Hydrodynamic modeling of heavy ion collisions
–
status and open issues*

THE OHIO STATE UNIVERSITY DEPARTMENT OF PHYSICS

Ulrich Heinz

Department of Physics
The Ohio State University
191 West Woodruff Avenue
Columbus, OH 43210

Workshop on

Flow and Dissipation in Ultrarelativistic Heavy Ion Collisions

ECT*, Trento, September 14 - 18, 2009

---

*Supported by the U.S. Department of Energy (DOE)

Office of Nuclear Physics
Exploring Nuclear Matter - Quarks to Stars

1. **Formalism**

2. **Numerical implementation and results**

Starting point: The conservation laws

$$\partial_\mu N^\mu = 0$$ charge conservation

$$\partial_\mu T^{\mu\nu} = 0$$ energy-momentum conservation

$$\partial_\mu S^\mu \geq 0$$ 2^{nd} law of thermodynamics

Ideal fluid decomposition

Ideal fluid dynamics $\Longleftrightarrow$ local thermal equilibrium $f(x,p) = f_{\rm eq}(x,p)$
$\Longleftrightarrow$ collision time scale $\ll$ macroscopic time scales $\Longleftrightarrow$ **strong coupling**

$$N^\mu = \frac{1}{(2\pi)^3}\int \frac{d^3p}{E} p^\mu f(x,p) = n\, u^\mu \qquad n = \text{(net) charge density}$$

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{(2\pi)^3}\int \frac{d^3p}{E} p^\mu p^\nu f(x,p) && e = \text{energy density} \\ &= (e+p)\, u^\mu u^\nu - p\, g^{\mu\nu} && p = \text{pressure} \\ &= e\, u^\mu u^\nu - p\, \Delta^{\mu\nu} && \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \\ S^\mu &= s\, u^\mu && s = \text{entropy density} \end{aligned}$$

First law of thermodynamics: $Ts = p - \mu n + e$

$$\partial_\mu N^\mu = \partial_\mu T^{\mu\nu} = 0 \Longrightarrow \partial_\mu S^\mu = 0$$

(in absence of shock discontinuities, entropy is conserved)

Ideal fluid equations (in comoving frame)

Convective and transverse derivative: $\partial_\mu = u^\mu D + \nabla^\mu$

$$D \equiv u^\nu \partial_\nu, \quad \nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$$

$$\dot{n} = -n\,\theta$$

$$\dot{e} = -(e+p)\,\theta$$

$$\dot{u}^\mu = \frac{\nabla^\mu p}{e+p} = \frac{c_s^2}{1+c_s^2}\frac{\nabla^\mu e}{e}$$

$$p = p(n,e)$$

$\dot{f} = u^\mu \partial_\mu f \equiv Df =$ time derivative in local rest frame

$\theta \equiv \partial \cdot u =$ local expansion rate

$c_s^2(T) = \frac{\partial p}{\partial e} = (\text{speed of sound})^2$

equation of state (EOS)

6 equations for 6 unknowns: $n,\ e,\ p,\ u^\mu$

Non-ideal fluid decomposition

$$f(x,p) = f_{\rm eq}(x,p) + \delta f(x,p)$$

$$\begin{aligned} N^\mu &= n\, u^\mu + V^\mu \\ &= N^\mu_{\rm eq} + \delta N^\mu \\ T^{\mu\nu} &= e\, u^\mu u^\nu - p\Delta^{\mu\nu} \\ &\quad - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} \\ &\quad + W^\mu u^\nu + W^\nu u^\mu \\ &= T^{\mu\nu}_{\rm eq} + \delta T^{\mu\nu} \\ S^\mu &= s\, u^\mu + \Phi^\mu \\ &= S^\mu_{\rm eq} + \delta S^\mu \end{aligned}$$

$n = u_\mu N^\mu$

$V^\mu = \Delta^{\mu\nu} N_\nu =$ charge flow in l.r.f.

$e = u_\mu T^{\mu\nu} u_\nu$

$\Pi = -\frac{1}{3}\Delta_{\mu\nu}T^{\mu\nu} - p =$ viscous bulk pressure

$W^\mu = u^\nu T_{\nu\lambda}\Delta^{\lambda\mu} =$ energy flow in l.r.f.

$= q^\mu + \frac{e+p}{n} V^\mu \quad q^\mu =$ heat flow in l.r.f.

$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}$

$\equiv \left[\frac{1}{2}(\Delta^{\mu\sigma}\Delta^{\nu\tau} + \Delta^{\mu\tau}\Delta^{\nu\sigma}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\tau\sigma}\right] T_{\tau\sigma}$

$=$ viscous shear pressure tensor $(\pi^\mu_\mu = 0)$

$s = u_\mu S^\mu$

$\Phi^\mu = \Delta^{\mu\nu} S_\nu =$ entropy flow in l.r.f.

Frame choice and matching conditions

The local equilibrium distribution $f_{\rm eq}(x,p)$ (with local temperature $T(x)$ and chemical potential $\mu(x)$) that best matches the non-equilibrium $f(x,p)$ is defined by the matching conditions

$$u_\mu\, \delta T^{\mu\nu}\, u_\nu = u_\mu\, \delta N^\mu = 0$$

Local rest frame is ambiguous:

Eckart frame: $u^\mu = \frac{N^\mu}{\sqrt{N\cdot N}}$: $V^\mu = 0,\ q^\mu = W^\mu$

Landau frame: $u^\mu = \frac{T^{\mu\nu}u_\nu}{\sqrt{u_\alpha T^{\alpha\beta}T_{\beta\gamma}u^\gamma}}$: $W^\mu = 0,\ q^\mu = -\frac{e+p}{n}V^\mu$

(Intermediate frames also possible.)

$\Longrightarrow$ Need $1+3+5=9$ additional equations for Π, q^μ, $\pi^{\mu\nu}$ from underlying transport theory.

Non-ideal fluid equations

$$\begin{aligned}
\dot{n} &= -n\,\theta - \nabla\cdot V + V\cdot\dot{u} \\
\dot{e} &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\sigma^{\mu\nu} - \nabla\cdot W + 2\,W\cdot\dot{u} \\
(e{+}p{+}\Pi)\,\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu \\
&\quad -[\Delta^{\mu\nu}\dot{W}_\nu + W^\mu\theta + (W\cdot\nabla)u^\mu]
\end{aligned}$$

Here $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu}u^{\nu\rangle}$ is the velocity shear tensor.

Depending on frame, can set either $V^\mu = 0$ or $W^\mu = 0$. In Landau frame ($W^\mu = 0$) and for baryon-free systems ($n = 0$, no heat conduction) equations simplify to:

$$\begin{aligned}
\dot{e} &= -(e+p+\Pi)\,\theta + \pi_{\mu\nu}\sigma^{\mu\nu} \\
(e{+}p{+}\Pi)\,\dot{u}^\mu &= \nabla^\mu(p+\Pi) - \Delta^{\mu\nu}\nabla^\sigma\pi_{\nu\sigma} + \pi^{\mu\nu}\dot{u}_\nu
\end{aligned}$$

Need 6 extra equations for bulk and shear viscous pressures Π, $\pi^{\mu\nu}$ $\Longrightarrow$ different paths (Navier-Stokes, Israel-Stewart, Öttinger-Grmela, BRSSS, . . .)

Here we follow Chapman-Enskog strategy: write $f(x,p) = f_{\rm eq}\big(p\cdot u(x); T(x), \mu(x)\big) + \delta f(x,p)$ and assume that $\delta f \ll f$ (and thus δN^μ and $\delta T^{\mu\nu}$) can be expanded in gradients of equilibrium parameters T, μ, u_μ.

The second law of thermodynamics (I)

In equilibrium the identity $Ts = p - \mu n + e$ can be rewritten as

$$S^\mu_{\rm eq} = p(\alpha,\beta)\beta^\mu - \alpha N^\mu_{\rm eq} + \beta_\nu T^{\mu\nu}_{\rm eq}$$

where $\alpha \equiv \mu/T$, $\beta \equiv 1/T$, and $\beta^\mu \equiv u^\mu/T$.

The most general off-equilibrium generalization is (Israel & Stewart 1979)

$$S^\mu = p(\alpha,\beta)\beta^\mu - \alpha N^\mu + \beta_\nu T^{\mu\nu} + Q^\mu(\delta N^\mu, \delta T^{\mu\nu})$$

where Q^μ is second and higher order in the off-equilibrium deviations δN^μ and $\delta T^{\mu\nu}$.

The Gibbs-Duhem relation $dp = s\,dT + n\,d\mu$ can be recast as

$$\partial_\mu(p(\alpha,\beta)\beta^\mu) = N^\mu_{\rm eq}\partial_\mu\alpha - T^{\mu\nu}_{\rm eq}\partial_\mu\beta_\nu$$

Using also the conservation laws, the entropy production rate takes the form

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu\alpha + \delta T^{\mu\nu}\partial_\mu\beta_\nu + \partial_\mu Q^\mu$$

The second law of thermodynamics (II)

In the Chapman-Enskog spirit, one now postulates linear relations between the off-equilibrium flows δN^μ, $\delta T^{\mu\nu}$ and the thermodynamic forces $\partial^\mu \alpha$, $\partial^{(\mu}\beta^{\nu)}$, consistent with the second law

$$\partial_\mu S^\mu = -\delta N^\mu \partial_\mu \alpha + \delta T^{\mu\nu}\partial_\mu \beta_\nu + \partial_\mu Q^\mu \geq 0$$

These relations depend on the choice of Q^μ. Standard dissipative relativistic fluid dynamics assumes $Q^\mu = 0$. In this case

$$T\partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\langle\mu\nu\rangle} \equiv \frac{\Pi^2}{\zeta} - \frac{q^\mu q_\mu}{2\lambda T} + \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0,$$

with thermodynamic forces $X \equiv -\nabla \cdot u = -\theta$, $X^\mu \equiv \frac{\nabla^\mu T}{T} - \dot{u}^\mu = -\frac{nT}{e+p}\nabla^\mu \left(\frac{\mu}{T}\right)$ and $X_{\langle\mu\nu\rangle} \equiv \sigma_{\mu\nu} = \nabla_{\langle\mu} u_{\nu\rangle}$, can be satisfied by setting

$$\Pi = -\zeta\theta, \quad q^\mu = -\lambda \frac{nT^2}{e+p}\nabla^\mu \left(\frac{\mu}{T}\right), \quad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

with positive transport coeffcients $\zeta \geq 0$, $\lambda \geq 0$, and $\eta \geq 0$ (relativistic Navier-Stokes theory).
Unfortunately, plugging these equations for Π, q^μ, and $\pi^{\mu\nu}$ directly into the non-ideal hydro equations leads to acausal signal propagation.

The second law of thermodynamics (III)

Causal relativistic fluid dynamics requires keeping Q^μ in the entropy flux, at least up to terms of second order in the irreversible flows.

$$S^\mu = s u^\mu + \frac{q^\mu}{T} + Q^\mu$$

Setting $q^\nu = 0$ ($n = 0$) for simplicity, we get up to second order

$$S^\mu = s u^\mu - (\beta_0 \Pi^2 + \beta_2 \pi_{\nu\lambda} \pi^{\nu\lambda}) \frac{u^\mu}{2T}$$

This yields (after some algebra)

$$\begin{aligned} T\partial_\mu S^\mu &= \Pi \left[-\theta - \beta_0 \dot{\Pi} - \Pi\, T\partial_\mu \left(\frac{\beta_0 u^\mu}{2T} \right) \right] \\ &+ \pi^{\alpha\beta} \left[\sigma_{\alpha\beta} - \beta_2 \dot{\pi}_{\alpha\beta} - \pi_{\alpha\beta}\, T\partial_\mu \left(\frac{\beta_2 u^\mu}{2T} \right) \right] \\ &\overset{!}{=} \frac{\Pi^2}{\zeta} + \frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta} \geq 0 \end{aligned}$$

The thermodynamic forces $-\theta$, $\sigma_{\alpha\beta}$ are seen to be self-consistently modified by the irreversible flows Π, $\pi_{\alpha\beta}$. This leads to dynamical ("transport") equations for Π, $\pi_{\alpha\beta}$.

Transport equations for the irreversible flows

The resulting transport equations for Π, $\pi_{\alpha\beta}$ are (Israel & Stewart 1979, Muronga 2002, 2004)

$$\dot{\Pi} = -\frac{1}{\tau_\Pi}\left[\Pi + \zeta\theta + \Pi\zeta T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)\right] = -\frac{1}{\tau'_\Pi}\left[\Pi + \zeta'\theta\right]$$

$$\begin{aligned}\Delta_{\alpha\mu}\Delta_{\beta\nu}\dot{\pi}^{\mu\nu} &= -\frac{1}{\tau_\pi}\left[\pi_{\alpha\beta} - 2\eta\sigma_{\alpha\beta} + \pi_{\alpha\beta}\eta T\partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)\right] \\ &\quad + \text{terms that don't generate entropy} \\ &= -\frac{1}{\tau'_\pi}\left[\pi_{\alpha\beta} - 2\eta'\sigma_{\alpha\beta}\right] + \dots\end{aligned}$$

Here we introduced the relaxation times $\tau_\Pi = \zeta\beta_0,\ \tau_\pi = 2\eta\beta_2$, and $\tau'_\Pi = \frac{\tau_\Pi}{1+\zeta\gamma_\Pi},\ \zeta' = \frac{\zeta}{1+\eta\gamma_\Pi}$, $\tau'_\pi = \frac{\tau_\pi}{1+\eta\gamma_\pi},\ \eta' = \frac{\eta}{1+\eta\gamma_\pi}$, where $\gamma_\Pi \equiv T\partial_\mu\left(\frac{\tau_\Pi u^\mu}{2\zeta T}\right)$ and $\gamma_\pi \equiv T\partial_\mu\left(\frac{\tau_\pi u^\mu}{2\eta T}\right)$.

(UH, Song, Chaudhuri 2006)

$\zeta,\ \eta$ and $\tau_\Pi,\ \tau_\pi$ should be calculated from the underlying microscopic theory. This has been done by KSS and BRSSS for infinitely strongly coupled SYM theory, and by AMY and YM for weakly coupled QCD in Boltzmann transport theory (see below).

The purple terms kick in wherever the expansion rate gets large and then effectively reduce the viscosities and relaxation times. – The viscous pressures $\Pi, \pi^{\mu\nu}$ relax exponentially towards their (flow-modified) Navier-Stokes limits on (flow-modified) microscopic relaxation time scales τ'_Π, τ'_π.

For discussion of bulk viscosity and bulk viscous relaxation time see Huichao Song's talk on Thursday.

Concentrate here on shear viscosity.

More second-order terms . . .

Analyzing the second law of thermodynamics misses second-order terms that don't contribute to 2nd order entropy production but may still affect the evolution of flow.

For systems with conformal symmetry ($\Pi{=}0$) and vanishing chemical potentials ($q^\mu{=}0$) BRSSS (Baier, Romatschke, Son, Starinets, Stephanov, JHEP 04 (2006) 100) found 5 possible second-order terms:

$$\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \tau_\pi \left[\Delta^{\mu\alpha}\Delta^{\nu\beta}\dot{\pi}_{\alpha\beta} + \frac{4}{3}\theta\pi^{\mu\nu}\right] - \frac{\lambda_1}{2\eta^2}\pi^{\langle\mu}{}_\alpha\pi^{\nu\rangle\alpha} - \frac{\lambda_2}{2\eta}\pi^{\langle\mu}{}_\alpha\omega^{\nu\rangle\alpha} - \frac{\lambda_3}{2}\omega^{\langle\mu}{}_\alpha\omega^{\nu\rangle\alpha} + \frac{\kappa}{2}\left[R^{\langle\mu\nu\rangle} + 2u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right]$$

where $\omega_{\mu\nu} = \frac{1}{2}(\nabla_\mu u_\nu - \nabla_\nu u_\mu)$ is the vorticity and $R^{\alpha\mu\nu\beta}$, $R^{\mu\nu}$ are the Riemann and Ricci tensors, respectively.

Now we have **5** second order coefficients τ_π, λ_1, λ_2, λ_3, κ, in addition to η.

Betz, Henkel and Rischke (arXiv:0812.1440 [nucl-th]) generalized this to include heat conduction and bulk viscosity $\Longrightarrow$ even more coefficients . . .

Weak and strong coupling limits of shear viscosity and second order coefficients

(zero masses and chemical potentials)

Ratio	pQCD ($N_f = 3$) (AMY '00,'03; YM '08)	SYM (PSS '01; KSS '05; BRSSS '08)
$\frac{\eta}{s}$	$\frac{46.1}{N_c^2 g^4 \ln(4.17/g\sqrt{N_c})} \approx 1.7 \; (g=2)$	$\frac{1}{4\pi} \approx 0.08$
$\frac{(e+p)\tau_\pi}{\eta}$	5 to 5.9	$4 - 2\ln 2 \approx 2.6137$
$\frac{(e+p)\lambda_1}{\eta^2}$	5.2 to 4.1	2
$\frac{(e+p)\lambda_2}{\eta^2}$	$-2\eta\frac{(e+p)\tau_\pi}{\eta^2} = -10$ to -11.8	$-4\ln 2 \approx -2.7726$
$\frac{(e+p)\lambda_3}{\eta^2}$	0	0
$\frac{(e+p)\kappa}{\eta^2}$	0	4

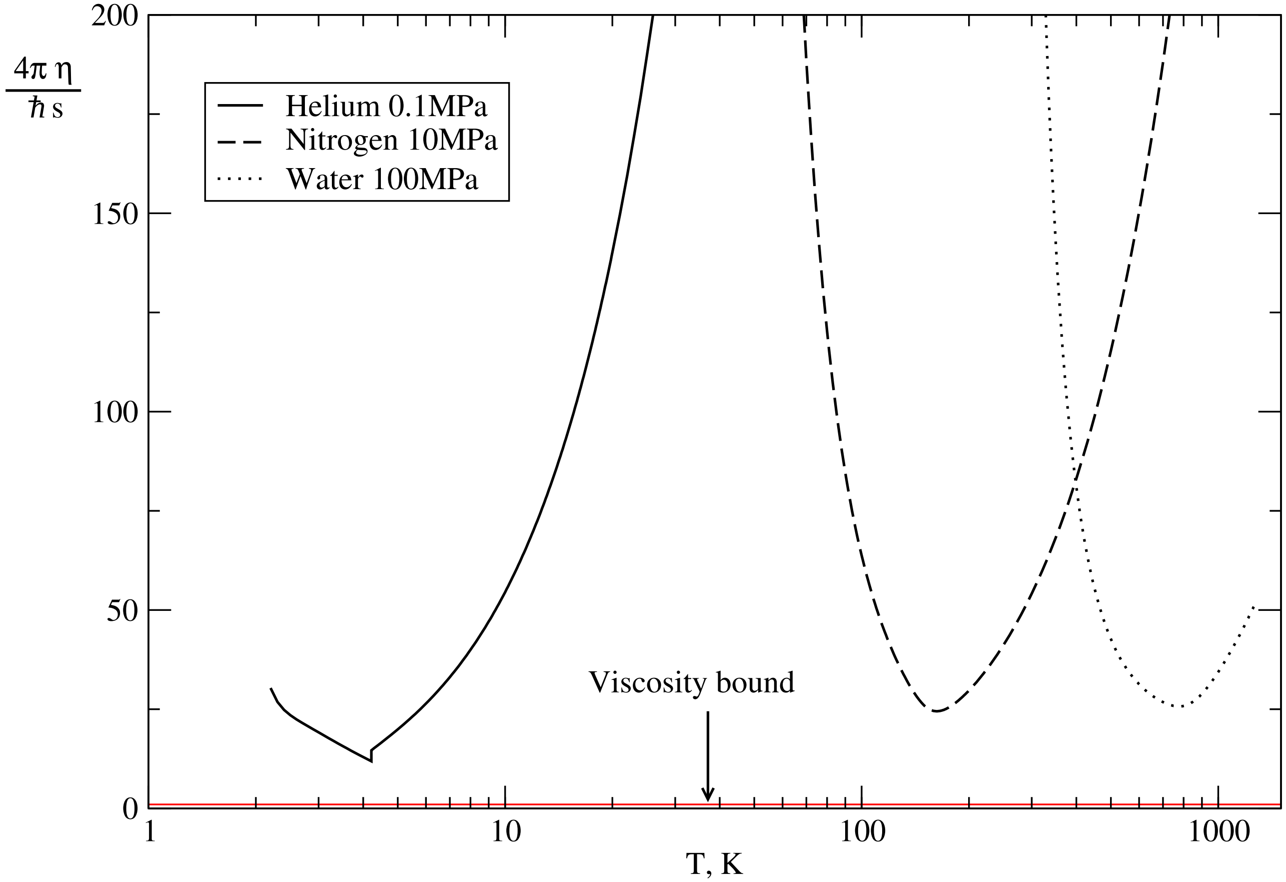
AMY = Arnold/Moore/Yaffe; YM = York & Moore; PSS = Policastro/Son/Starinets; KSS = Kovtun/Son/Starinets; BRSSS = Baier/Romatschke/Son/Starinets/Stephanov

Fortunately, terms $\sim \lambda_{1,2,3}$ appear to be numerically unimportant for hydro evolution.

QGP – the most perfect fluid ever observed?

AdS/CFT universal lower viscosity bound conjecture: $\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$

Kovtun, Son, Starinets, PRL 94 (2005) 111601



Upper limit for QGP viscosity from various recent estimates are close to this bound!

But: quantitative constraint on η/s requires viscous hydrodynamics code.

1. Formalism

2. **Numerical implementation and results**

(2+1)-d viscous hydrodynamics: status March 2009

- Romatschke & Romatschke, PRL 99 ('07); Luzum & Romatschke, PRC 78 ('08) & arXiv:0901.4588:
  full Isreal-Stewart eqn., EOS I, EOS L*
  Au+Au, $T_{\rm dec} = 150\,{\rm MeV}$ (EOS L* is quasiparticle EOS based on Lattice QCD)

- Song & Heinz, PLB 658 ('08), PRC 77 ('08), PRC 78 ('08), QM08, SQM2008:
  simplified I-S eqn. & full I-S eqn., EOS I, SM-EOS Q, EOS L
  Cu+Cu & Au+Au, $T_{\rm dec} = 130\,{\rm MeV}$

- Dusling & Teaney, PRC 77 ('08):
  Öttinger-Grmela eqn., EOS I
  Au+Au, kinetic decoupling by scattering cross section

- Huovinen & Molnar, QM08:
  full I-S eqn., EOS I
  comparison of viscous hydro with parton cascade

- Chaudhuri, arXiv:0704.0134, 0708.1252, 0801.3180, 0901.0460, 0901.4181, PLB 672 ('09), QM08:
  Au+Au, EOS I, EOS Q

- Pratt & Vredevoogd, PRC 78 ('08):
  non-linearly modified I-S eqns., modified EOS Q, **Au+Au**
  viscous hydro + hadron cascade

- Denicol, Kodama et al., arXiv:0903.3595:
  simplified I-S eqn., EOS I, EOS II, EOS III (lattice motivated), **Au+Au**, $T_{\rm dec} = 130\,{\rm MeV}$
  bulk viscosity only

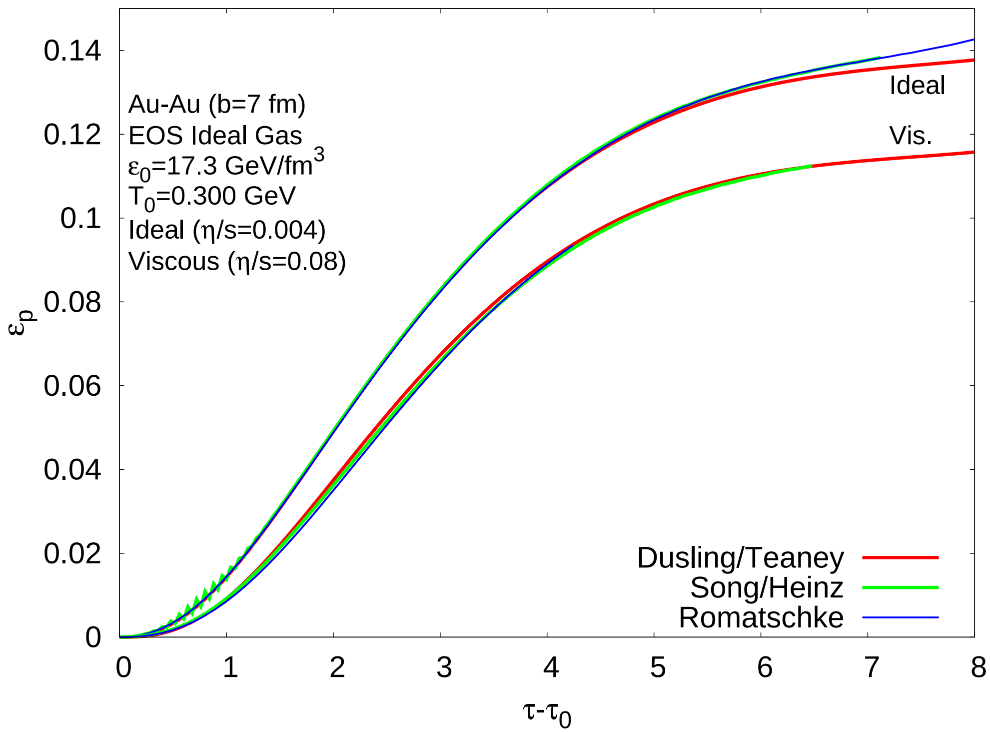
(3+1)-d viscous hydrodynamics: first news here?

(Molnar, Niemi, Rischke, arXiv:0907.2583; Vredevoogd, talk on Friday)

Verification of (2+1)-d viscous codes (TECHQM)

Three codes (UVH2+1/Seattle, VISH2+1/OSU, Dusling/Teaney Stony Brook) were compared with each other under identical conditions:

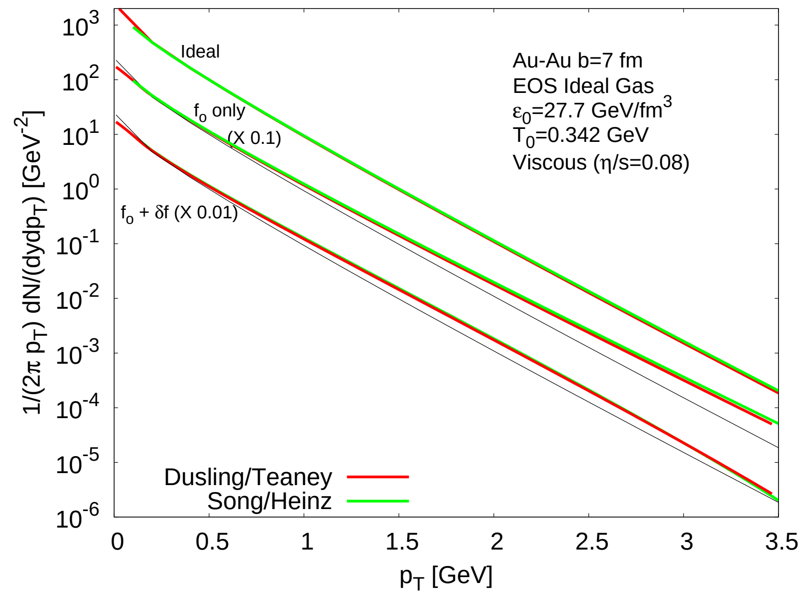
momentum anisotropy



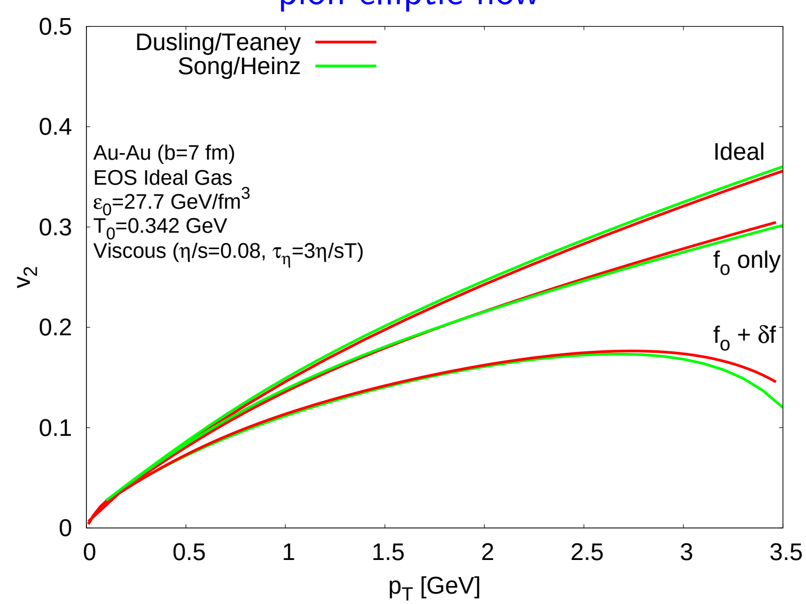
Excellent agreement!

(All codes should undergo verification)

p_T spectra (black: ideal hydro f.c.)



pion elliptic flow



The three codes solve slightly different I-S equations which differ by second-order terms

$\Longrightarrow$ **very small effects!** (Still, a systematic study of second-order terms should be done.)

Generic viscous effects
in heavy-ion collisions

(Concentrate on shear viscous effects; for discussion of bulk viscous effects see Huichao Song's talk on Thursday.)

(1+1)-d viscous hydrodynamic equations

(Muronga & Rischke 2004, Chaudhuri & Heinz 2005)

Azimuthally symmetric transverse dynamics with long. boost invariance:

Use (τ, r, ϕ, η) coordinates and solve

- hydrodynamic equations for $T^{\tau\tau} = (e+\mathcal{P})\gamma_r^2 - \mathcal{P}, \quad T^{\tau r} = (e+\mathcal{P})\gamma_r^2 v_r$ (with "effective pressure" $\mathcal{P} = p + \Pi - r^2\pi^{\phi\phi} - \tau^2\pi^{\eta\eta}$) together with
- kinetic relaxation equations for $\pi^{\phi\phi},\ \pi^{\eta\eta},\ \Pi$:

$$\frac{1}{\tau}\partial_\tau\left(\tau T^{\tau\tau}\right) + \frac{1}{r}\partial_r\left(r(T^{\tau\tau}+\mathcal{P})v_r\right) = -\frac{p+\tau^2\pi^{\eta\eta}}{\tau},$$

$$\frac{1}{\tau}\partial_\tau\left(\tau T^{\tau r}\right) + \frac{1}{r}\partial_r\left(r(T^{\tau r}v_r+\mathcal{P})\right) = +\frac{p+r^2\pi^{\phi\phi}}{r},$$

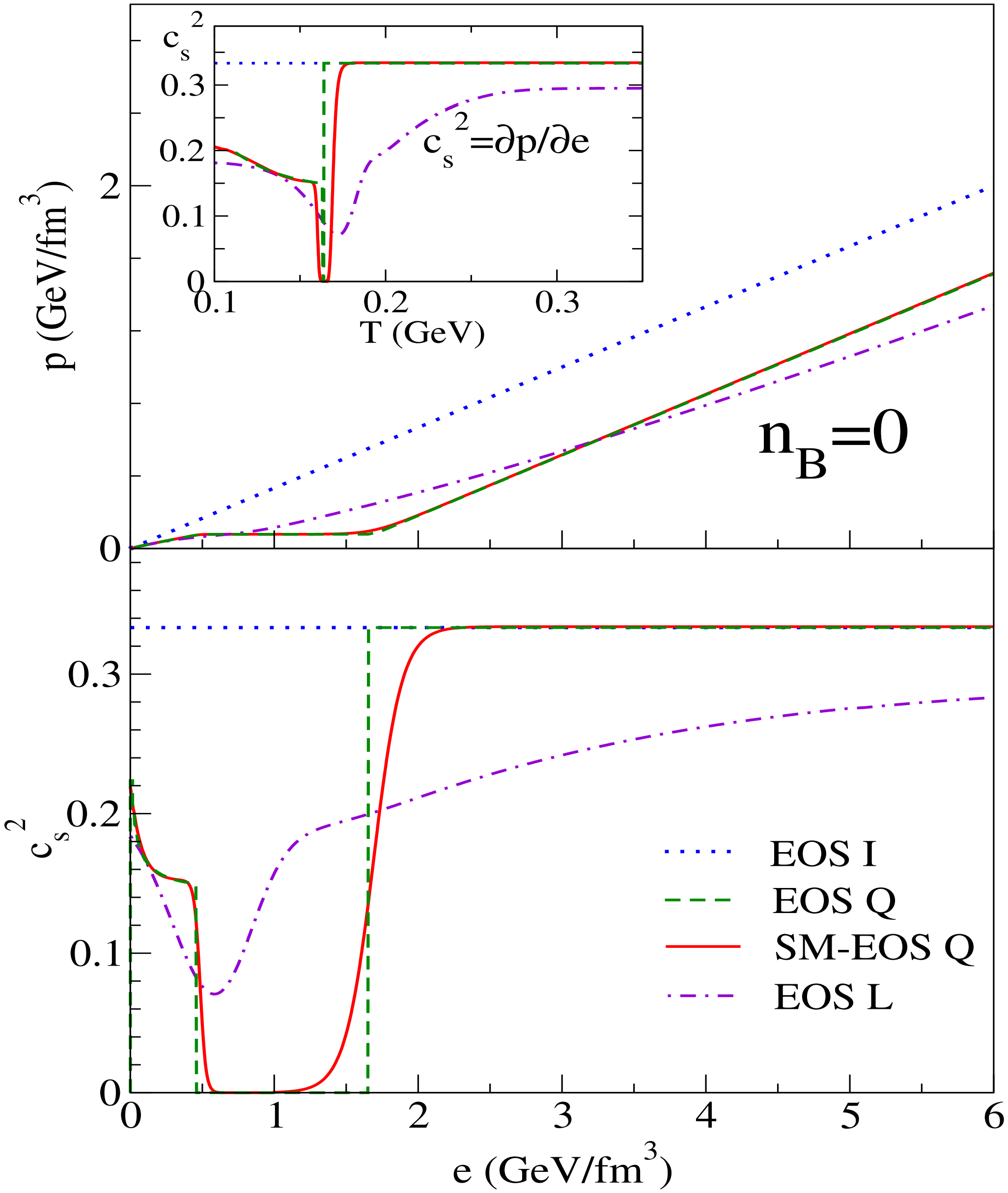
$$\left(\partial_\tau + v_r\partial_r\right)\pi^{\eta\eta} = -\frac{1}{\gamma_r\tau'_\pi}\left[\pi^{\eta\eta} - \frac{2\eta'}{\tau^2}\left(\frac{\theta}{3} - \frac{\gamma_r}{\tau}\right)\right],$$

$$\left(\partial_\tau + v_r\partial_r\right)\pi^{\phi\phi} = -\frac{1}{\gamma_r\tau'_\pi}\left[\pi^{\phi\phi} - \frac{2\eta'}{r^2}\left(\frac{\theta}{3} - \frac{\gamma_r v_r}{r}\right)\right],$$

$$\left(\partial_\tau + v_r\partial_r\right)\Pi = -\frac{1}{\gamma_r\tau'_\Pi}\left[\Pi + \zeta'\theta\right].$$

Close equations with EOS $p(e)$ where $e = T^{\tau\tau} - v_r T^{\tau r}$ and $v_r = T^{\tau r}/(T^{\tau\tau}+\mathcal{P})$.

Equations of state (EOS)



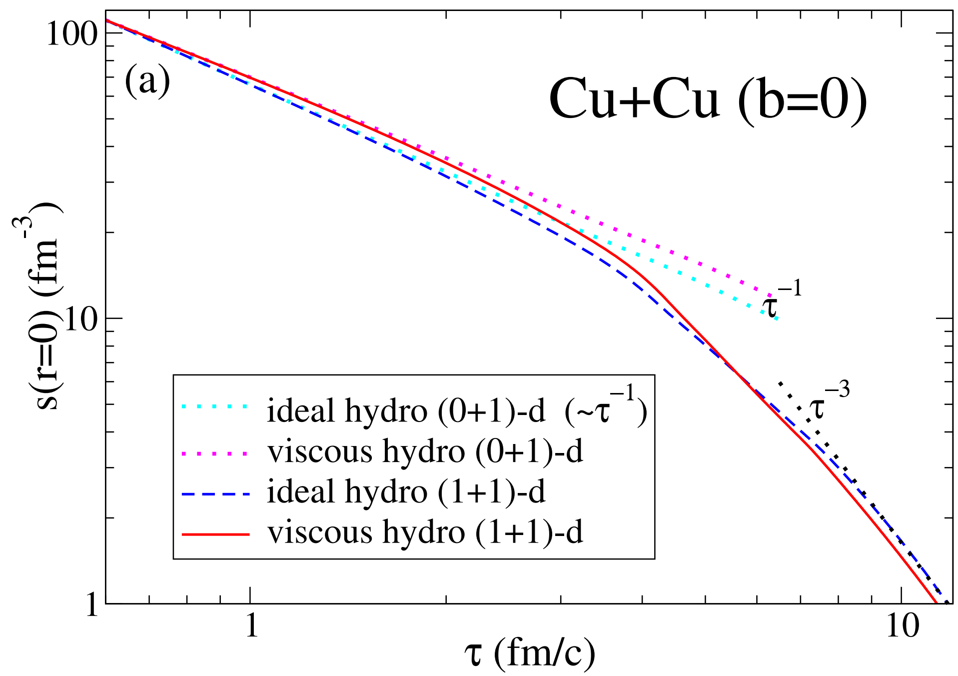
For most up-to-date EOS based on LQCD data and implementing chemical freeze-out at $T_{\rm chem} = 165\,$MeV see P. Prtreczky's talk on Friday.

(1+1)-d viscous hydro: less longitudinal work, more radial flow

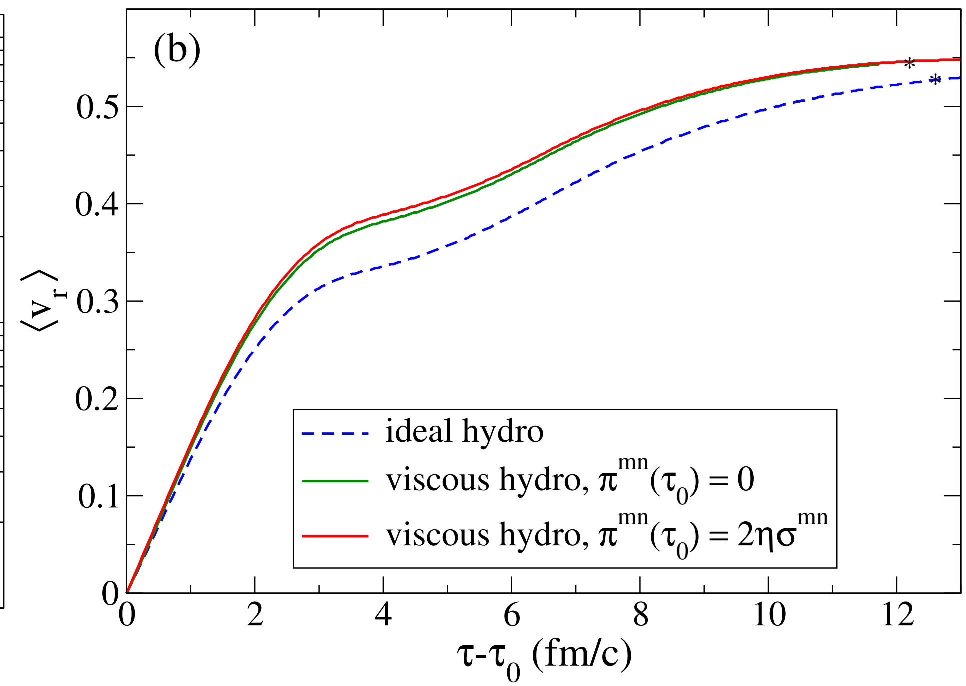
Cu+Cu @ $b = 0$, SM-EOS Q (Song & Heinz, PLB 658 ('08))

$\tau_0 = 0.6\,\frac{\text{fm}}{c}$, $e_0 = 30\,\frac{\text{GeV}}{\text{fm}^3}$, $\frac{\eta}{s} = \frac{1}{4\pi}$, $\tau_\pi = 0.24\left(\frac{200\,\text{MeV}}{T}\right)\frac{\text{fm}}{c}$, $T_{\text{dec}} = 130\,\text{MeV}$

Entropy density vs. time



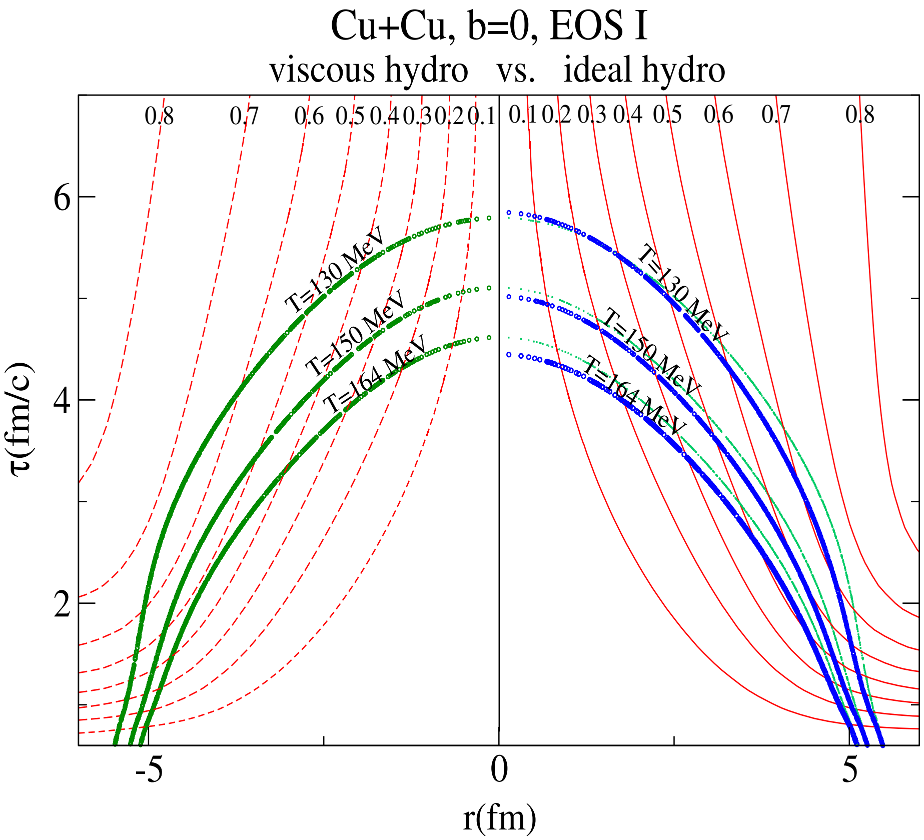
avg. radial flow vs. time

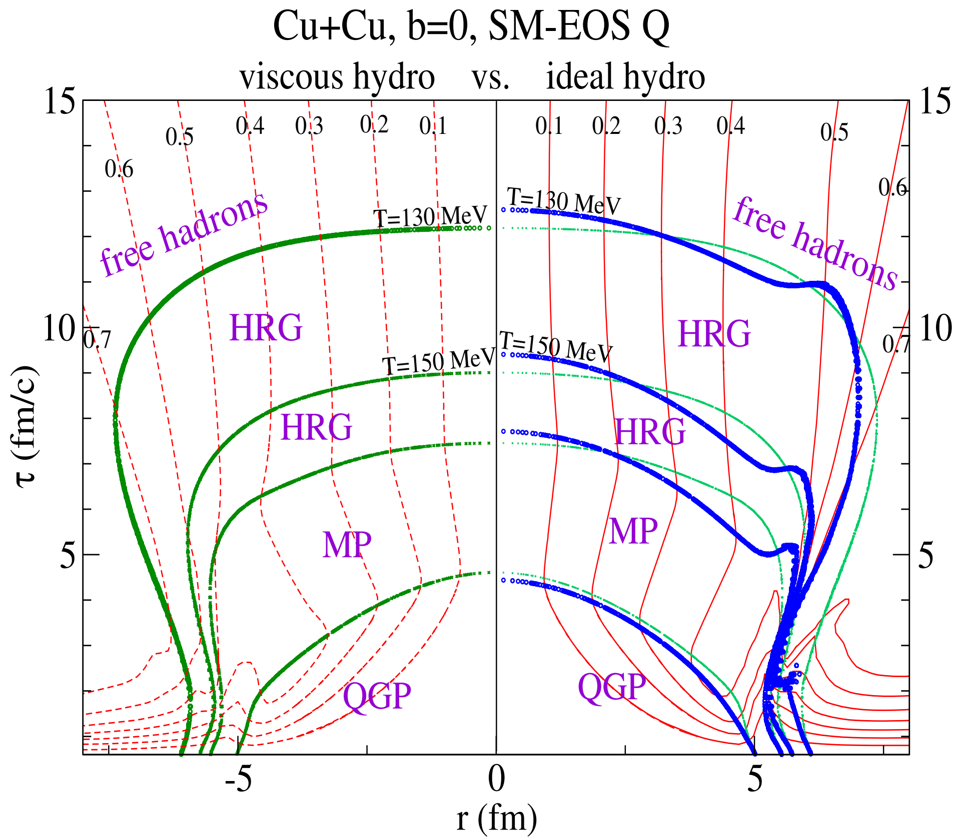


- Radial flow develops much faster, expansion turns 3-dimensional more abruptly
- Shear viscosity initially reduces the cooling due to longitudinal work, but then leads to faster cooling in the fireball center than for ideal fluid later, due to stronger radial flow (seen also by Teaney 2004, Chaudhuri 2006,2007; Romatschke et al. 2006,2007)

Central Cu+Cu (b=0): ideal vs. viscous hydro

$\tau_0 = 0.6\,\frac{\text{fm}}{c},\ e_0 = 30\,\frac{\text{GeV}}{\text{fm}^3},\ \frac{\eta}{s} = \frac{1}{4\pi},\ \tau_\pi = 0.24\left(\frac{200\,\text{MeV}}{T}\right)\frac{\text{fm}}{c},\ T_{\text{dec}} = 130\,\text{MeV}$



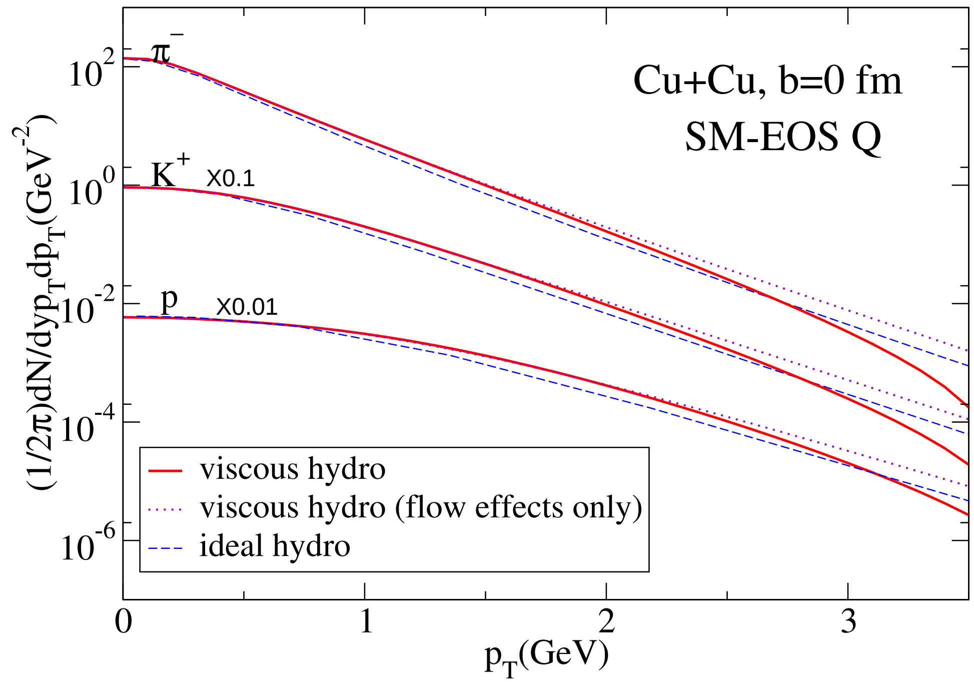


- Viscous hydro smoothes out phase transition structures
- Viscous hydro cools more slowly than ideal hydro, except for the center where cooling is accelerated at late times by faster radial expansion in the viscous case
- Viscous effects increase QGP lifetime, but viscous pressure gradients in the mixed phase shorten the mixed phase lifetime

(1+1)-d viscous hydro: more radial flow $\Longrightarrow$ flatter spectra

hadron p_T-spectra:

$$E\frac{dN}{d^3p} = \int_\Sigma \frac{p\cdot d^3\sigma(x)}{(2\pi)^3}\Big[f_{\rm eq}(x,p) + \delta f(x,p)\Big] = \int_\Sigma \frac{p\cdot d^3\sigma(x)}{(2\pi)^3}\, f_{\rm eq}(x,p)\left(1 + \frac{1}{2}\frac{p^\alpha p^\beta}{T^2(x)}\frac{\pi_{\alpha\beta}(x)}{(e+p)(x)}\right)$$



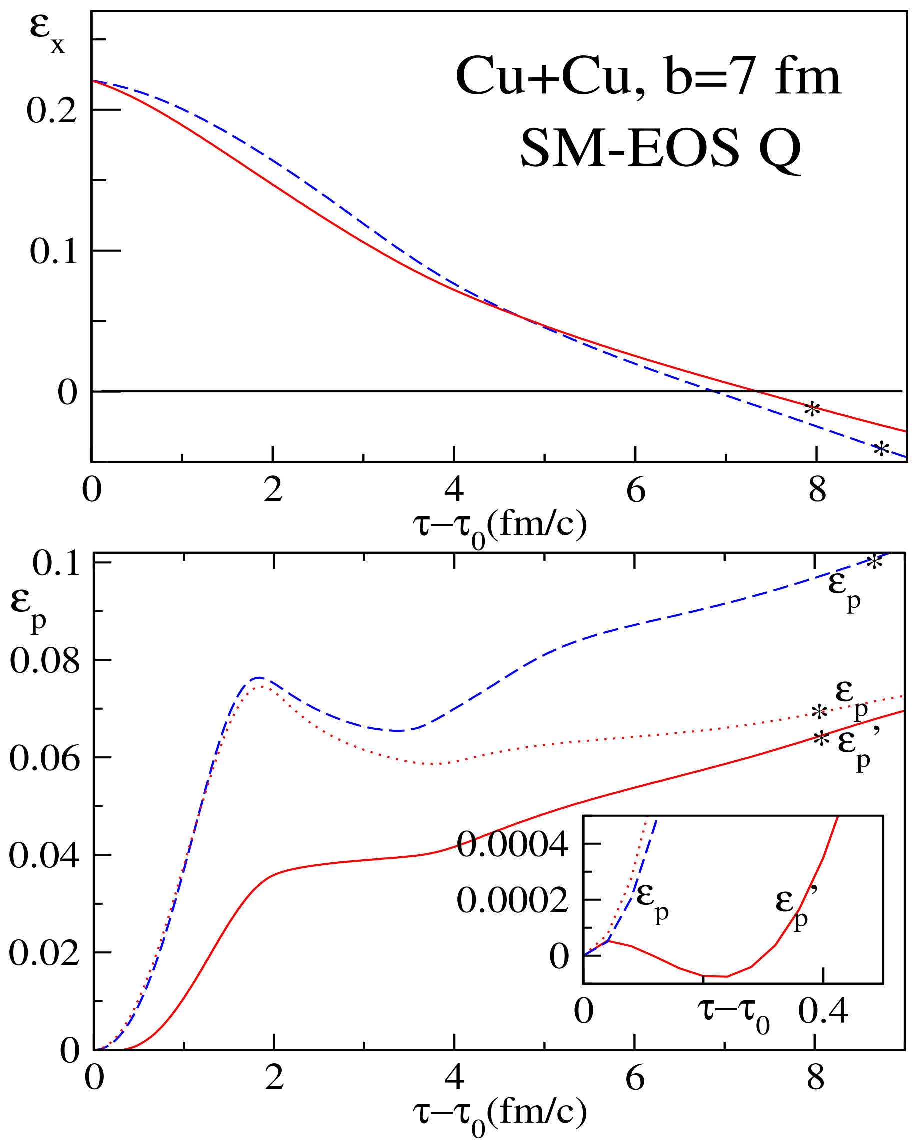
$$\tau_0 = 0.6\,\frac{\rm fm}{c},\ e_0 = 30\,\frac{\rm GeV}{\rm fm^3},\ \frac{\eta}{s} = \frac{1}{4\pi},\ \tau_\pi = 0.24\left(\frac{200\,\rm MeV}{T}\right)\frac{\rm fm}{c},\ T_{\rm dec} = 130\,\rm MeV$$

- For identical initial and freeze-out conditions, viscous evolution yields more radial flow and flatter spectra (as previously seen by Chaudhuri 2006,2007; Romatschke 2007)
- Effect on $b = 0$ spectra can be largely absorbed by starting viscous hydro later with lower initial density (Romatschke et al., 2006,2007)
- See A. Monnai's and D. Teaney's Wednesday talks for generalized and alternate forms for δf.

(2+1)-d viscous hydro: less momentum anisotropy

Cu+Cu @ $b = 7\,\mathrm{fm}$, SM-EOS Q, $\frac{\eta}{s} = \frac{1}{4\pi}$, same initial and final conditions

spatial eccentricity and momentum anisotropy



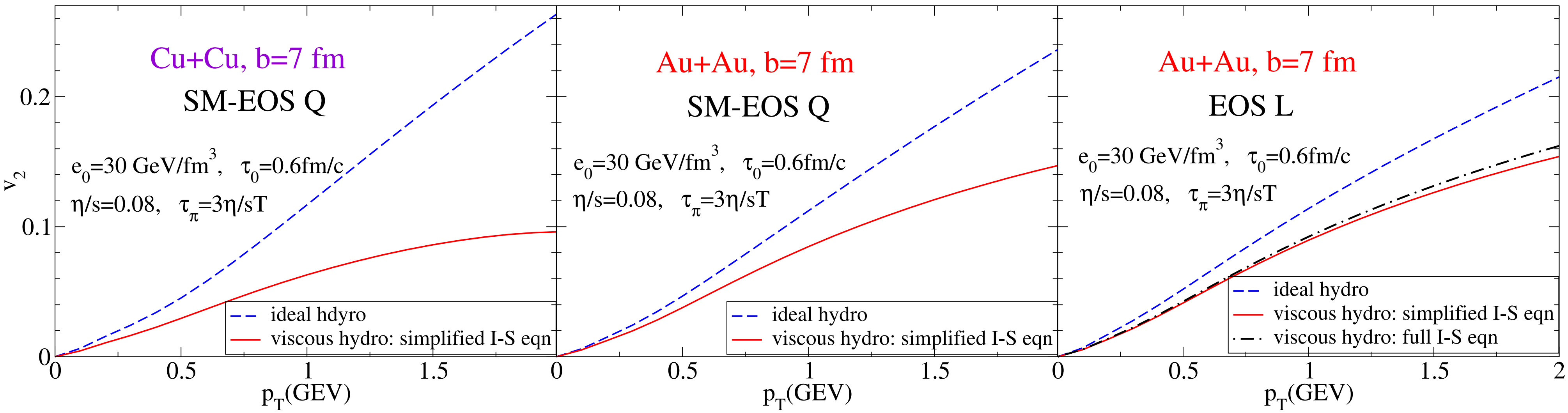
- - - ideal hydro, ——, · · · viscous hydro

- Source eccentricity $\epsilon_x = \frac{\langle y^2 - x^2 \rangle}{\langle y^2 + x^2 \rangle}$ decays initially faster, but later more slowly;
- Flow anisotropy $\epsilon_p = \frac{\langle T_0^{xx} - T_0^{yy} \rangle}{\langle T_0^{xx} + T_0^{yy} \rangle}$ develops faster initially, but soon drops significantly below ideal fluid values;
- during the first 3-4 fm/c viscous pressure components $\pi^{\mu\nu}$ contribute strong out-of-plane (i.e. negative) momentum anisotropy in the local fluid rest frame; inhibits build-up of flow anisotropy and delays local momentum isotropization
- Total momentum anisotropy $\epsilon'_p = \frac{\langle T^{xx} - T^{yy} \rangle}{\langle T^{xx} + T^{yy} \rangle}$ is reduced by almost 50% relative to ideal fluid.

Pion elliptic flow from 2D+1 viscous hydrodynamics

Dependence on system size, EOS, and 2nd order formalism used:

$$\frac{\eta}{s} = \frac{1}{4\pi},\ \tau_\pi = 0.24\left(\frac{200\,\mathrm{MeV}}{T}\right)\,\mathrm{fm}/c$$

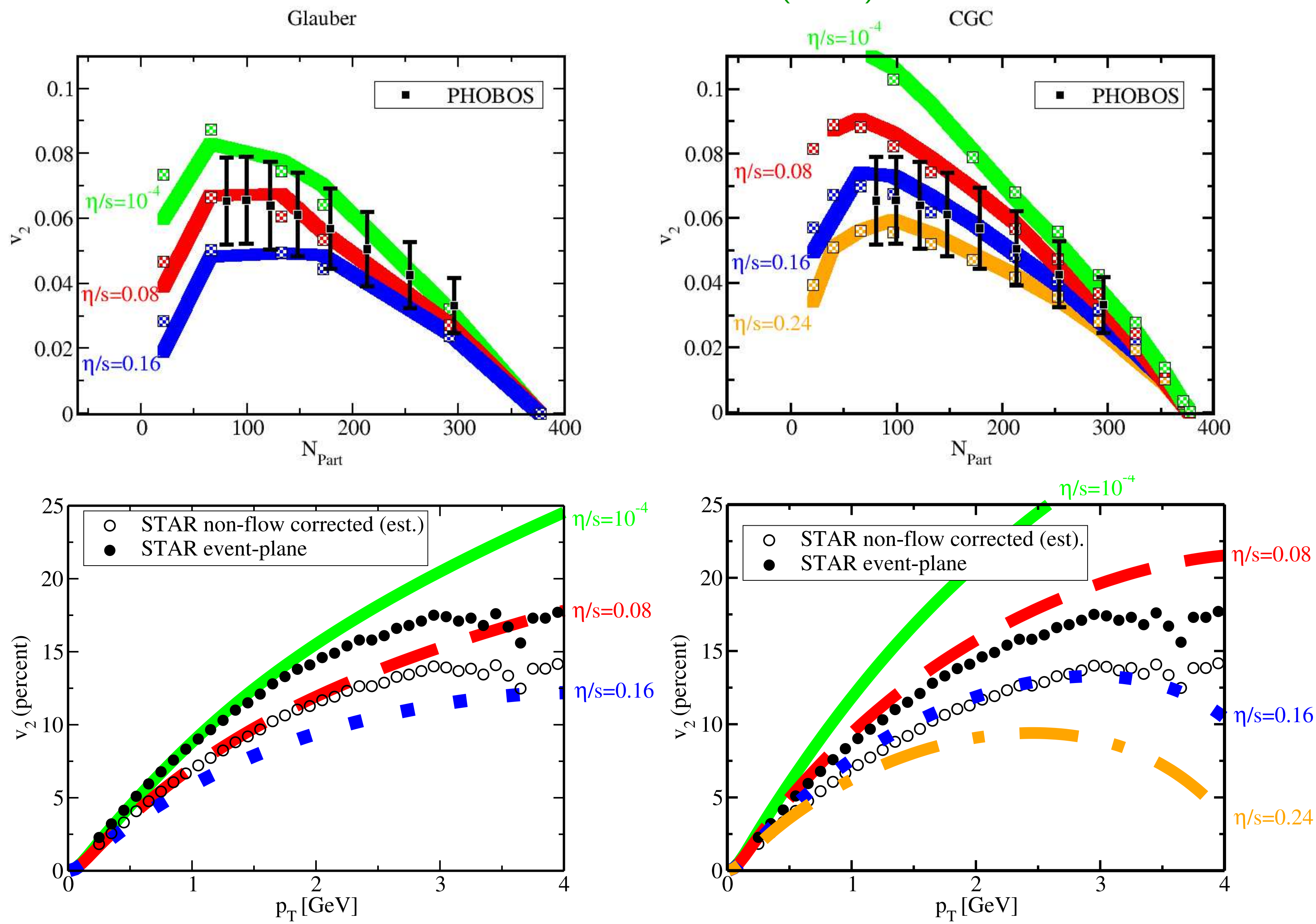


- **Elliptic flow very sensitive to even minimal shear viscosity!**
- Viscous v_2 suppression significantly larger in smaller collision systems
- Difference in v_2 suppression between SM-EOS Q and EOS L $\sim 25-30\%$
- **Suggests two ways to extract η/s:** (i) suppression of v_2 below ideal fluid baseline; (ii) system size dependence of v_2 suppression.

Towards comparison with experiment

Constraining η/s from charged hadron elliptic flow data

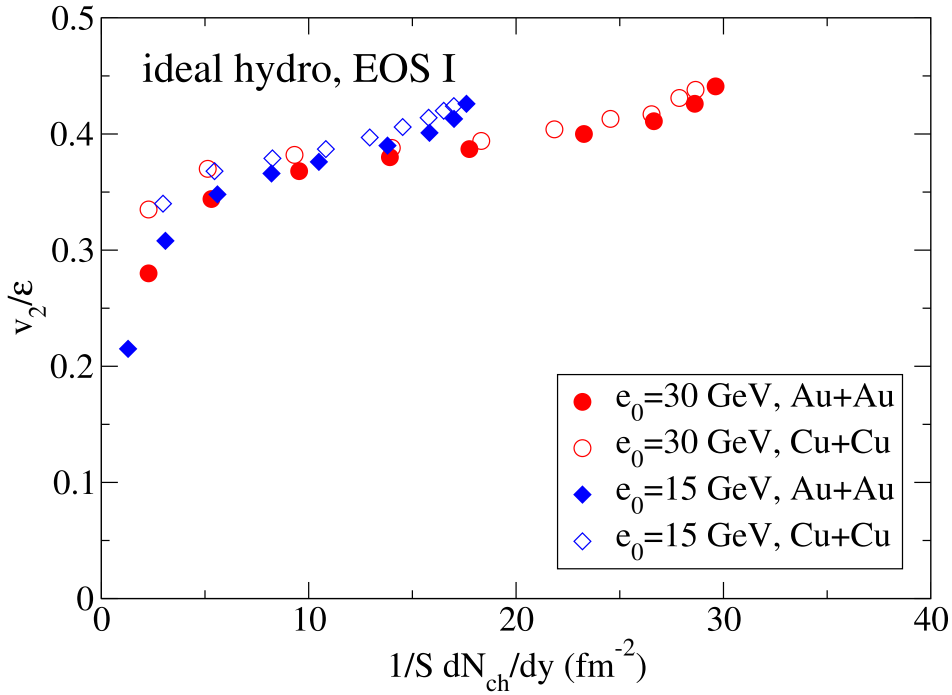
Luzum & Romatschke, PRC 78 (2008) 034915

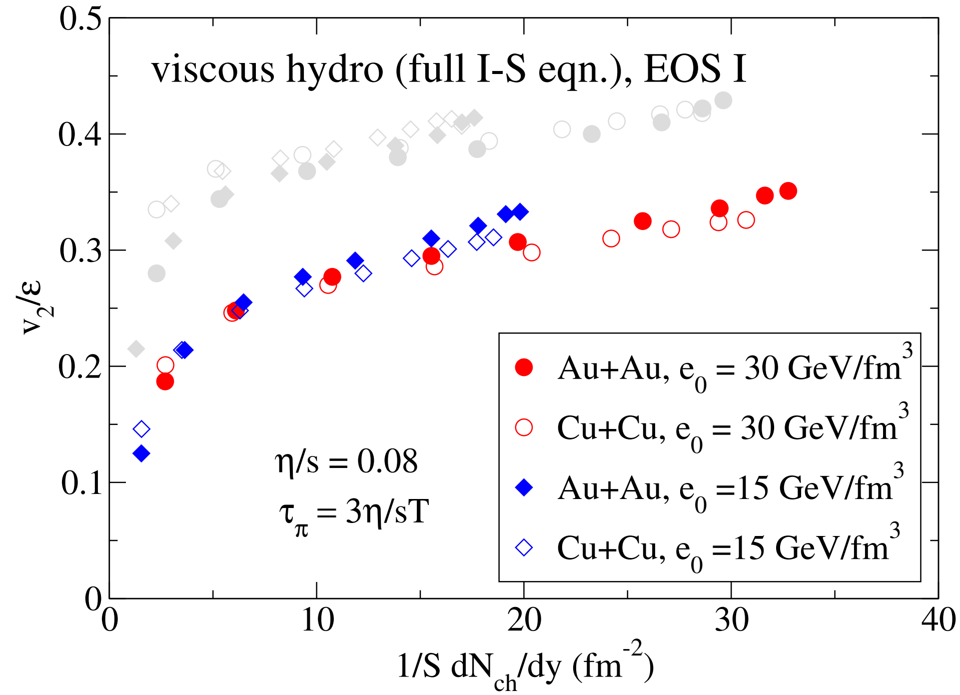


- **Largest uncertainty ($\sim 100\%$): initial source eccentricity!**
- Others: EOS near T_c ($\sim 25-30\%$); chemical comp. below T_c (??); no pre-equilibrium transverse flow; late hadronic viscous effects not subtracted; bulk viscous v_2 suppression not subtracted
- "Conservative" upper limit: $\frac{\eta}{s} < \frac{6}{4\pi} \approx 0.5$ (Luzum & Romatschke '08)

Multiplicity scaling of the normalized elliptic flow v_2/ε_x (I)

Song & Heinz, PRC 78 (2008) 024902



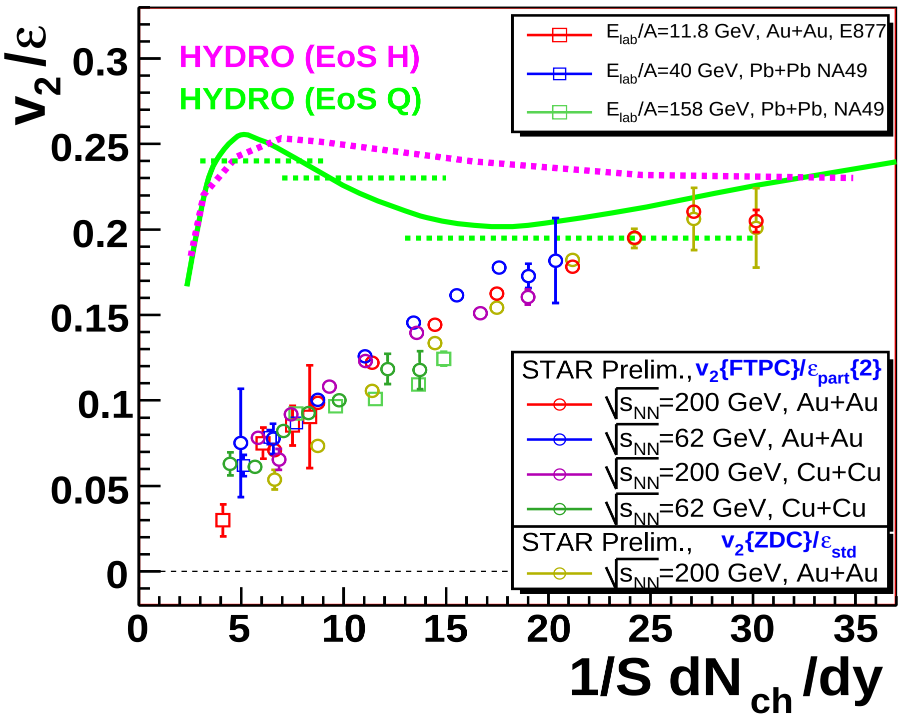


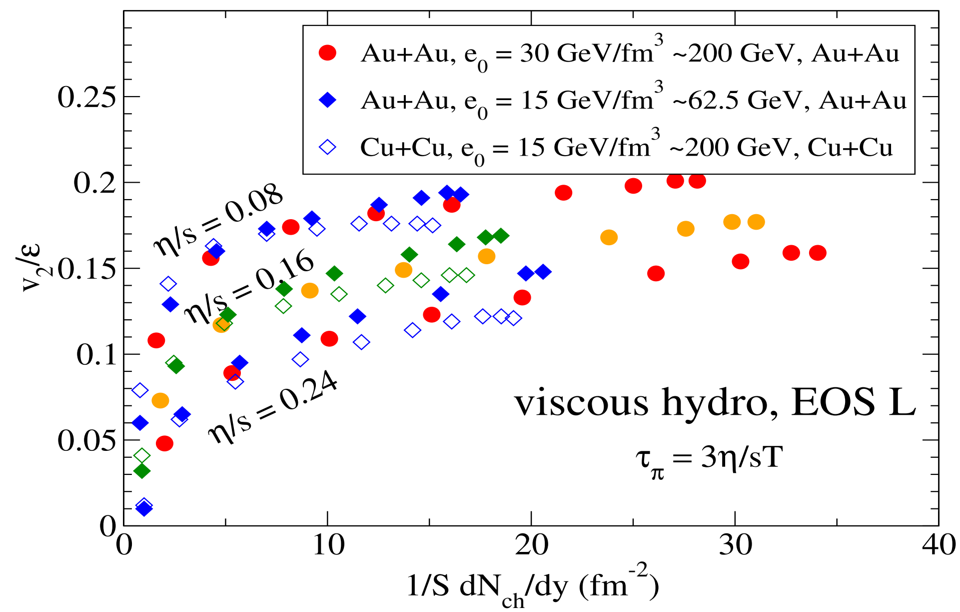
- Freeze-out at constant $e_{\rm dec}$ introduces time scale, breaking the scale invariance of ideal hydro and cutting short the build-up of elliptic flow before it saturates
- At the same $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$, collisions between smaller nuclei and more peripheral collisions freeze out earlier, with less elliptic flow v_2/ϵ_x
- This breaks the multiplicity scaling with $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$ even for ideal hydro
- For larger than minimal η/s this scaling is broken more strongly in viscous hydro

  At fixed $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$, smaller collision systems and more peripheral collisions show more viscous suppression of v_2/ϵ_x than more central collisions or collisions of larger nuclei

Multiplicity scaling of the normalized elliptic flow v_2/ε_x (II)

A case study with fixed specific viscosity η/s:





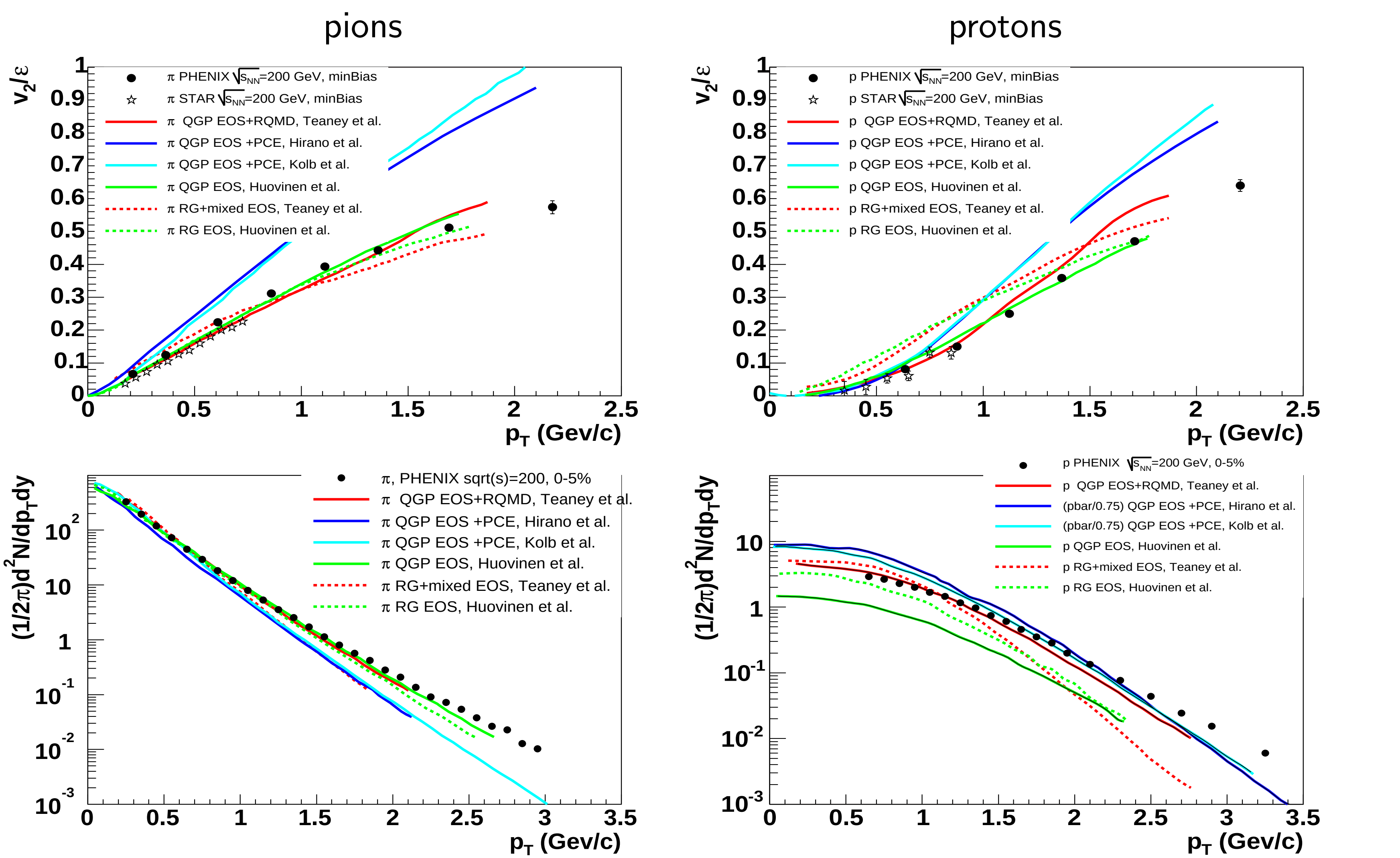
- General tendency of experimental data consistent with viscous effects
- **At low $(1/S)(dN_{\rm ch}/dy)$ the data require more than minimal shear viscosity** (because of the highly viscous late hadron gas stage)
- Search for scale-breaking effects requires more accurate data
- Realistic modeling must account for T-dependence of shear and bulk viscosity, especially near $T_{\rm c}$

Open Issues:

1. Hadronic viscosity and non-equilibrium chemistry
2. Initial fireball eccentricity
3. Early pre-equilibrium flow
4. Equation of State
5. Bulk viscosity

At RHIC, hadronic viscosity & chemical non-equilibrium matter:

PHENIX White Paper, NPA 757 (2005) 184



All theory curves use the same hydrodynamics and EOS in QGP phase!

How we deal with the hadron phase makes all the difference . . .

The only model that simultaneously fits all data is hydro+RQMD

(**Teaney & Shuryak 2001**)

Open Issues:

1. **Hadronic viscosity and non-equilibrium chemistry**

2. **Initial fireball eccentricity**

3. **Early pre-equilibrium flow**

4. **Equation of State**

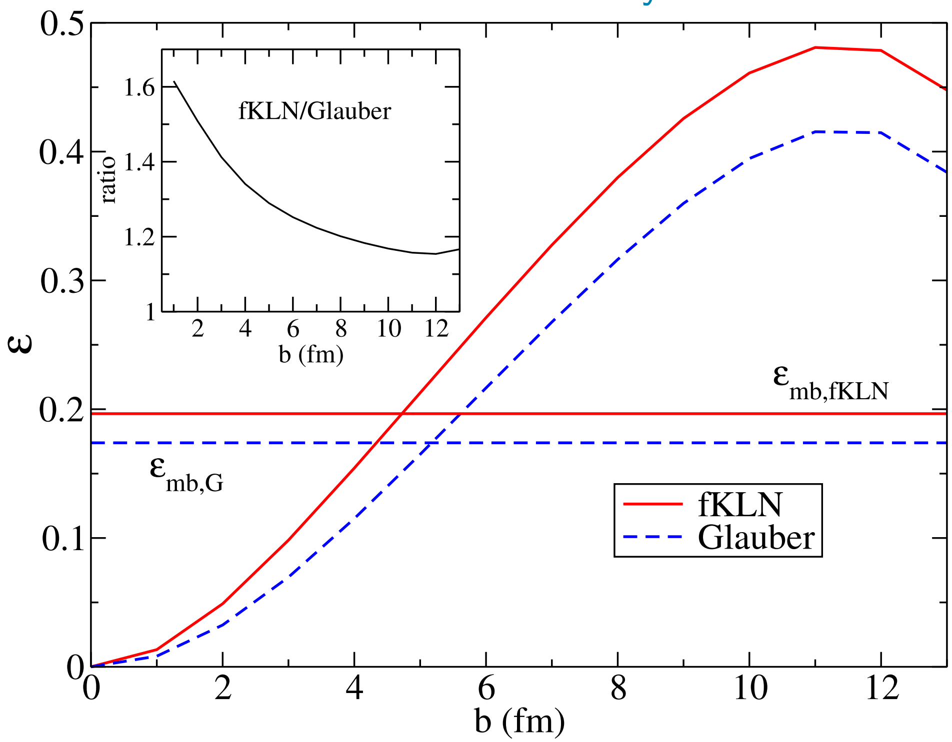
5. **Bulk viscosity**

Initial source eccentricity and viscous v_2 suppression

(UH, Moreland, Song, arXiv:0908.2617)

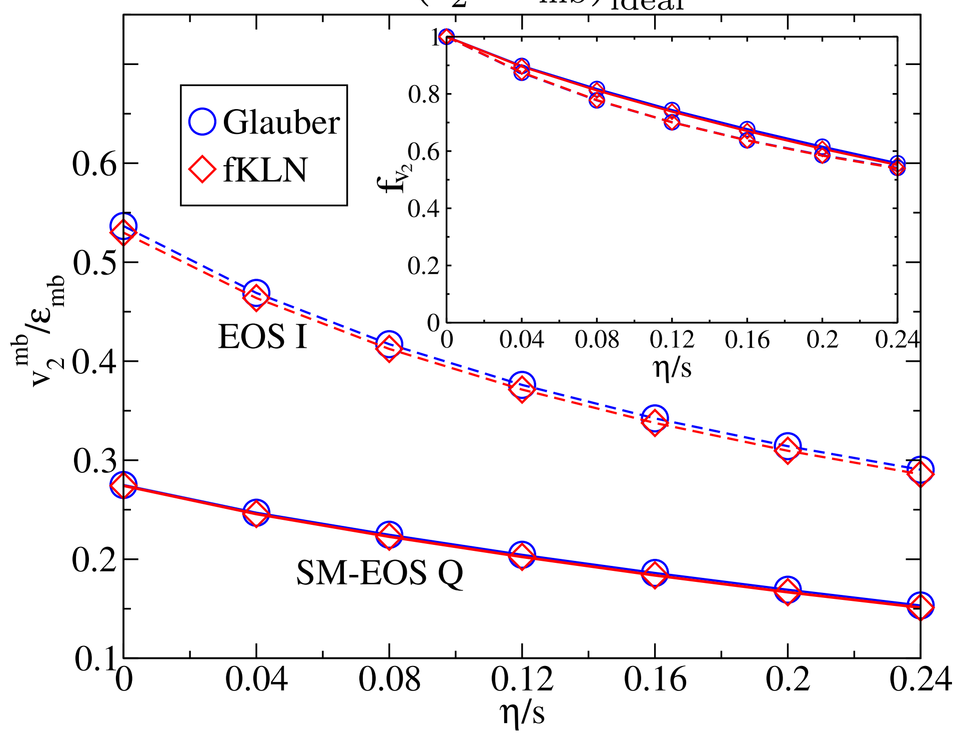
fKLN model: H.-J. Drescher et al., PRC 74 (2006) 044905

Initial source eccentricity vs. b



Scaled elliptic flow vs. η/s

$$f_{v_2} = \frac{\left(v_2^{mb}/\varepsilon_{mb}\right)_{viscous}}{\left(v_2^{mb}/\varepsilon_{mb}\right)_{ideal}}$$

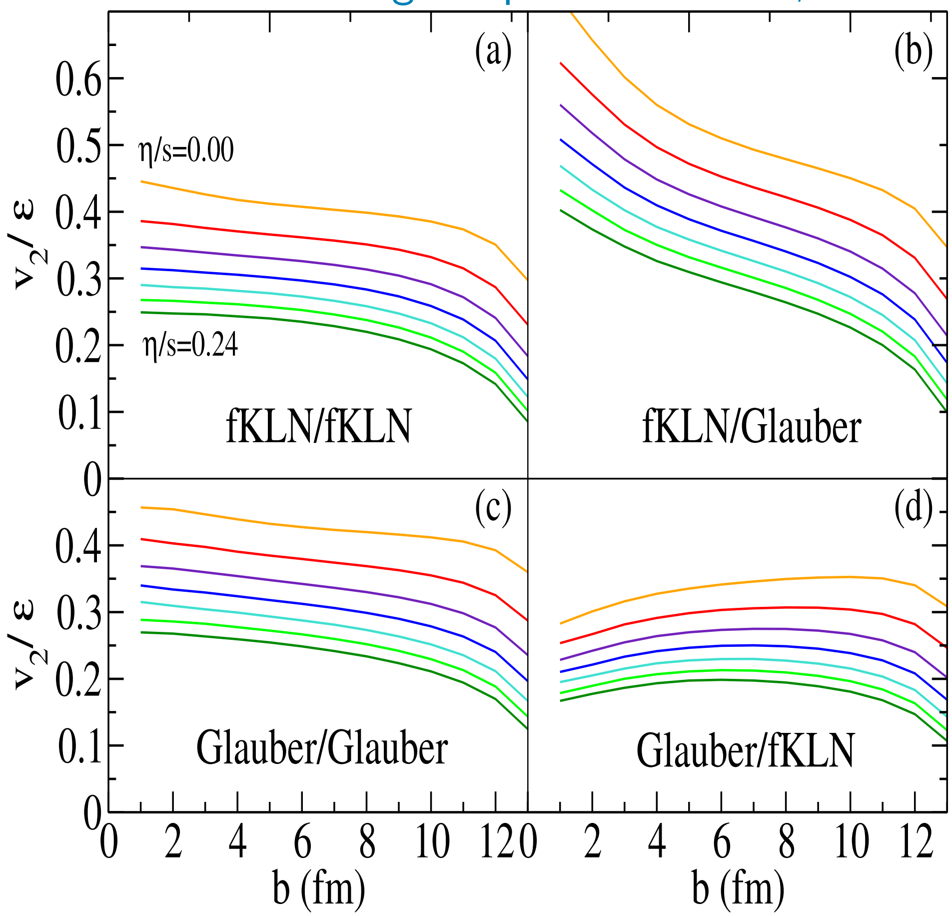


- CGC/fKLN model gives larger source eccentricity than Glauber model, but excess is strongly centrality dependent!
- Minimum bias eccentricity 12% larger for fKLN than Glauber $\Longrightarrow$ $\mathcal{O}(100\%)$ uncertainty for η/s.
- Viscous v_2 suppression relative to ideal hydro is a unique function of η/s, independent of EOS.

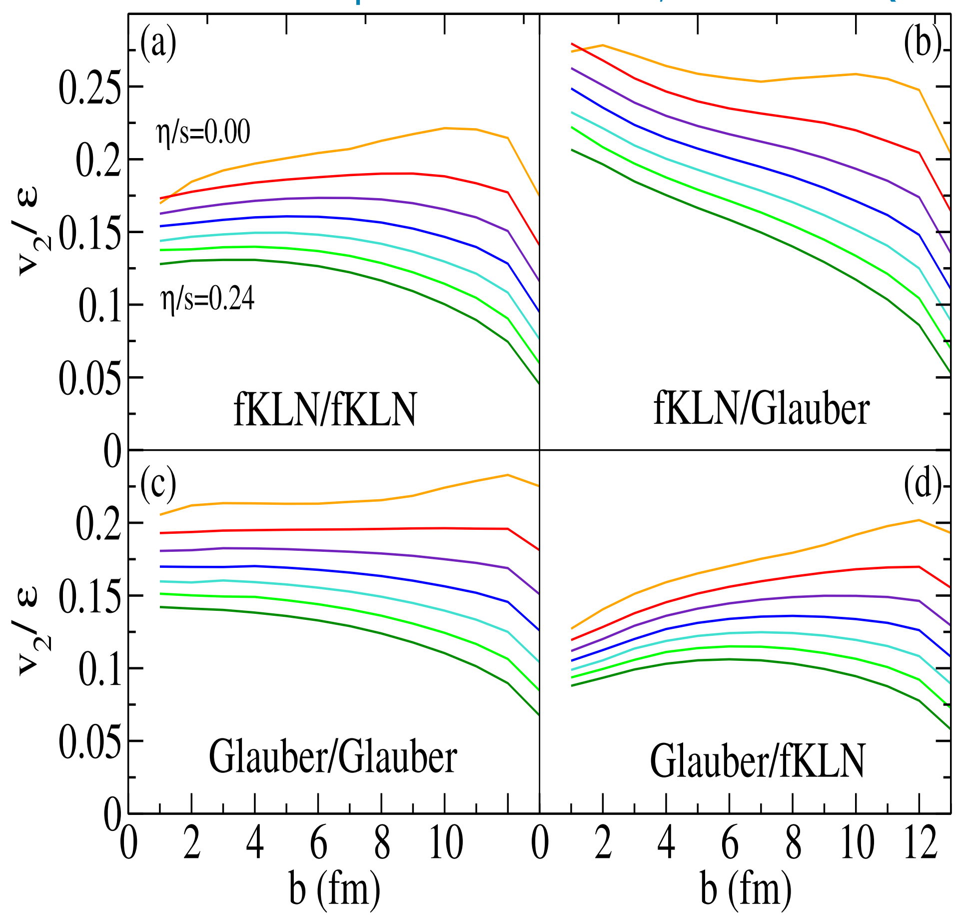
Centrality dependence of viscous suppression of v_2/ε

(UH, Moreland, Song, arXiv:0908.2617)

Massless ideal gas equation of state, EOS I



EOS with phase transition, SM-EOS Q



- If Nature provides Glauber initial conditions, but we normalize v_2 by fKLN eccentricity, scaled elliptic flow **increases** with impact parameter!
- Robust signal, does not depend on EOS.
- This is not seen in experiment $\Longrightarrow$ Glauber initial state model eliminated by data.

Open Issues:

1. **Hadronic viscosity and non-equilibrium chemistry**

2. **Initial fireball eccentricity**

3. **Early pre-equilibrium flow:** see S. Pratt's talk

4. **Equation of State**

5. **Bulk viscosity**

Open Issues:

1. **Hadronic viscosity and non-equilibrium chemistry**

2. **Initial fireball eccentricity**

3. **Early pre-equilibrium flow**

4. **Equation of State:** see P. Petreczky's talk

5. **Bulk viscosity**

Open Issues:

1. **Hadronic viscosity and non-equilibrium chemistry**
2. **Initial fireball eccentricity**
3. **Early pre-equilibrium flow**
4. **Equation of State**
5. **Bulk viscosity: see Huichao Song's talk**

Summary

- Shear viscosity reduces the longitudinal pressure but increases the transverse pressure in heavy ion collision
  $\Longrightarrow$ slower cooling by longitudinal work initially, but faster cooling by stronger transverse expansion later

- For same initial conditions, viscous hydro leads to flatter p_T-spectra (increased radial flow)

- While viscous pressure effects on angle-averaged p_T-spectra (radial flow) can be largely absorbed by changing the initial conditions (starting the transverse expansion later and with lower initial energy density), this increases the destructive effects of shear viscosity on the buildup of elliptic flow.

- The effects of shear viscosity on elliptic flow are **large! RHIC data seem to require $\frac{\eta}{s} < 0.5$**. Largest uncertainty stems from initial source eccentricity. Can be constrained by centrality dependence of v_2/ϵ.

- With full I-S system, sensitivities to initial values and kinetic relaxation time for $\pi^{\mu\nu}$ are weak and can be neglected. I-S and Ö-G appear to give very similar results.

- Multiplicity scaling of normalized elliptic flow v_2/ϵ_x weakly broken by freeze-out in ideal hydro and slightly more strongly broken by shear viscosity in viscous hydro. Experimentally observed scaling requires **larger η/s in hadronic matter than in QGP**.

- Extraction of QGP viscosity from elliptic flow data requires full simulation model: early pre-equilibrium model $\oplus$ viscous hydro $\oplus$ hadron cascade, plus careful modeling of EOS and temperature dependence of η/s and ζ/s. **NO SHORTCUTS!**

Supplements

(2+1)-d viscous hydrodynamic equations

Heinz, Song & Chaudhuri, PRC 73 (2006) 034904

Transverse dynamics w/o azimuthal symmetry, but with long. boost invariance: Use (τ, x, y, η) coordinates and solve

- hydrodynamic equations for $T^{\tau\tau} = (e{+}p)\gamma_r^2 - p + \pi^{\tau\tau}$, $T^{\tau x} = (e{+}p)\gamma_\perp^2 v_x + \pi^{\tau x}$, $T^{\tau y} = (e{+}p)\gamma_\perp^2 v_y + \pi^{\tau y}$:

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau\tau}\Big) + \partial_x\Big(v_x T^{\tau\tau}\Big) + \partial_y\Big(v_y T^{\tau\tau}\Big) = \mathcal{S}^{\tau\tau}[v_x, v_y, \pi^{\eta\eta}, \pi^{\tau\tau}, \pi^{\tau x}, \pi^{\tau y}]$$

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau x}\Big) + \partial_x\Big(v_x T^{\tau x}\Big) + \partial_y\Big(v_y T^{\tau x}\Big) = \mathcal{S}^{\tau x}[v_x, v_y, \pi^{xx}, \pi^{xy}, \pi^{\tau x}]$$

$$\frac{1}{\tau}\partial_\tau\Big(\tau T^{\tau y}\Big) + \partial_x\Big(v_x T^{\tau y}\Big) + \partial_y\Big(v_y T^{\tau y}\Big) = \mathcal{S}^{\tau y}[v_x, v_y, \pi^{yy}, \pi^{xy}, \pi^{\tau y}]$$

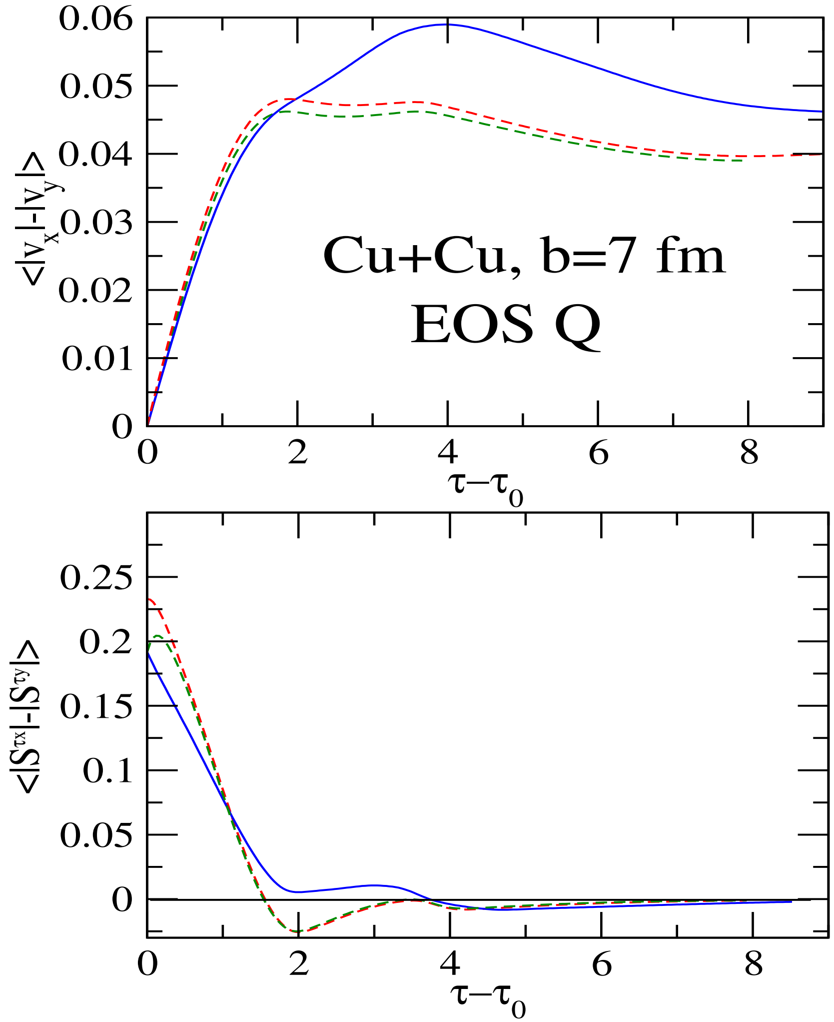
- kinetic relaxation equations for $\pi^{\tau\tau}$, $\pi^{\tau x}$, $\pi^{\tau y}$, and $\pi^{\eta\eta}$ (4, not 3!).

Close equations with EOS $p(e)$ where $e = M_0 - v_\perp M$ and $v_\perp = M/(M_0 + p(e))$ (again one implicit scalar equation!), with the definitions $(M_0, M_x, M_y) \equiv (T^{\tau\tau} - \pi^{\tau\tau},\ T^{\tau x} - \pi^{\tau x},\ T^{\tau y} - \pi^{\tau y})$ and $M = \sqrt{M_x^2 + M_y^2}$, and the relations $v_x = M_x/M,\ v_y = M_y/M$.

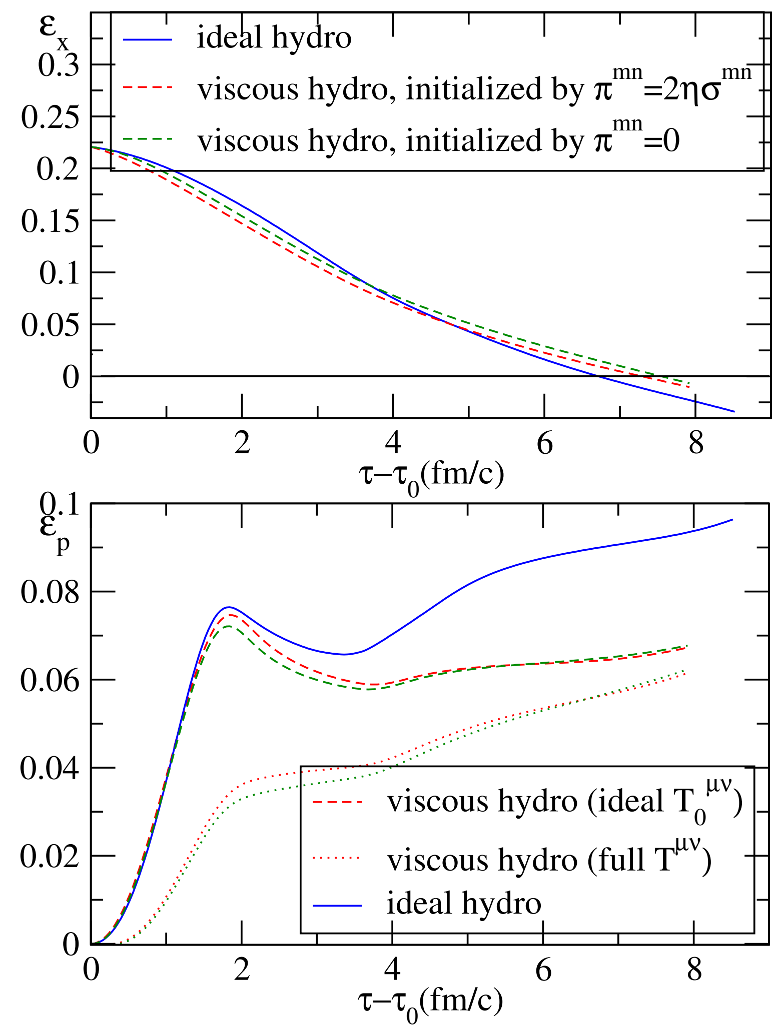
Sensitivity to initial values for viscous pressure tensor

Romatschke & Romatschke 2007 seem to find much smaller viscous effects than we do. But they initialize their evolution with $\pi^{mn} = 0$. Could this be the origin of the discrepancy? **No!**

flow anisotropy vs. time



spatial eccentricity and momentum anisotropy

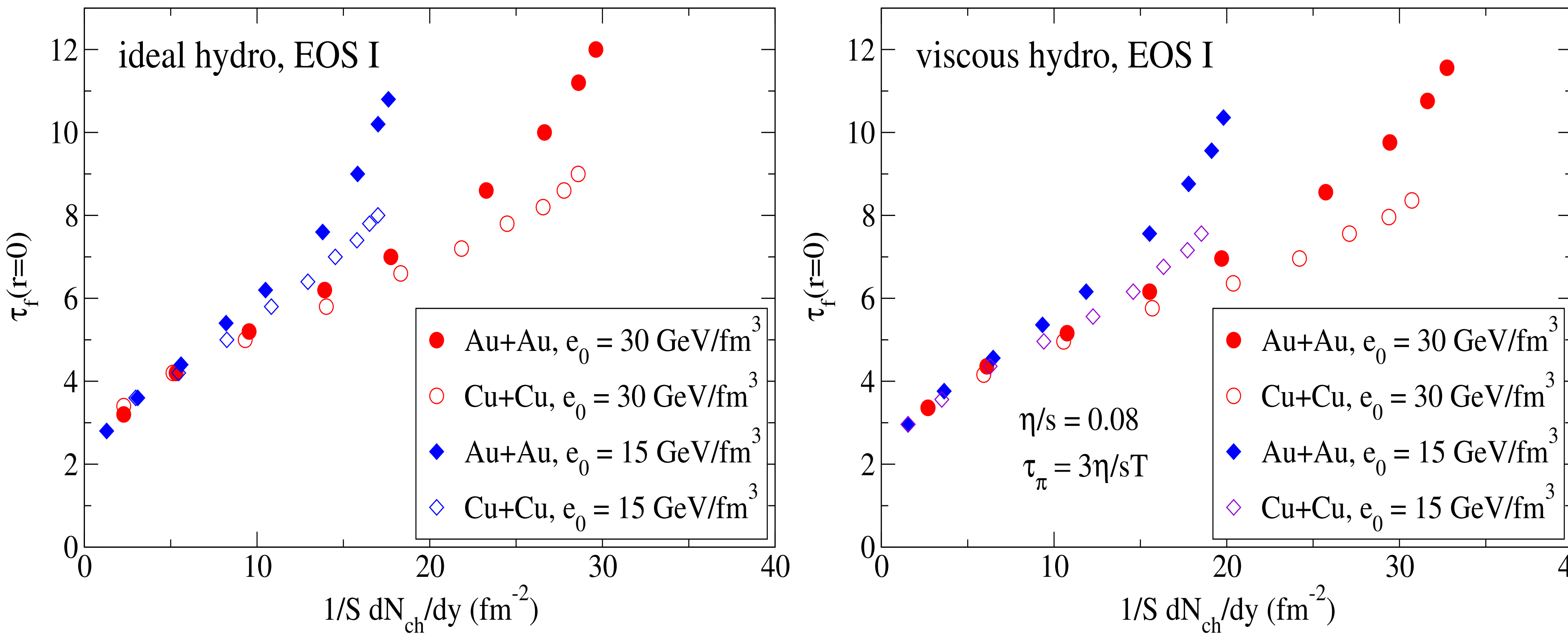


Green lines show results for $\pi_0^{mn} = 0$, with otherwise identical parameters

$\Longrightarrow$ weak sensitivity to initial conditions for viscous pressure tensor.

Central freeze-out times for different collisions systems and centralities

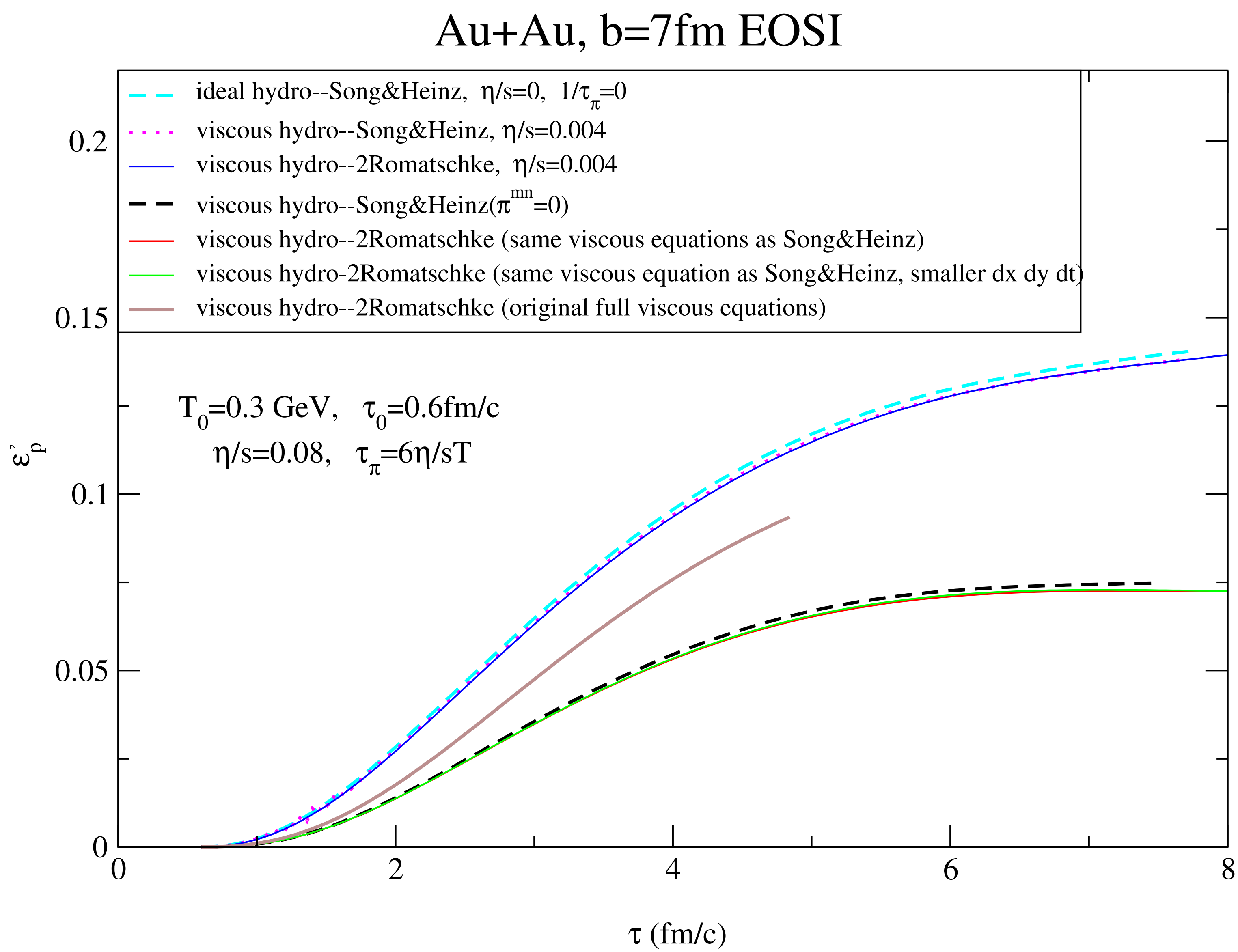
Heinz & Song, JPG 35 (2008) 104126



- At the same $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$, collisions between larger nuclei and more central collisions take longer to freeze out

Comparison between VISH2+1 and Romatschkes' code

Evolution of total momentum anisotropy ϵ'_p, Au+Au with EOS I



Sensitivity to parameters

$$(\tau_\pi,\ \pi^{\mu\nu}(\tau_0))$$

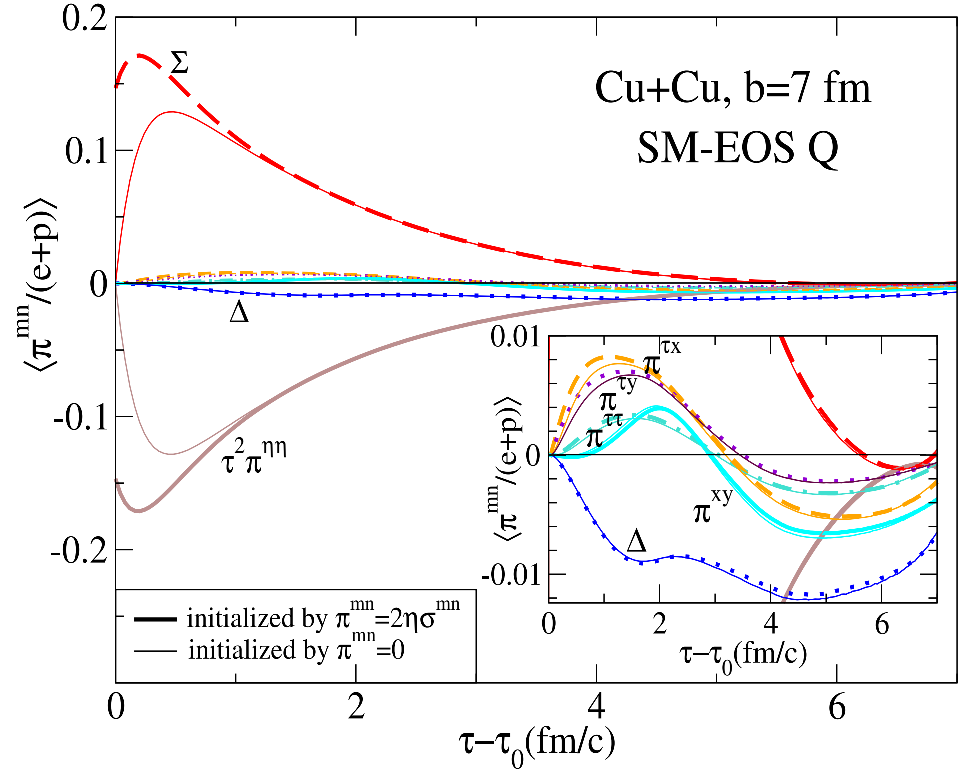
Sensitivity to initial values for viscous pressure tensor

Thin lines: $\pi_0^{mn} = 0$; Thick lines: $\pi_0^{mn} = 2\eta\sigma^{mn} \equiv 2\eta\nabla^{\langle m}u^{n\rangle}$.

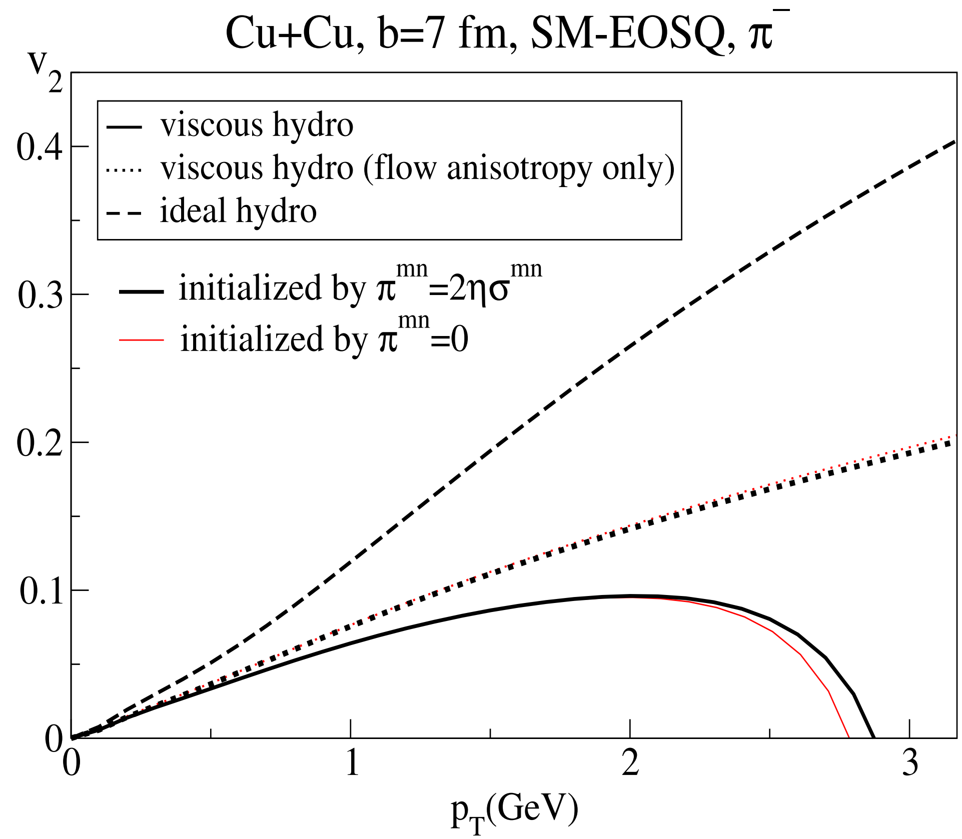
$\tau_0 = 0.6\,\frac{\text{fm}}{c},\ e_0 = 30\,\frac{\text{GeV}}{\text{fm}^3},\ \frac{\eta}{s} = \frac{1}{4\pi},\ \tau_\pi = 0.24\left(\frac{200\,\text{MeV}}{T}\right)\frac{\text{fm}}{c},\ T_{\text{dec}} = 130\,\text{MeV}$

largest viscous pressure components vs. time

$\Sigma = \pi^{xx}+\pi^{yy},\quad \Delta = \pi^{xx}-\pi^{yy}$

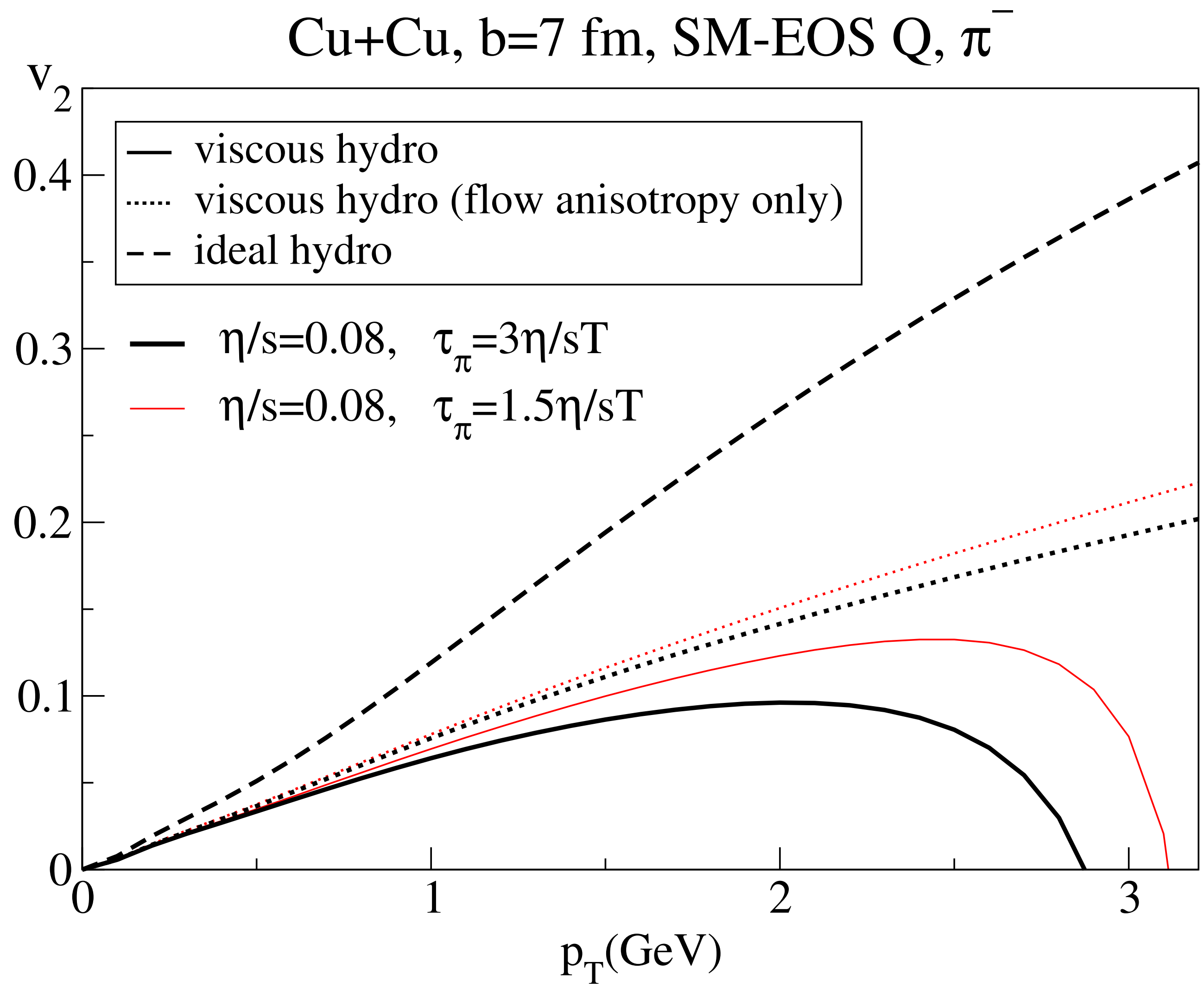


pion elliptic flow



- For fixed η/s, viscous pressure components become small at late times $\longrightarrow$ ideal hydro
- After $\tau \sim 1\,\text{fm}/c \sim 5\tau_\pi$, viscous pressure tensor has lost all memory of initial conditions!
- Effects of initial π^{mn} on final v_2 are small

Sensitivity to kinetic relaxation time τ_π:



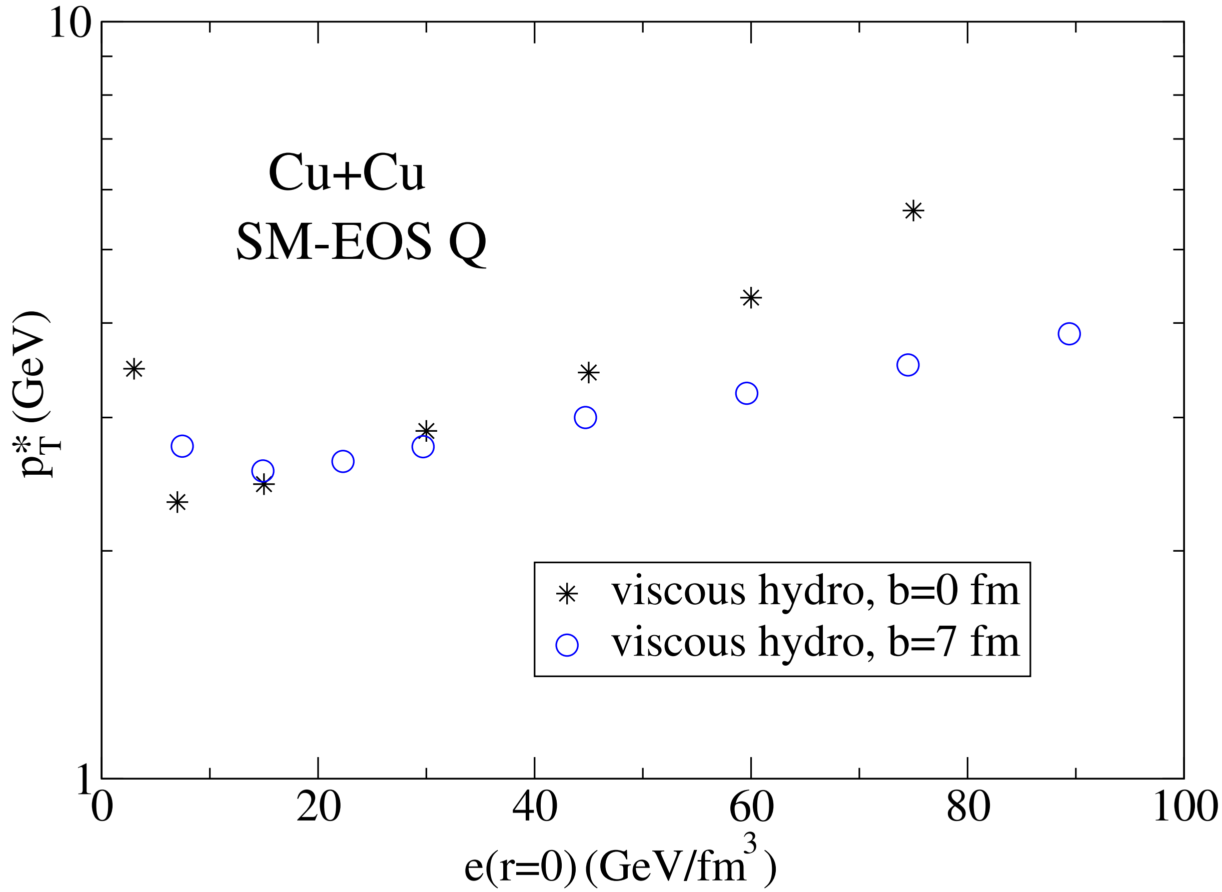
- **Faster kinetic relaxation at fixed η/s reduces viscous effects $\longrightarrow$ Janik 2007**
- **larger τ_π $\rightarrow$ larger $\frac{\pi^{\mu\nu}}{e+p}$ at early times, and more deviation from ideal hydro!**

Limits of viscous hydrodynamics:

The limits of viscous hydrodynamics

At sufficiently large p_T, viscous corrections become large even if η/s is small.

$|\delta N(p)| > \frac{1}{2}|N_0(p)|$ indicates breakdown of the assumptions:

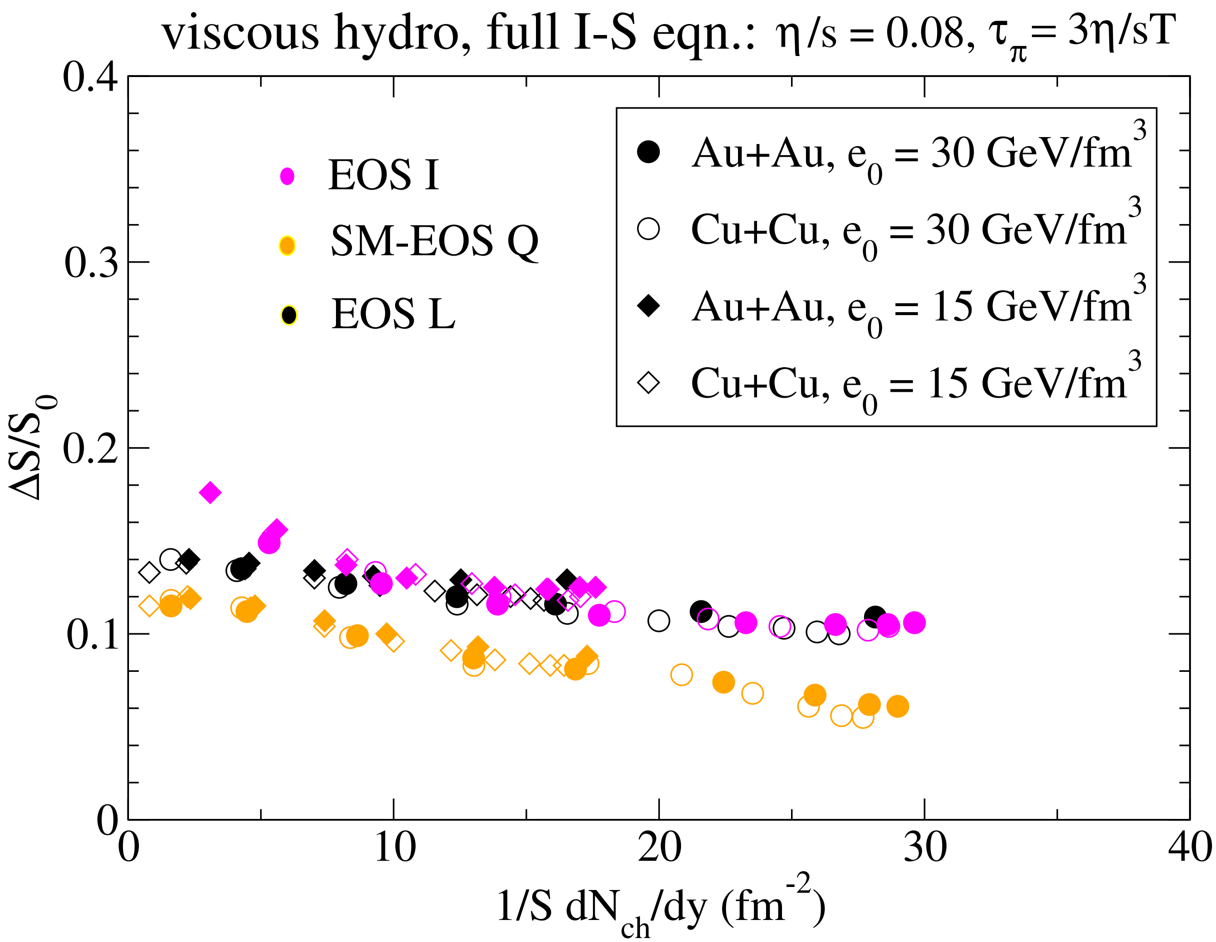


- **For larger initial energy densities, p_T-range increases where viscous hydro can be applied to describe hadron spectra.**

Tests of the viscous hydro code VISH2+1

- $\eta \to 0 \quad \longrightarrow$ ideal fluid code AZHYDRO (test hydro evolution algorithm)
- $\nabla_\perp p = 0$, $\tau_\pi \to 0 \quad \Longrightarrow$ reproduce analytic soln. of boost-invariant Navier-Stokes
- η, τ_π small $\quad \Longrightarrow$ Israel-Stewart $\mapsto$ Navier-Stokes (tests kinetic evolution algorithm for $\pi^{\mu\nu}$)
- $\pi^\mu_\mu = 0$, $u_\mu \pi^{\mu\nu} = 0$ to better than 2%
- Evolution of e, u^μ, $\pi^{\mu\nu}$ by VISH2+1 tested against Romatschkes' code:
  - excellent agreement for identical initial conditions, EOS, kinetic evolution equations
  - large difference in published $v_2(p_T)$ due to extra terms in $D\pi^{\mu\nu} = \dots$ used by the Romatschkes

Viscous entropy production

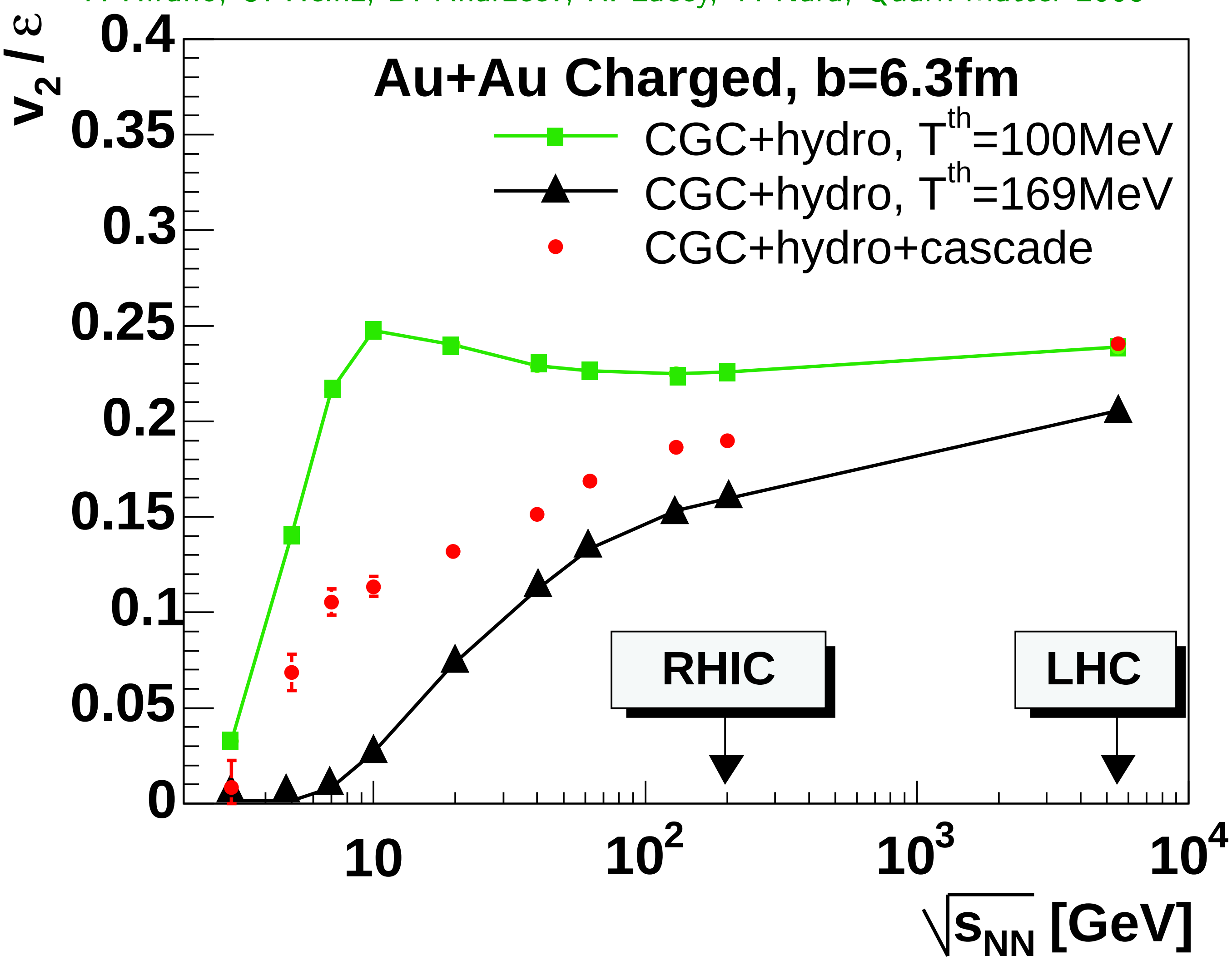


EOS I: ideal gas of massless partons
SM-EOS Q: 1st order QGP-HRG phase transition
EOS L: smooth crossover from lattice QCD data above T_c to HRG below T_c.

- Viscous entropy production larger for faster-expanding fireballs
- Entropy production scales approximately with charged multiplicity density per unit area, $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$
- Entropy production fraction is larger for smaller $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$ (lower-energy and more peripheral collisions)
- At the same $\frac{1}{S}\frac{dN_{\rm ch}}{dy}$, collisions between larger nuclei or more central collisions take longer to freeze out, generating slightly more entropy
- In full Israel-Stewart approach, $\Delta\mathcal{S}/\mathcal{S}_0$ is (almost) τ_π-independent

Hadronic dissipation effects disappear at the LHC:

T. Hirano, U. Heinz, D. Kharzeev, R. Lacey, Y. Nara, Quark Matter 2008



At LHC, all momentum anisotropy is created in QGP phase
$\Longrightarrow$ hadronic dissipation effects become negligible
Late hadronic evolution still important for final distribution of momentum anisotropy over particle species (e.g. pions vs. protons)