

# Complete second-order dissipative relativistic fluid dynamics

**Dirk H. Rischke**

Institut für Theoretische Physik and  
Frankfurt Institute for Advanced Studies

Johann Wolfgang Goethe-Universität  
Frankfurt am Main



**with:**

**Barbara Betz, Tomoi Koide, Harri Niemi**

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## Preliminaries (I)

**Tensor decomposition of net charge current and energy-momentum tensor:**

1. **Net charge current:**

$$\boxed{N^\mu = n u^\mu + \nu^\mu}$$

$u^\mu$  **fluid 4-velocity**,  $u^\mu u_\mu = u^\mu g_{\mu\nu} u^\nu = 1$

$g_{\mu\nu} \equiv \text{diag}(+, -, -, -)$  (**West coast!!**) **metric tensor**,

$n \equiv u^\mu N_\mu$  **net charge density in fluid rest frame**

$\nu^\mu \equiv \Delta^{\mu\nu} N_\nu$  **diffusion current** (flow of net charge relative to  $u^\mu$ ),  $\nu^\mu u_\mu = 0$

$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$  **projector onto 3-space orthogonal to  $u^\mu$** ,  $\Delta^{\mu\nu} u_\nu = 0$

2. **Energy-momentum tensor:**

$$\boxed{T^{\mu\nu} = \epsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + 2 q^{(\mu} u^{\nu)} + \pi^{\mu\nu}}$$

$\epsilon \equiv u^\mu T_{\mu\nu} u^\nu$  **energy density in fluid rest frame**

$p$  **pressure in fluid rest frame**

$\Pi$  **bulk viscous pressure**,  $p + \Pi \equiv -\frac{1}{3} \Delta^{\mu\nu} T_{\mu\nu}$

$q^\mu \equiv \Delta^{\mu\nu} T_{\nu\lambda} u^\lambda$  **heat flux current** (flow of energy relative to  $u^\mu$ ),  $q^\mu u_\mu = 0$

$\pi^{\mu\nu} \equiv T^{<\mu\nu>}$  **shear stress tensor**,  $\pi^{\mu\nu} u_\mu = \pi^{\mu\nu} u_\nu = 0$ ,  $\pi^\mu{}_\mu = 0$

$a^{(\mu\nu)} \equiv \frac{1}{2} (a^{\mu\nu} + a^{\nu\mu})$  **symmetrized tensor**

$a^{<\mu\nu>} \equiv \left( \Delta_\alpha^{(\mu} \Delta^{\nu)}_\beta - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$  **symmetrized, traceless spatial projection**

## Preliminaries (II)

### Fluid dynamical equations:

1. **Net charge** (e.g., baryon, strangeness, etc.) **conservation:**

$$\partial_\mu N^\mu = 0 \iff \dot{n} + n\theta + \partial \cdot \nu = 0$$

$\dot{a} \equiv u^\mu \partial_\mu a$  convective (comoving) derivative

(fluid rest frame,  $u_{\text{RF}}^\mu \equiv g^\mu_0 \implies$  time derivative,  $\dot{a}_{\text{RF}} \equiv \partial_t a$ )

$\theta \equiv \partial_\mu u^\mu$  expansion scalar

2. **Energy-momentum conservation:**

$$\partial_\mu T^{\mu\nu} = 0 \iff \text{energy conservation:}$$

$$u_\nu \partial_\mu T^{\mu\nu} = \dot{\epsilon} + (\epsilon + p + \Pi)\theta + \partial \cdot q - q \cdot \dot{u} - \pi^{\mu\nu} \partial_\mu u_\nu = 0$$

acceleration equation:

$$\Delta^{\mu\nu} \partial^\lambda T_{\nu\lambda} = 0 \iff$$

$$(\epsilon + p)\dot{u}^\mu = \nabla^\mu(p + \Pi) - \Pi\dot{u}^\mu - \Delta^{\mu\nu}\dot{q}_\nu - q^\mu\theta - q^\nu\partial_\nu u^\mu - \Delta^{\mu\nu}\partial^\lambda\pi_{\nu\lambda}$$

$\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$  3-gradient (spatial gradient in fluid rest frame)

## Preliminaries (III)

### Problem:

5 equations, **but** 15 unknowns (for given  $u^\mu$ ):  $\epsilon$ ,  $p$ ,  $n$ ,  $\Pi$ ,  $\nu^\mu$  (3),  $q^\mu$  (3),  $\pi^{\mu\nu}$  (5)

### Solution:

1. **clever choice of frame** (Eckart, Landau,...): eliminate  $\nu^\mu$  or  $q^\mu$ 
  - $\implies$  does not help! Promotes  $u^\mu$  to dynamical variable!
2. **ideal fluid limit**: all dissipative terms vanish,  $\Pi = \nu^\mu = q^\mu = \pi^{\mu\nu} = 0$ 
  - $\implies$  6 unknowns:  $\epsilon$ ,  $p$ ,  $n$ ,  $u^\mu$  (3) (not quite there yet...)
  - $\implies$  fluid is in local thermodynamical equilibrium
  - $\implies$  provide **equation of state (EOS)**  $p(\epsilon, n)$  to close system of equations
3. **provide additional equations for dissipative quantities**
  - $\implies$  **dissipative** relativistic fluid dynamics
    - (a) **First-order** theories: e.g. generalization of **Navier-Stokes (NS)** equations to the relativistic case (Eckart, Landau-Lifshitz)
    - (b) **Second-order** theories: e.g. **Israel-Stewart (IS)** equations

## Preliminaries (IV)

Navier-Stokes (NS) equations:

1. **bulk viscous pressure:**  $\Pi_{\text{NS}} = -\zeta \theta$

$\zeta$  **bulk viscosity**

2. **heat flux current:**

$$q_{\text{NS}}^{\mu} = \frac{\kappa}{\beta} \frac{n}{\beta(\epsilon + p)} \nabla^{\mu} \alpha$$

$\beta \equiv 1/T$  **inverse temperature,**

$\alpha \equiv \beta \mu$ ,  $\mu$  **chemical potential,**

$\kappa$  **thermal conductivity**

3. **shear stress tensor:**  $\pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$

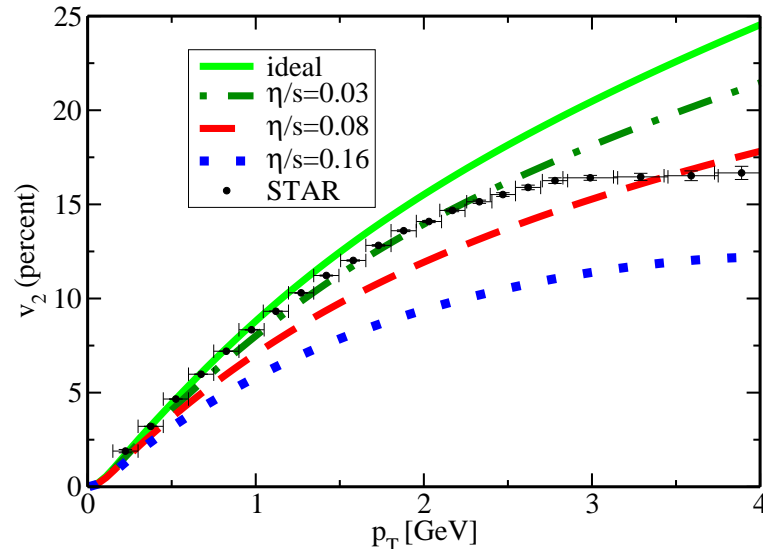
$\eta$  **shear viscosity,**

$\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$  **shear tensor**

$\Rightarrow$  algebraic expressions in terms of thermodynamic and fluid variables

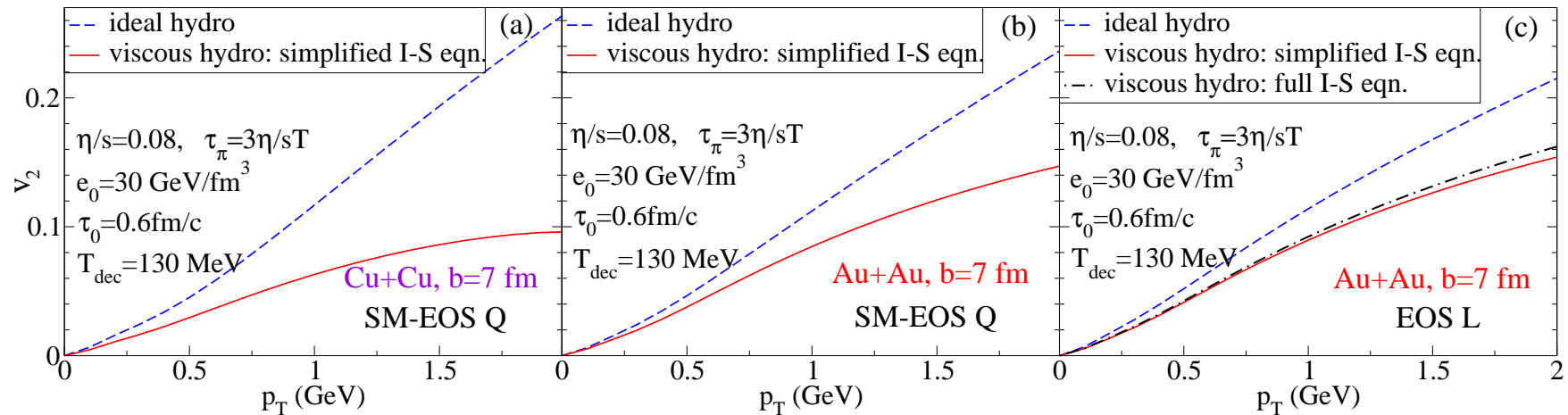
$\Rightarrow$  simple... **but: unstable and acausal** equations of motion!!

## Motivation (I)



P. Romatschke, U. Romatschke, PRL 99 (2007) 172301  
 Au+Au @  $\sqrt{s} = 200$  AGeV  
 charged particles, min. bias

H. Song, U.W. Heinz, PRC 78 (2008) 024902



## Motivation (II)

**Israel-Stewart (IS) equations: second-order, dissipative relativistic fluid dynamics**

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

**“Simplified” IS equations: e.g. shear stress tensor**

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = \pi_{\text{NS}}^{\mu\nu}$$

⇒ **dynamical** (instead of **algebraic**) equations for dissipative terms!

⇒  $\pi^{\mu\nu}$  relaxes to its **NS** value  $\pi_{\text{NS}}^{\mu\nu}$  on the time scale  $\tau_\pi$

⇒ **stable** and **causal** fluid dynamical equations of motion!

**“Full” IS equations:**

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = \pi_{\text{NS}}^{\mu\nu} - \frac{\eta}{2\beta} \pi^{\mu\nu} \partial_\lambda \left( \frac{\tau_\pi}{\eta} \beta u^\lambda \right) + 2 \tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda}$$

$$\omega^{\mu\nu} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha) \quad \text{vorticity}$$

## Motivation (III)

⇒ Difference between “simplified” and “full” IS equations:  
the latter include higher-order terms?

For instance, if  $\frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$ ,  $\tau_\pi \omega^{\mu\nu} \sim \delta \ll 1 \implies \tau_\pi \omega_\lambda^{<\mu} \pi^{\nu>\lambda} \frac{1}{\epsilon} \sim \delta^2$

⇒ **Goals:**

1. What are the correct equations of motion for the dissipative quantities?  
⇒ develop consistent **power counting scheme**
2. Generalization to  $\mu \neq 0$  (relevant for FAIR physics!)  
⇒ include **heat flux**  $q^\mu$
3. Generalization to non-conformal fluids (relevant near  $T_c$ !)  
⇒ include **bulk viscous pressure**  $\Pi$

## Results (I)

### Power counting:

3 length scales: 2 microscopic, 1 macroscopic

- thermal wavelength  $\lambda_{\text{th}} \sim \beta \equiv 1/T$
- mean free path  $\ell_{\text{mfp}} \sim (\langle \sigma \rangle n)^{-1}$   
 $\langle \sigma \rangle$  averaged cross section,  $n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$
- length scale over which macroscopic fluid fields vary  $L_{\text{hydro}}$ ,  $\partial_\mu \sim L_{\text{hydro}}^{-1}$

**Note:** since  $\eta \sim (\langle \sigma \rangle \lambda_{\text{th}})^{-1} \implies$

$$\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{1}{\langle \sigma \rangle n} \frac{1}{\lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\lambda_{\text{th}}^3}{\langle \sigma \rangle \lambda_{\text{th}}} \sim \frac{\eta}{s}$$

$s$  entropy density,  $s \sim n \sim T^3 = \beta^{-3} \sim \lambda_{\text{th}}^{-3}$

$\implies \frac{\eta}{s}$  solely determined by the 2 microscopic length scales!

**Note:** similar argument holds for  $\frac{\zeta}{s}$ ,  $\frac{\kappa}{\beta s}$

## Results (II)

### 3 regimes:

● **dilute gas limit**  $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \gg 1 \iff \langle \sigma \rangle \ll \lambda_{\text{th}}^2 \implies$  **weak-coupling limit**

● **viscous fluids**  $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \sim 1 \iff \langle \sigma \rangle \sim \lambda_{\text{th}}^2$

interactions happen on the scale  $\lambda_{\text{th}} \implies$  **moderate coupling**

● **ideal fluid limit**  $\frac{\ell_{\text{mfp}}}{\lambda_{\text{th}}} \sim \frac{\eta}{s} \ll 1 \iff \langle \sigma \rangle \gg \lambda_{\text{th}}^2 \implies$  **strong-coupling limit**

**gradient (derivative) expansion:**

$$\ell_{\text{mfp}} \partial_{\mu} \sim \frac{\ell_{\text{mfp}}}{L_{\text{hydro}}} \equiv K \sim \delta \ll 1$$

**$K$  Knudsen number**

$\implies$  equivalent to  $k \ell_{\text{mfp}} \ll 1$ ,  $k$  typical momentum scale

R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100

$\implies$  separation of macroscopic fluid dynamics (large scale  $\sim L_{\text{hydro}}$ )  
from microscopic particle dynamics (small scale  $\sim \ell_{\text{mfp}}$ )

## Results (III)

**Primary quantities:**  $\epsilon, p, n, s$   $\iff$  **Dissipative quantities:**  $\Pi, q^\mu, \pi^{\mu\nu}$

$$\text{If } K \sim \ell_{\text{mfp}} \partial_\mu \sim \delta \ll 1, \text{ then } \frac{\Pi}{\epsilon} \sim \frac{q^\mu}{\epsilon} \sim \frac{\pi^{\mu\nu}}{\epsilon} \sim \delta \ll 1$$

**Dissipative quantities** are small compared to **primary quantities**

$\implies$  small deviations from local thermodynamical equilibrium!

**Note:** statement independent of value of  $\frac{\zeta}{s}, \frac{\kappa}{\beta s}, \frac{\eta}{s}$  !

**Proof:** Gibbs relation:  $\epsilon + p = Ts + \mu n \implies \beta\epsilon \sim s$  !

Estimate dissipative terms by their **Navier-Stokes values:**

$$\Pi \sim \Pi_{\text{NS}} = -\zeta \theta, \quad q^\mu \sim q_{\text{NS}}^\mu = \frac{\kappa}{\beta} \frac{n}{\beta(\epsilon + p)} \nabla^\mu \alpha, \quad \pi^{\mu\nu} \sim \pi_{\text{NS}}^{\mu\nu} = 2\eta \sigma^{\mu\nu}$$

$$\implies \frac{\Pi}{\epsilon} \sim -\frac{\zeta}{\beta\epsilon} \beta \theta \sim -\frac{\zeta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \theta \sim \ell_{\text{mfp}} \partial_\mu u^\mu \sim \delta,$$

$$\frac{q^\mu}{\epsilon} \sim \frac{\kappa}{\beta} \frac{1}{\beta\epsilon} \frac{n}{\beta(\epsilon + p)} \beta \nabla^\mu \alpha \sim \frac{\kappa}{\beta s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \nabla^\mu \alpha \sim \ell_{\text{mfp}} \nabla^\mu \alpha \sim \delta,$$

$$\frac{\pi^{\mu\nu}}{\epsilon} \sim 2 \frac{\eta}{\beta\epsilon} \beta \sigma^{\mu\nu} \sim 2 \frac{\eta}{s} \frac{\beta}{\lambda_{\text{th}}} \frac{\lambda_{\text{th}}}{\ell_{\text{mfp}}} \ell_{\text{mfp}} \sigma^{\mu\nu} \sim \ell_{\text{mfp}} \nabla^{\langle\mu} u^{\nu\rangle} \sim \delta, \quad \text{q.e.d.}$$

## Results (IV)

IS equations:

$$\begin{aligned}
 \tau_{\Pi} \dot{\Pi} + \Pi &= \Pi_{\text{NS}} + \tau_{\Pi q} \mathbf{q} \cdot \dot{\mathbf{u}} - \ell_{\Pi q} \partial \cdot \mathbf{q} \\
 &\quad - \tau_{\Pi} \frac{\hat{\zeta}_1}{\zeta} \Pi^2 - \tau_{\Pi} \frac{\hat{\zeta}_2 \beta}{\kappa} \mathbf{q} \cdot \mathbf{q} - \tau_{\Pi} \frac{\hat{\zeta}_3}{2\eta} \pi^{\mu\nu} \pi_{\mu\nu} \\
 \tau_q \Delta^{\mu\nu} \dot{q}_{\nu} + q^{\mu} &= q_{\text{NS}}^{\mu} - \tau_{q\Pi} \Pi \dot{u}^{\mu} - \tau_{q\pi} \pi^{\mu\nu} \dot{u}_{\nu} \\
 &\quad + \ell_{q\Pi} \nabla^{\mu} \Pi - \ell_{q\pi} \Delta^{\mu\nu} \partial^{\lambda} \pi_{\nu\lambda} + \tau_q \omega^{\mu\nu} q_{\nu} \\
 &\quad - \tau_q \frac{\hat{\kappa}_1}{\zeta} q^{\mu} \Pi - \tau_q \frac{\hat{\kappa}_2}{2\eta} \pi^{\mu\nu} q_{\nu} \\
 \tau_{\pi} \dot{\pi}^{<\mu\nu>} + \pi^{\mu\nu} &= \pi_{\text{NS}}^{\mu\nu} + 2 \tau_{\pi q} q^{<\mu} \dot{u}^{\nu>} + 2 \ell_{\pi q} \nabla^{<\mu} q^{\nu>} + 2 \tau_{\pi} \pi_{\lambda}^{<\mu} \omega^{\nu>\lambda} \\
 &\quad - 2 \tau_{\pi} \frac{\hat{\eta}_1}{2\eta} \pi_{\lambda}^{<\mu} \pi^{\nu>\lambda} - 2 \tau_{\pi} \frac{\hat{\eta}_2 \beta}{\kappa} q^{<\mu} q^{\nu>} - 2 \tau_{\pi} \frac{\hat{\eta}_3}{\zeta} \Pi \pi^{\mu\nu}
 \end{aligned}$$

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341

A. Muronga, PRC 76 (2007) 014909 (and parts of  $\hat{\zeta}_1$ ,  $\hat{\kappa}_1$ ,  $\hat{\eta}_3$ )

B. Betz, D. Henkel, DHR, Prog. Part. Nucl. Phys. 62 (2009) 556

B. Betz, T. Koide, H. Niemi, DHR, in preparation

## Results (V)

### Remarks:

1. **Structure of second-order terms** follows exclusively from **Lorentz covariance**
2. **Coefficients** can be computed from **kinetic theory** and **Grad's 14-moment method**    B. Betz, H. Niemi, T. Koide, DHR, in preparation
3. R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100: second-order fluid dynamics for **conformal** fluids (AdS/CFT correspondence)

⇒ second-order terms (in flat space):

$$4 \lambda_1 \sigma_\lambda^{<\mu} \sigma^{\nu>\lambda} - 2 \lambda_2 \sigma_\lambda^{<\mu} \omega^{\nu>\lambda} + \lambda_3 \omega_\lambda^{<\mu} \omega^{\nu>\lambda}$$

to 2nd order: may use  $\pi^{\mu\nu} \simeq \pi_{\text{NS}}^{\mu\nu} = 2 \eta \sigma^{\mu\nu}$  to replace  $\sigma^{\mu\nu}$  with  $\pi^{\mu\nu}$

⇒ in kinetic theory:  $\lambda_1 \equiv -\tau_\pi \eta \hat{\eta}_1$ ,  $\lambda_2 \equiv -2\tau_\pi \eta$ ,  $\lambda_3 \equiv 0$

Note: second-order terms from collision integral ⇒  $\hat{\eta}_1 \neq 1!$

cf. M.A. York, G.D. Moore, arXiv:0811.0729

## Results (VI)

### 4. P. Romatschke, arXiv:0906.4787:

second-order fluid dynamics for **nonconformal** fluids without conserved charges

⇒ second-order terms (in flat space):

$$4\xi_1 \sigma^{\mu\nu} \sigma_{\mu\nu} + \xi_2 \theta^2 + \xi_3 \omega^{\mu\nu} \omega_{\mu\nu} + \xi_4 \nabla^\mu \ln s \nabla_\mu \ln s$$

to 2nd order: may use  $\pi^{\mu\nu} \simeq 2\eta \sigma^{\mu\nu}$ ,  $\Pi \simeq -\zeta \theta$

and 0th order acceleration eq.  $\dot{u}^\mu \simeq \frac{\nabla^\mu p}{\epsilon + p} = c_s^2 \nabla^\mu \ln s$

⇒ in kinetic theory:  $\xi_1 \equiv -\frac{1}{2} \tau_\Pi \eta \hat{\zeta}_3$ ,  $\xi_2 \equiv -\tau_\Pi \zeta \hat{\zeta}_1$ ,  $\xi_3 \equiv \xi_4 \equiv 0$

### 5. Coefficients $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\zeta}_3, \hat{\kappa}_1, \hat{\kappa}_2, \hat{\eta}_1, \hat{\eta}_2, \hat{\eta}_3$ are (complicated) **dimensionless** functions of $\alpha, \beta$

### 6. Viscosities and thermal conductivity $\zeta, \eta, \kappa$ , relaxation times $\tau_\Pi, \tau_q, \tau_\pi$ , coefficients $\tau_{\Pi q}, \tau_{q\Pi}, \tau_{q\pi}, \tau_{\pi q}, \ell_{\Pi q}, \ell_{q\Pi}, \ell_{q\pi}, \ell_{\pi q}$ are (complicated) functions of $\alpha, \beta$ , divided by **tensor coefficients of second moment of collision integral**: $\sim \chi_i(\alpha, \beta) / \langle \sigma \rangle \rightarrow 0$ as cross section $\sigma \rightarrow \infty$ (“strong coupling limit”!)

⇒  $\Pi = q^\mu = \pi^{\mu\nu} \rightarrow 0$  **ideal fluid limit!**

## Results (VII)

7. **IS** equations are **formally** independent of calculational frame (Eckart, Landau,...), **but ...**

8. **Values** of coefficients are **frame dependent!** We have analyzed:

(a) **Eckart (N)** or (net) charge frame:

$$\nu^\mu = 0, \quad \epsilon = \epsilon_0, \quad n = n_0$$

$\epsilon_0, n_0$ : **energy density** and **charge density** in local thermodyn. equilibrium

(b) **Landau (E)** or energy frame:

$$q^\mu = 0, \quad \epsilon = \epsilon_0, \quad n = n_0$$

**Note:** in **IS** equations  $q^\mu \equiv -\frac{\epsilon + p}{n} \nu^\mu$

(c) **Tsumura-Kunihiro-Ohnishi (TKO)** frame:

$$\nu^\mu = 0, \quad \epsilon = \epsilon_0 - 3\Pi, \quad n = n_0$$

We have checked agreement with the results of **IS** for most coefficients computed by **IS**...

9. R.h.s.: all terms except **NS** terms are of second order,  $\sim \delta^2$

$\implies t < \tau_\Pi \sim \tau_q \sim \tau_\pi$  : dissipative terms relax towards their **NS** values,

$t > \tau_\Pi \sim \tau_q \sim \tau_\pi$  : last terms on r.h.s. and **NS** terms on l.h.s. largely cancel, second-order terms govern evolution!

## Conclusions and open problems

1. Derived **Israel-Stewart (IS)** equations from **kinetic theory** via **Grad's 14-moment method**  $\implies$  **new second-order terms!**
2. **Results consistent with**  
R. Baier, P. Romatschke, D.T. Son, A.O. Starinets, M.A. Stephanov, JHEP 0804 (2008) 100  
M.A. York, G.D. Moore, arXiv:0811.0729
3. **Coefficients of terms in IS equations are frame dependent**  
 $\implies$  **have (not yet completely) been computed in various frames**  
(Eckart, Landau, TKO)
4. **Generalization to a system of various particle species**  
(done: quarks, antiquarks, gluons), **various conserved charges**  
cf. M. Prakash, M. Prakash, R. Venugopalan, G. Welke, Phys. Rept. 227 (1993) 321  
G. Denicol, DHR, in preparation
5. **Numerical implementation**  
E. Molnar, H. Niemi, DHR, in preparation

## Derivation: Grad's 14-moment method (I)

### 1. history: derivation of IS equations from kinetic theory

- H. Grad, Commun. Pure Appl. Math. 2 (1949) 381
- J.M. Stewart, Proc. Roy. Soc. London Ser. A 357 (1977) 59
- W. Israel, J.M. Stewart, Ann. Phys. 118 (1979) 341
- DHR, PhD thesis, 1992 (unpublished)
- R. Baier, P. Romatschke, U.A. Wiedemann, PRC 73 (2006) 064903
- A. Muronga, PRC 76 (2007) 014909, 014910

### 2. single-particle distribution function in local thermodynamical equilibrium:

$$f_0 = (e^{-y_0} + a)^{-1}, \quad y_0 = \alpha_0 - \beta_0 K \cdot u$$

$a = \pm 1, 0$  for fermions/bosons, Boltzmann particles

**local:**  $\alpha_0 \equiv \alpha_0(X)$ ,  $\beta_0 \equiv \beta_0(X)$ ,  $u^\mu \equiv u^\mu(X)$ ,  $X^\mu = (t, \vec{x})$

### single-particle distribution function in local non-equilibrium:

$$f = (e^{-y} + a)^{-1}, \quad y = \alpha - \beta K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu}$$

$$v \cdot u = 0, \quad w^\mu{}_\mu = 0$$

## Derivation: Grad's 14-moment method (II)

### 3. assume small deviations from equilibrium:

$$y - y_0 = \alpha - \alpha_0 - (\beta - \beta_0) K \cdot u - K \cdot v + K^\mu K^\nu w_{\mu\nu} \sim \delta \ll 1$$

$$\iff \alpha - \alpha_0 \sim \beta - \beta_0 \sim v^\mu \sim w^{\mu\nu} \sim \delta$$

$\implies$  expand  $f$  around  $f_0$  to linear order in  $y - y_0$ :

$$f \simeq f_0 + f_0(1 - a f_0)(y - y_0) + O(\delta^2)$$

### 4. compute $N^\mu$ and $T^{\mu\nu}$ as moments of $f$ :

$$\begin{aligned} N^\mu &= \int d\tilde{K} K^\mu f = I_0^\mu + (\alpha - \alpha_0) J_0^\mu - (\beta - \beta_0) J_0^{\mu\nu} u_\nu - J_0^{\mu\nu} v_\nu + J_0^{\mu\nu\lambda} w_{\nu\lambda} \\ T^{\mu\nu} &= \int d\tilde{K} K^\mu K^\nu f = I_0^{\mu\nu} + (\alpha - \alpha_0) J_0^{\mu\nu} - (\beta - \beta_0) J_0^{\mu\nu\lambda} u_\lambda - J_0^{\mu\nu\lambda} v_\lambda + J_0^{\mu\nu\lambda\rho} w_{\lambda\rho} \end{aligned}$$

where  $\int d\tilde{K} \equiv \frac{g}{(2\pi)^3} \int \frac{d^3\vec{k}}{E}$ ,  $E = \sqrt{k^2 + m^2}$ ,  $g$  no. of internal d.o.f.'s, and

$$\begin{aligned} I_0^{\alpha_1 \dots \alpha_n} &\equiv \int d\tilde{K} K^{\alpha_1} \dots K^{\alpha_n} f_0, \\ J_0^{\alpha_1 \dots \alpha_n} &\equiv \int d\tilde{K} K^{\alpha_1} \dots K^{\alpha_n} f_0(1 - a f_0) \end{aligned}$$

Note:  $I_0^\mu \equiv N_0^\mu$  net charge current in local thermodyn. equilibrium  
 $I_0^{\mu\nu} \equiv T_0^{\mu\nu}$  energy-momentum tensor in local thermodyn. equilibrium

### Derivation: Grad's 14-moment method (III)

5. tensor decomposition of  $I_0^{\alpha_1 \dots \alpha_n}$  up to  $n = 3$  and of  $J_0^{\alpha_1 \dots \alpha_n}$  up to  $n = 5$ :

$$\begin{aligned}
 I_0^\mu &= I_{10} u^\mu, & J_0^\mu &= J_{10} u^\mu, \\
 I_0^{\mu\nu} &= I_{20} u^\mu u^\nu + I_{21} \Delta^{\mu\nu}, & J_0^{\mu\nu} &= J_{20} u^\mu u^\nu + J_{21} \Delta^{\mu\nu}, \\
 I_0^{\mu\nu\lambda} &= I_{30} u^\mu u^\nu u^\lambda + 3 I_{31} u^{(\mu} \Delta^{\nu\lambda)} & J_0^{\mu\nu\lambda} &= J_{30} u^\mu u^\nu u^\lambda + 3 J_{31} u^{(\mu} \Delta^{\nu\lambda)} \\
 & & J_0^{\mu\nu\lambda\rho} &= J_{40} u^\mu u^\nu u^\lambda u^\rho + 6 J_{41} u^{(\mu} u^\nu \Delta^{\lambda\rho)} + 3 J_{42} \Delta^{\mu(\nu} \Delta^{\lambda\rho)} \\
 & & J_0^{\mu\nu\lambda\rho\sigma} &= J_{50} u^\mu u^\nu u^\lambda u^\rho u^\sigma + 10 J_{51} u^{(\mu} u^\nu u^\lambda \Delta^{\rho\sigma)} + 15 J_{52} u^{(\mu} \Delta^{\nu\lambda} \Delta^{\rho\sigma)}
 \end{aligned}$$

**Note:**  $I_{10} \equiv n_0$ ,  $I_{20} \equiv \epsilon_0$ ,  $I_{21} \equiv -p_0$

**In general:**  $I_{nq} \equiv \frac{1}{(2q+1)!!} \int d\tilde{K} E^{n-2q} (-k^2)^q f_0$

$J_{nq} \equiv \frac{1}{(2q+1)!!} \int d\tilde{K} E^{n-2q} (-k^2)^q f_0 (1 - a f_0)$

6. insert tensor decomposition for  $I_0^{\alpha_1 \dots \alpha_n}$ ,  $J_0^{\alpha_1 \dots \alpha_n}$ , as well as

$$w_{\mu\nu} = w \left( u_\mu u_\nu - \frac{1}{3} \Delta_{\mu\nu} \right) + 2 w_{(\mu} u_{\nu)} + \tilde{w}_{\mu\nu}$$

$$w_\mu u^\mu = \tilde{w}_{\mu\nu} u^\nu = \tilde{w}_{\mu\nu} u^\mu = 0; \quad \tilde{w}_{\langle\mu\nu\rangle} = \tilde{w}_{\mu\nu}$$

into expansion for  $N^\mu$ ,  $T^{\mu\nu}$ , cf. 4.

### Derivation: Grad's 14-moment method (IV)

7. choose specific frame, i.e., impose matching conditions (cf. Results (VII), 8.)

$$\Rightarrow \alpha - \alpha_0 = w \chi_\alpha(\alpha, \beta), \quad \beta - \beta_0 = w \chi_\beta(\alpha, \beta), \quad v^\mu = w^\mu \chi_v(\alpha, \beta)$$

8. re-insert into tensor decomposition for  $N^\mu$ ,  $T^{\mu\nu}$

$$\Rightarrow w = \Pi \chi_\Pi(\alpha, \beta), \quad w^\mu = q^\mu \chi_q(\alpha, \beta), \quad \tilde{w}^{\mu\nu} = \pi^{\mu\nu} \chi_\pi(\alpha, \beta)$$

9. compute third moment of  $f$ :

$$\begin{aligned} S^{\mu\nu\lambda} &= \int d\tilde{K} K^\mu K^\nu K^\lambda f \\ &= S_1 u^\mu u^\nu u^\lambda + 3 S_2 u^{(\mu} \Delta^{\nu\lambda)} + 3 \psi_1 q^{(\mu} u^\nu u^\lambda) + 3 \psi_2 q^{(\mu} \Delta^{\nu\lambda)} + 3 \psi_3 \pi^{(\mu\nu} u^\lambda) \end{aligned}$$

$$S_1 = I_{30} + \psi_4 \Pi, \quad S_2 = I_{31} - \frac{1}{3} \psi_4 \Pi, \quad \psi_i = \psi_i(\alpha, \beta), \quad i = 1, \dots, 4$$

10. compute second moment of collision integral  $\mathcal{C}[f]$ : to first order in  $\delta$

$$C^{\nu\lambda} = \int d\tilde{K} K^\nu K^\lambda \mathcal{C}[f] = A \Pi \left( u^\nu u^\lambda - \frac{1}{3} \Delta^{\nu\lambda} \right) + B \pi^{\nu\lambda} + 2 C u^{(\nu} q^{\lambda)}$$

$$A, B, C \sim \langle \sigma \rangle$$

## Derivation: Grad's 14-moment method (V)

11. compute second moment of Boltzmann equation,  $K \cdot \partial f = \mathcal{C}[f]$ :

$\Rightarrow$   $\partial_\mu S^{\mu\nu\lambda} = C^{\nu\lambda}$  or, tensor-decomposed:

$$\begin{aligned}
 A \Pi - \psi_4 \dot{\Pi} &= -\beta J_{41} \theta + \dot{I}_{30} + \dot{\psi}_4 \Pi + \frac{5}{3} \psi_4 \Pi \theta \\
 &\quad + \psi_1 \partial \cdot q - q \cdot \nabla \psi_1 - 2 \psi_1 q \cdot \dot{u} - 2 \psi_3 \pi^{\mu\nu} \sigma_{\mu\nu} \\
 C q^\mu - \psi_1 \Delta^{\mu\nu} q_\nu &= -\beta J_{41} \dot{u}^\mu + \nabla^\mu J_{31} - \frac{1}{3} \psi_4 \nabla^\mu \Pi - \frac{1}{3} \Pi \nabla^\mu \psi_4 + \frac{5}{3} \psi_4 \Pi \dot{u}^\mu \\
 &\quad + (\psi_1 - 2 \psi_2) q_\nu \sigma^{\mu\nu} - \psi_1 q_\nu \omega^{\mu\nu} + q^\mu \left[ \dot{\psi}_1 + \frac{1}{3} (4 \psi_1 - 5 \psi_2) \theta \right] \\
 &\quad + \psi_3 \Delta^{\mu\nu} \partial^\lambda \pi_{\nu\lambda} + \pi^{\mu\nu} (\nabla_\nu \psi_3 - \psi_3 \dot{u}_\nu) \\
 B \pi^{\mu\nu} - \psi_3 \dot{\pi}^{\langle\mu\nu\rangle} &= 2 I_{31} \sigma^{\mu\nu} - \frac{2}{3} \psi_4 \Pi \sigma^{\mu\nu} \\
 &\quad + 2 \psi_2 \nabla^{\langle\mu} q^{\nu\rangle} + 2 q^{\langle\mu} \nabla^{\nu\rangle} \psi_2 + 2 (\psi_1 - \psi_2) q^{\langle\mu} \dot{u}^{\nu\rangle} \\
 &\quad + 2 \psi_3 \pi_\lambda^{\langle\mu} \sigma^{\nu\rangle\lambda} - 2 \psi_3 \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} + \pi^{\mu\nu} \left( \dot{\psi}_3 + \frac{5}{3} \psi_3 \theta \right)
 \end{aligned}$$

12. substitute  $\dot{I}_{30}$ ,  $-\beta J_{41} \dot{u}^\mu$ ,  $\nabla^\mu I_{31}$ ,  $\dot{\psi}_i$ ,  $\nabla^\mu \psi_i$ ,  $i = 1, \dots, 4$

with the help of energy conservation and acceleration equation

$\Rightarrow$  IS equations with explicit (frame-dependent) expressions for coefficients