# What Can We Learn from Off-Shell Green Functions

**Stefan Scherer**

Institut für Kernphysik, Johannes Gutenberg-Universität Mainz

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## 1. Motivation

**To what extent is the off-shell behavior of Green functions unique?**

| | Fundamental theory | Effective hadronic theories |
|---|---|---|
| | QCD | Effective field theory,<br>e.g., ChPT<br>Models |
| dof | quarks & gluons | Goldstone bosons<br>(+ other hadrons) |
| parameters | $g_3$ + quark masses | ($\infty$ # of ) LECs<br>+ quark masses |

## Effective field theory

... if one writes down the most general possible Lagrangian, including all terms consistent with assumed symmetry principles, and then calculates matrix elements with this Lagrangian to any given order of perturbation theory, the result will simply be the most general possible S−matrix consistent with analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles. ... [1]

... if we include in the Lagrangian all of the infinite number of interactions allowed by symmetries, then there will be a counterterm available to cancel every ultraviolet divergence. In this sense, as said earlier, non-renormalizable theories are just as renormalizable as renormalizable theories, as long as we include all possible terms in the Lagrangian. ... [2]

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[1] S. Weinberg, Physica A 96, 327 (1979)

[2] S. Weinberg, The Quantum Theory of Fields, Vol. I, 1995, Chap. 12

## Simplified analogies between multipole expansion and EFT

| Multipole expansion | EFT |
|---|---|
| $\lvert\vec{x}\,\rvert \gg R$ | $q \ll \Lambda_\chi$ |
| $\Phi(\vec{x}) = \sum_{lm}$ $q_{lm}$ $\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ | $\mathcal{L}_{\rm eff} = \sum_{lm}$ $c_{lm}$ $\mathcal{L}_{lm}$ |
| multipole moment $q_{lm}$ | LEC $c_{lm}$ |
| $\frac{1}{2l+1}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$ | Structures $\mathcal{L}_{lm}$ |

- In principle, infinite number of terms.
  Actual calculation: Truncation at finite order.

- Systematic improvement possible.

## 2. Pion-pion scattering and pion-pion bremsstrahlung

To what extent is the off-shell behavior of Green functions unique?

[Simplified version of H. W. Fearing, Phys. Rev. Lett. 81, 758 (1998)]

- Simple example: $\pi^+ + \pi^0 \to \pi^+ + \pi^0 + \gamma$

- Chiral perturbation theory at lowest order

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\text{Tr}\left[[D_\mu U(D^\mu U)^\dagger\right] + \frac{F_\pi^2 M_\pi^2}{4}\text{Tr}(U + U^\dagger).$$

  - $M_\pi^2 = 2B\hat{m}$.

  - $U$ is an SU(2) matrix containing the pion fields.

- $F_\pi = 92.4\,$MeV.
- Covariant derivative generates interaction with e.m. field

$$D_\mu U = \partial_\mu U + ieA_\mu[Q,U], \quad Q = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$$

- **Alternative parameterizations of $U$:**

$$\begin{aligned} U(x) &= \frac{1}{F_\pi}\left[\sigma(x)1_{2\times 2} + i\vec{\tau}\cdot\vec{\pi}(x)\right] \\ &= \frac{1}{F_\pi}\begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi_1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix}, \\ \sigma(x) &= \sqrt{F_\pi^2 - \vec{\pi}^2(x)}, \end{aligned}$$

$$
\begin{aligned}
U(x) &= \exp\left[i\frac{\vec{\tau}\cdot\vec{\phi}(x)}{F_\pi}\right] \\
&= \cos\left(\frac{\phi}{F_\pi}\right) 1_{2\times 2} + i\vec{\tau}\cdot\hat{\phi}\sin\left(\frac{\phi}{F_\pi}\right).
\end{aligned}
$$

- **Interpretation: "field transformation" or "change of variables"**

$$
\frac{\vec{\pi}}{F_\pi} = \hat{\phi}\sin\left(\frac{\phi}{F_\pi}\right) = \frac{\vec{\phi}}{F_\pi}\left(1 - \frac{1}{6}\frac{\vec{\phi}^2}{F_\pi^2} + \cdots\right).
$$

- **Analogy**

$$\begin{aligned}\vec{x} &= (x, y, z)\\ &= (r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\theta))\end{aligned}$$

change of variables from cartesian to spherical coordinates

$$\begin{aligned}L &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z),\\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2(\theta)\dot{\varphi}^2)\\ &\quad - \underbrace{V(r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\theta))}_{V'(r,\theta,\varphi)}\end{aligned}$$

- **Interaction Lagrangians**

$$\mathcal{L}_1^{4\pi} = \frac{1}{2F_\pi^2}\partial_\mu\vec{\pi}\cdot\vec{\pi}\partial^\mu\vec{\pi}\cdot\vec{\pi} - \frac{M_\pi^2}{8F_\pi^2}(\vec{\pi}^2)^2,$$

$$\mathcal{L}_2^{4\phi} = \frac{1}{6F_\pi^2}(\partial_\mu\vec{\phi}\cdot\vec{\phi}\partial^\mu\vec{\phi}\cdot\vec{\phi} - \vec{\phi}^2\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}) + \frac{M_\pi^2}{24F_\pi^2}(\vec{\phi}^2)^2.$$

**Observe that the two interaction Lagrangians depend differently on the respective pion fields.**

- **Simple example**

$$\mathcal{L}(\phi, \partial_\mu\phi) = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

**field transformation** $\quad \phi = \chi f(\chi),$

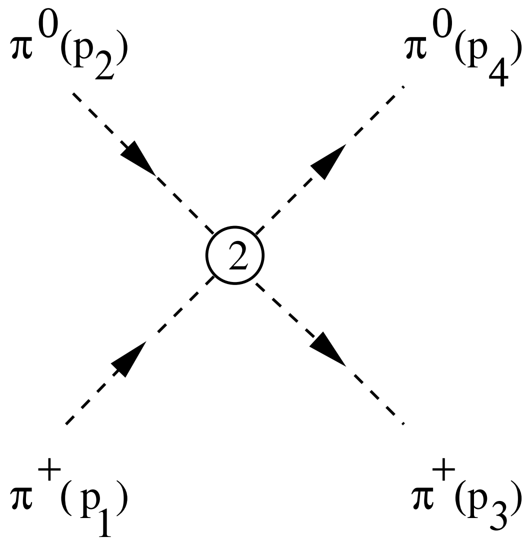
where

$$f(x) = 1 + ax + bx^2 + \cdots$$

$$\mathcal{L}'(\chi, \partial_\mu \chi) = \mathcal{L}_0 + \mathcal{L}'_{\text{int}}$$

$$\left[ \quad \text{e.g.} \quad f(x) = \frac{a}{x} + \frac{b}{x^2} \cdots \quad \text{will not do} \quad \right]$$

## Feynman rule for the $\pi^+\pi^0$ scattering amplitude



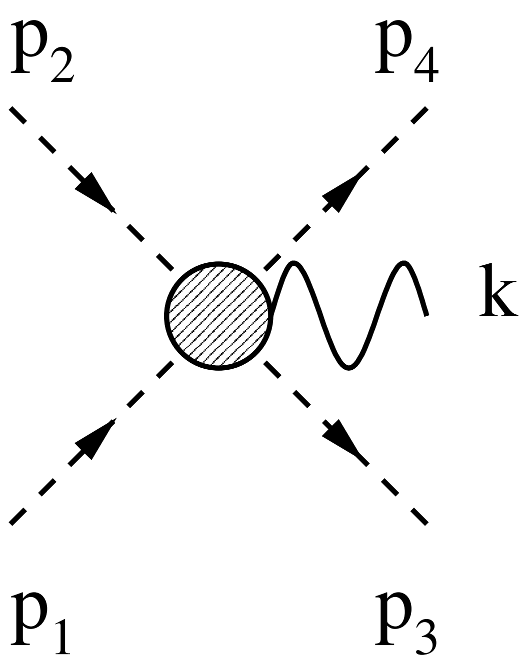

$$\mathcal{M}_1 = \frac{i}{F_\pi^2} T_0(p_1, p_3),$$

$$\mathcal{M}_2 = \frac{i}{F_\pi^2} \left[ T_0(p_1, p_3) - \frac{1}{3}(\Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4) \right],$$

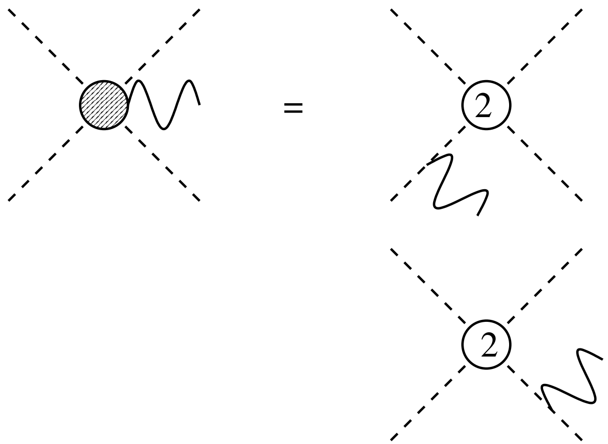
$$T_0(p_1, p_3) = (p_3 - p_1)^2 - M_\pi^2, \quad \Lambda_i = p_i^2 - M_\pi^2$$

**The same on-shell scattering amplitude but different off-mass-shell behavior!**

**Apply to** $\pi^+(p_1) + \pi^0(p_2) \to \pi^+(p_3) + \pi^0(p_4) + \gamma(k)$



## Parameterization 1

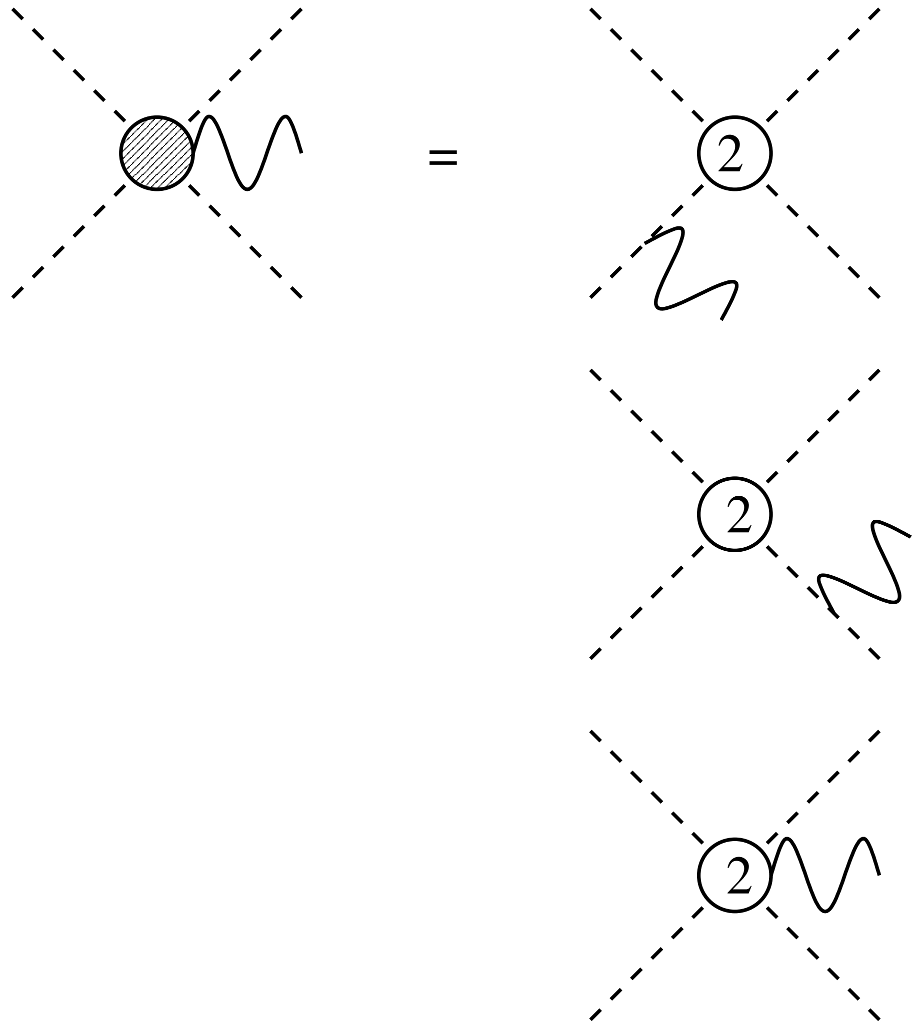



$$\mathcal{M}_1 = \frac{i}{F_\pi^2} T_0(p_1 - k, p_3) \frac{i}{(p_1 - k)^2 - M_\pi^2} (-2iep_1 \cdot \epsilon)$$

$$-2iep_3 \cdot \epsilon \frac{i}{(p_3 + k)^2 - M_\pi^2} \frac{i}{F_\pi^2} T_0(p_1, p_3 + k)$$

$$= \left( \frac{p_3 \cdot \epsilon}{p_3 \cdot k} - \frac{p_1 \cdot \epsilon}{p_1 \cdot k} \right) \frac{ie}{F_\pi^2} [T_0(p_1, p_3) - 2(p_1 - p_3) \cdot k]$$

**Parameterization 2**



$$
\begin{aligned}
\mathcal{M}_2 &= \frac{i}{F_\pi^2}\left\{T_0(p_1-k,p_3)\boxed{-\frac{1}{3}\left[(p_1-k)^2-M_\pi^2\right]}\right\} \\
&\quad \times\frac{i}{(p_1-k)^2-M_\pi^2}(-2iep_1\cdot\epsilon) \\
&\quad -2iep_3\cdot\epsilon\frac{i}{(p_3+k)^2-M_\pi^2} \\
&\quad \times\frac{i}{F_\pi^2}\left\{T_0(p_1,p_3+k)\boxed{-\frac{1}{3}\left[(p_3+k)^2-M_\pi^2\right]}\right\} \\
&\quad +\frac{2ie}{3F_\pi^2}\epsilon\cdot(p_1+p_3) \\
&= \mathcal{M}_1
\end{aligned}
$$

- **Here: Complete cancellation of “off-shell” effects and contact interactions.**

- **In general: Two mechanisms are indistinguishable.**

- **Manifestation of the "equivalence theorem" of field theory:**
  **Lagrangians which are related by field transformations generate the same on-shell S-matrix elements and thus the same observables.**

- **Off-shell form functions not only model dependent but also representation dependent.**

$\cdots$ but

- No spin
- Chiral symmetry
- Gauge invariance (Low's theorem)
- No off-shell effects in the electromagnetic vertex
- $\cdots$

These issues have been addressed in H. W. Fearing and S. Scherer, Phys. Rev. C 62, 034003 (2000)

## 3. Off-shell extrapolations

- PCAC relation of Adler [3]

$$(\partial_\mu \delta_{ab} + e\mathcal{A}_\mu \epsilon_{3ab}) A_b^\mu = M_\pi^2 F_\pi \Phi_a$$

- Predictions of current algebra and PCAC relation involve so-called soft-pion limit:

$$\lim_{q_0 \to 0} \lim_{\vec{q} \to 0} [\cdots]$$

  However: Amplitudes for physical pions are to be taken at $q^2 = M_\pi^2$

- Q: Is the extrapolation unique?

- A: Depends on what we are referring to.

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[3] S. L. Adler, Phys. Rev. 139, B1638 (1965)

– Yes, if we refer exclusively to QCD Green functions

– No, if we talk about the off-shell behavior of "effective fields" (dependence on choice of fields).

- Illustration

  Two-flavor vector- and axial-vector currents in terms of (QCD) quark fields

$$V^{\mu,a} = \bar{q}\gamma^\mu \frac{\tau^a}{2} q, \quad A^{\mu,a} = \bar{q}\gamma^\mu\gamma_5 \frac{\tau^a}{2} q, \quad q = \begin{pmatrix} u \\ d \end{pmatrix}$$

  PCAC relation from QCD Lagrangian (without e.m.)

$$\partial_\mu A^{\mu,a} = \hat{m} i \bar{q}\gamma_5\tau^a q \equiv \hat{m} P^a$$

  Matrix elements between vacuum and single-pion state

$$\langle 0|A^{\mu,a}(x)|\pi^b(q)\rangle = i q^\mu F_\pi e^{-iq\cdot x}\delta^{ab}$$

$$\langle 0|P^a(x)|\pi^b(q)\rangle = G_\pi e^{-iq\cdot x}\delta^{ab}$$

PCAC (note $q^2 = M_\pi^2$) $\Rightarrow$

$$M_\pi^2 F_\pi = \hat{m} G_\pi$$

In other words,

$$\Phi^a(x) \equiv \frac{\hat{m} P^a(x)}{M_\pi^2 F_\pi}$$

serves as an interpolating pion field

Matrix element between single-nucleon states

$$\langle N(p')|A^{\mu,a}(0)|N(p)\rangle = \bar{u}(p')\left[\gamma^\mu \gamma_5 G_A(q^2) + \frac{q^\mu}{2m_N}\gamma_5 G_P(q^2)\right]\frac{\tau^a}{2}u(p)$$

$$\hat{m}\langle N(p')|P^a(0)|N(p)\rangle = \frac{M_\pi^2 F_\pi}{M_\pi^2 - q^2} G_{\pi N}(q^2) i\bar{u}(p')\gamma_5 \tau^a u(p)$$

Defines pion-nucleon form factor $G_{\pi N}(q^2)$ in terms of $\Phi^a(x)$

Pion-nucleon coupling constant [4]

$$g_{\pi N} = G_{\pi N}(M_\pi^2)$$

PCAC relation $\Rightarrow$

$$2m_N G_A(q^2) + \frac{q^2}{2m_N} G_P(q^2) = 2\frac{M_\pi^2 F_\pi}{M_\pi^2 - q^2} G_{\pi N}(q^2)$$

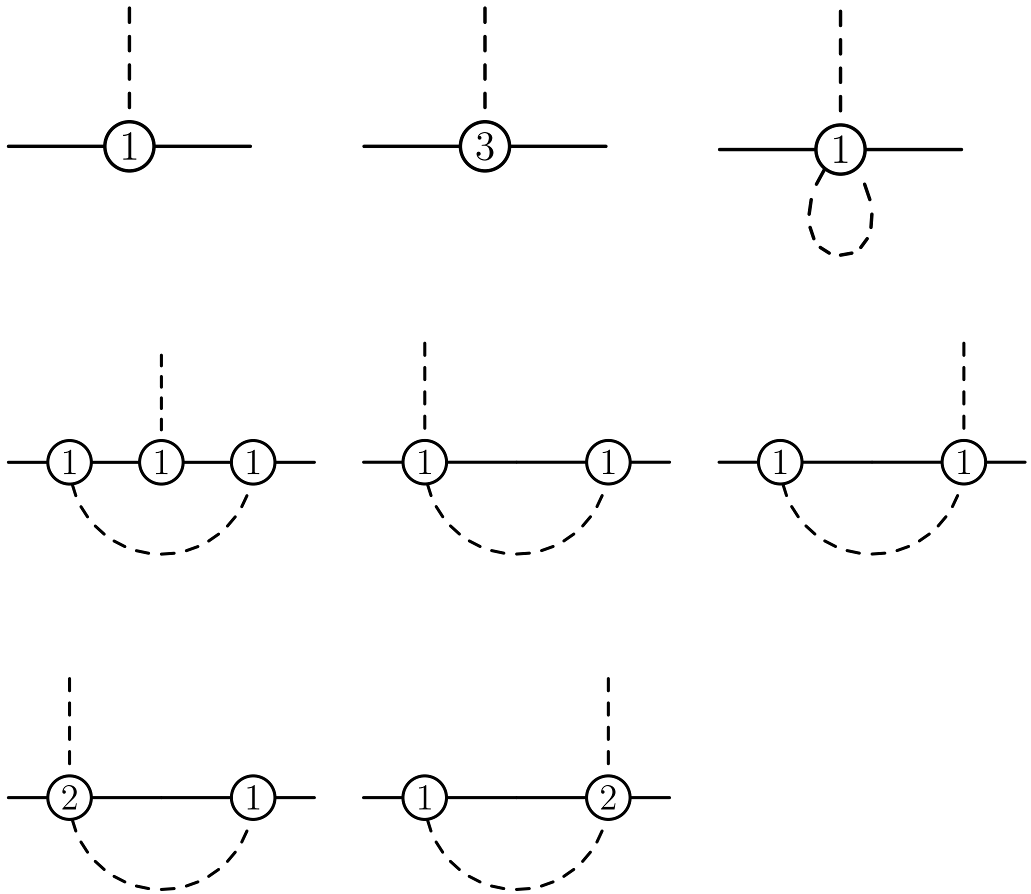
Exact relation, holds true for any value of $q^2$

Pion-nucleon vertex in ChPT

$$\Gamma(q^2)\gamma_5\tau^a$$

from diagrams contributing to the pion nucleon vertex up to order $\mathcal{O}(p^4)$

[4] M. R. Schindler, T. Fuchs, J. Gegelia, S. S., Phys. Rev. C 75, 025202 (2007)




## Pion-nucleon coupling constant

$$g_{\pi N} \;=\; Z_{\Psi}\sqrt{Z_{\pi}}\,\Gamma(M_{\pi}^2)$$

## Chiral expansion

$$g_{\pi N} \;=\; \mathrm{g}_{\pi N} + g_{\pi N}^{(1)} M^2 + g_{\pi N}^{(2)} M^2 \ln\left(\frac{M}{m}\right) + g_{\pi N}^{(3)} M^3 + \mathcal{O}(M^4)$$

$$
\begin{aligned}
g_{\pi N} &= \frac{g_A m}{F}\,, \\
g_{\pi N}^{(1)} &= -g_A \frac{l_4^r m}{F^3} - 4 g_A \frac{c_1}{F} + \frac{2(2d_{16} - d_{18})m}{F} - g_A^3 \frac{m}{16\pi^2 F^3}\,, \\
g_{\pi N}^{(2)} &= -g_A^3 \frac{m}{4\pi^2 F^3}\,, \\
g_{\pi N}^{(3)} &= g_A \frac{4 + g_A^2}{32\pi F^3} - g_A \frac{(c_3 - 2c_4)m}{6\pi F^3}
\end{aligned}
$$

$$
G_{\pi N}(M_\pi^2) = g_{\pi N} = Z_\Psi \sqrt{Z_\pi}\,\Gamma(M_\pi^2)
$$

But, for arbitrary $q^2$

$$
G_{\pi N}(q^2) \neq Z_\Psi \sqrt{Z_\pi}\,\Gamma(q^2)
$$

# Pion electroproduction near threshold and the axial radius

## Definition of relevant Green functions

$$\mathcal{M}_A^{\mu,a} = \langle N(p_f)| \boxed{A^{\mu,a}(0)} |N(p_i)\rangle$$

$$\mathcal{M}_{JA}^{\mu\nu,a} = \int d^4x\, e^{iq\cdot x}\langle N(p_f)|T\left[\boxed{J^\mu(0)A^{\nu,a}(x)}\right]|N(p_i)\rangle$$

$$\mathcal{M}_{JP}^{\mu,a} = \int d^4x\, e^{iq\cdot x}\langle N(p_f)|T\left[\boxed{J^\mu(0)P^a(x)}\right]|N(p_i)\rangle.$$

- Adler-Gilman (AG) relation (exact for all $q^2$) [5]

$$q_\nu \mathcal{M}_{JA}^{\mu\nu,a} = \underbrace{i\hat{m}\mathcal{M}_{JP}^{\mu,a}}_{\sim \text{ pion production}} + \epsilon_{3ab} \underbrace{\mathcal{M}_A^{\mu,b}}_{\text{axial-vector current}}$$

[5] S. L. Adler and F. J. Gilman, Phys. Rev. 152, 1460 (1966)

- Relation to pion electroproduction via LSZ reduction

$$\mathcal{M}^{\mu,a} = -i\frac{\hat{m}}{M_\pi^2 F_\pi}\lim_{q^2\to M_\pi^2}(q^2-M_\pi^2)\mathcal{M}_{JP}^{\mu,a}$$

$$\overset{\text{AG}}{=} \frac{1}{M_\pi^2 F_\pi}\lim_{q^2\to M_\pi^2}(q^2-M_\pi^2)(\epsilon_{3ab}\mathcal{M}_A^{\mu,b}-q_\nu\mathcal{M}_{JA}^{\mu\nu,a})$$

- Where does the off-shell extrapolation enter? Define for arbitrary $q$

$$\widetilde{\mathcal{M}}^{\mu,a}(q) = -i\frac{\hat{m}}{M_\pi^2 F_\pi}(q^2-M_\pi^2)\mathcal{M}_{JP}^{\mu,a}$$

Soft-pion limit

$$\lim_{q_0\to 0}\lim_{\vec{q}\to 0}\widetilde{\mathcal{M}}^{\mu,a}(q)$$

Pion electroproduction

$$\mathcal{M}^{\mu,a} = \widetilde{\mathcal{M}}^{\mu,a}(q)\Big|_{q^2=M_\pi^2}$$

- Notation

$$\vec{M}\Big|_{thr} = -\frac{4\pi W}{m_N} i \left[ i(\vec{\sigma} - \vec{\sigma}\cdot\hat{k}\hat{k})E_{0+}(k^2) + i\vec{\sigma}\cdot\hat{k}\hat{k}L_{0+}(k^2) \right]$$

- Contribution from pion loops [6]

  Modification of the $k^2$ dependence of $E_{0+}^{(-)}$ [at $\mathcal{O}(q^3)$]

$$E_{0+}^{(-)}(k^2) = \frac{e g_A}{8\pi F_\pi}[\underbrace{1 + \frac{k^2}{4m_N^2}\left(\kappa_V + \frac{1}{2}\right) + \boxed{\frac{k^2}{6} r_A^2}}_{\text{old}} + \underbrace{\frac{M_\pi^2}{8\pi^2 F_\pi^2} f\left(\frac{k^2}{M_\pi^2}\right)}_{\text{new}} + \cdots]$$

$$\text{new} = \frac{k^2}{128 F_\pi^2}\left(1 - \frac{12}{\pi^2}\right) + \cdots$$

- Conclusion: “new term” needs to be taken into account

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[6] V. Bernard, N. Kaiser, U.-G. Meißner, Phys. Rev. Lett. 69, 1877 (1992)

when extracting axial radius from pion electroproduction

- Where does the term $\sim \boxed{\frac{k^2}{6} r_A^2}$ come from?

- Originates from chiral Ward identities among QCD-Green functions:
  - Introduce into QCD Lagrangian coupling to external fields. [7]
  - Chiral Ward identities are encoded in the invariance of the generating functional under local chiral transformations.
  - Transport local invariance to EFT level

[7] J. Gasser and H. Leutwyler, Annals Phys. 158, 142 (1984)

- Generates

$$\mathcal{L}_{\rm eff}^{(3)} = \text{const.}\, \bar{\Psi}\gamma^\mu\gamma_5[D^\nu, f_{-\mu\nu}]\Psi + \cdots$$

with

$$\begin{aligned} f_{-\mu\nu} &= -2(\partial_\mu a_\nu - \partial_\nu a_\mu) + 2i\left([v_\mu, a_\nu] - [v_\nu, a_\mu]\right) \\ &\quad + \frac{i}{F}\left[\vec{\tau}\cdot\vec{\pi}, \partial_\mu v_\nu - \partial_\nu v_\mu\right] + \cdots \end{aligned}$$

$\Rightarrow$ Enters all three Green functions of the Adler-Gilman relation.

- Minimal coupling, in general, does not respect constraints due to chiral symmetry. [8]

[8] T. Fuchs and S. S., Phys. Rev. C 68, 055501 (2003)

## 4. Summary

1. Green functions of the effective theory depend on the choice of variables

2. Necessary condition for an observable: quantity is independent of representation

3. Corrections to current-algebra results (in the framework of ChPT)

4. Off-shell extrapolations unique if defined in terms of QCD Green functions