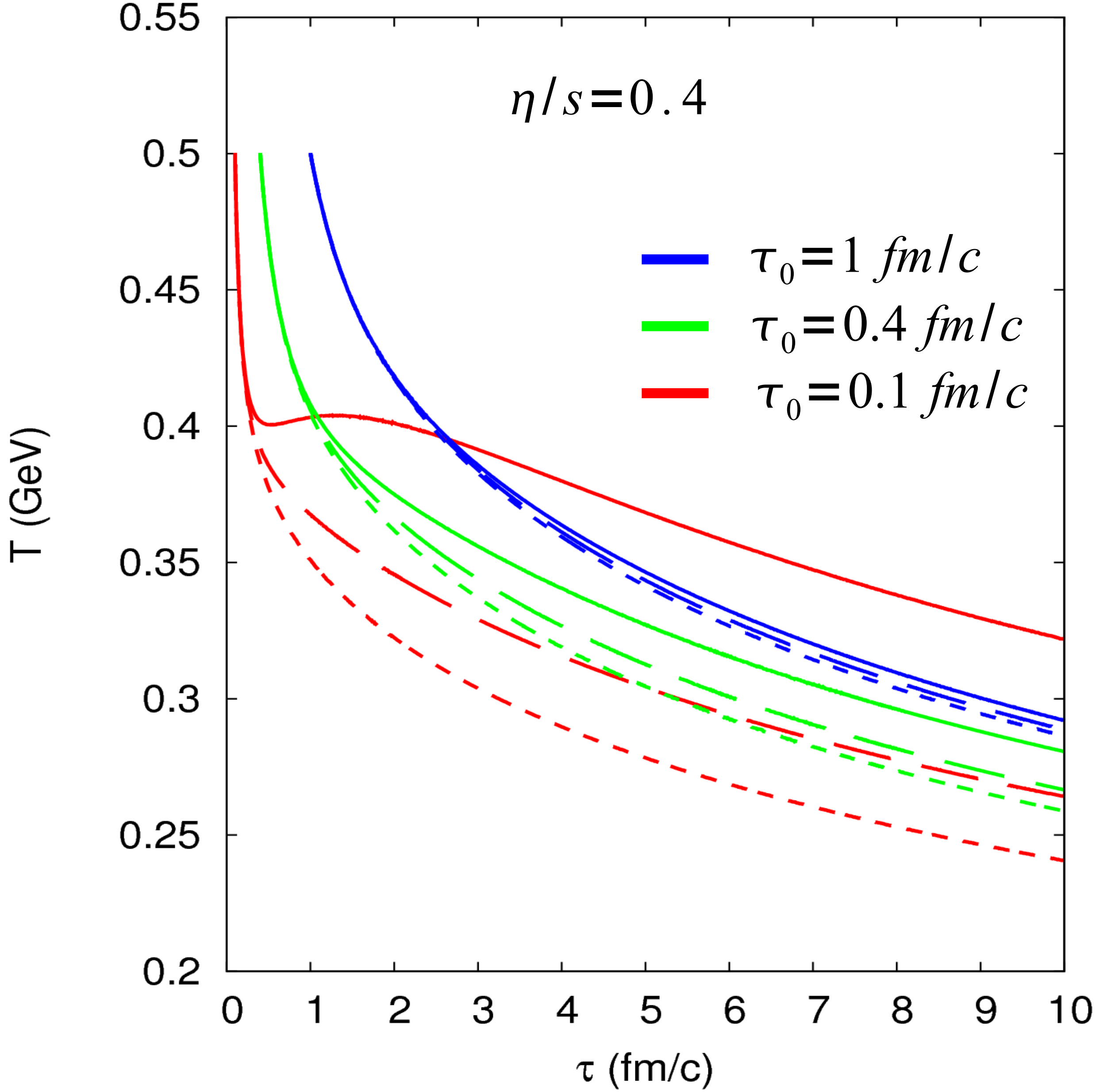
→ π cannot grow larger than p, no matter what η/s or τ_0 chosen

In Israel-Stewart eq. (x=0) it is possible



Third-order theory: Temperature

BACKUP



Dissipative hydrodynamics

$$T^{\mu\nu} = (e+p)u^\mu u^\nu - g^{\mu\nu} p + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu} - \Pi \Delta^{\mu\nu}$$

$$N^\mu = n u^\mu + h V^\mu$$

$$S^\mu = s_0 u^\mu + \Phi^\mu$$

$$\Pi = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu} - p$$

$$q^\mu = W^\mu - h V^\mu$$

$$\pi^{\mu\nu} = T^{<\mu\nu>}$$

Conservation laws:

$\partial_\mu T^{\mu\nu} = 0$ → Dynamical equation for energy density

$\partial_\mu N^\mu = 0$ → Dynamical equation for particle density

$\partial_\mu S^\mu \geq 0$ → Dynamical equation for shear pressure

BACKUP

Israel-Stewart theory

A.Muronga, Phys.Rev.C 69, 034903 (2004)

A.Muronga, Phys.Rev.Lett.88, 062302 (2002)

$$\dot{\pi}^{\mu\nu} = -\frac{\pi^{\mu\nu}}{\tau_\pi} - \frac{1}{2}\frac{T}{\beta_2}\partial_\alpha\left(\frac{\beta_2}{T}u^\alpha\right)\pi^{\mu\nu} + \frac{\sigma^{\mu\nu}}{\beta_2}$$

+

$$\partial_\mu T^{\mu 0} = 0, \quad \partial_\mu N^\mu = 0$$

for 1-Dim boost-invariant expansion :

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + \pi^{\mu\nu} = \begin{pmatrix} e & 0 & 0 & 0 \\ 0 & p+\pi/2 & 0 & 0 \\ 0 & 0 & p+\pi/2 & 0 \\ 0 & 0 & 0 & p-\pi \end{pmatrix}$$

→ energy density

→ transverse pressure

→ longitudinal pressure

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} \qquad \dot{e} = -\frac{4}{3}\frac{e}{\tau} + \frac{\pi}{\tau} \qquad \dot{n} = -\frac{n}{\tau}$$

Third-order theory: 1-Dim expansion

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau}$$

Israel-Stewart equation

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - 3\frac{\pi^2}{e\tau}$$

Third order equation

$$\dot{\pi} = -\frac{\pi}{\tau_\pi} - \frac{4}{3}\frac{\pi}{\tau} + \frac{8}{27}\frac{e}{\tau} - \frac{5}{3}\frac{\pi^2}{e\tau}$$

"*All orders*" equation

$$Kn(\tau) = \frac{6}{T\tau}\left(\frac{\eta}{s}\right)$$