Regimes of QCD

Ferenc Niedermayer (Bern)

SFB/TR 55 meeting

Trento, Jan. 7-10, 2009

Introduction

To test QCD it would be useful to do simulations in a very large volume, at physical quark masses, and at sufficiently small lattice spacing(s).

One is getting closer to this dream...

However, taking a non-physical situation, by varying the volume and the quark masses (including the chiral limit $m_q \to 0$) is also important –
it helps to determine the Low-Energy Constants
of the effective theory.

QCD action (for N_f degenerate masses, for simplicity)

$$\mathcal{A}_{\rm QCD}(A_\mu, \psi, \overline{\psi}; m_q) = \mathcal{A}_{\rm gauge}(A_\mu) + \overline{\psi}\,(\mathcal{D}(A) + m_q)\,\psi\,.$$

At $m_q = 0$ one has the spontaneous symmetry breaking

$$\mathrm{SU}(N_f)_{\rm L} \times \mathrm{SU}(N_f)_{\rm R} \to \mathrm{SU}(N_f)_{\rm V}$$

leading to $N_f^2 - 1$ Goldstone bosons (3 pions for $N_f = 2$)

The action of the low-energy effective theory:

$$\mathcal{A}_{\text{eff}}(U; F, m_q\Sigma, L_1, L_2, \ldots)$$

where the pion field $\Phi_a(x)$ enters through

$$U(x) = \mathrm{e}^{i\Phi_a(x)T_a/F} \in \mathrm{SU}(N_f)$$

For $N_f = 2$ it is equivalent to O(4) non-linear sigma model

$$U(x) = S_0(x) + iS_a(x)\sigma_a\,, \quad \mathbf{S}^2(x) = 1, \text{ O(4) vector}$$

$$\mathcal{A}_{\text{eff}}(S) = \frac{1}{2}F^2 \int d^4x\, (\partial_\mu \mathbf{S}(x))^2 - H \int d^4x S_0(x) + \ldots$$

with $H = m_q\Sigma \sim M_\pi^2 F^2$ (4d Heisenberg model)

The effect of the “magnetic field” (quark masses)

- $HV = m_q \Sigma V \gg 1$: $\sum_x \mathbf{S}(x) \parallel (1,0,0,0)$
- $HV = m_q \Sigma V \ll 1$: $\sum_x \mathbf{S}(x)$ rotates freely
  (the symmetry is restored in a finite volume for $m_q \to 0$)

The physical picture depends on the relations between pion mass and the size and shape of the lattice:

$$F\,,M\,,L_s\,,L_t$$

Different regimes of QCD

Different regimes

- $FL \ll 1$, $(L < 1\,\mathrm{fm})$: “femto-universe” $(g$, q, $\overline{q})$
- $FL_s \gg 1$, $L_t \sim 1/T_c$: finite temperature system
- p-regime (ChPT):
  $FL \gg 1$, $(L \gtrsim 2\,\mathrm{fm})$, $ML > 1$
  large volume, the infrared regulator is M

$$\frac{1}{V}\sum_p \frac{1}{p^2+M^2}\,,\ \text{one can take } p=0.$$

- ϵ regime (ChPT):
  $FL \gg 1$, $ML < 1$ (cubic geometry, $L_t \sim L_s$)
  small volume, the infrared regulator is L.
  The constant mode (the direction of $\sum_x \mathbf{S}(x)$) is integrated out exactly (leads to Bessel functions) and as a result the $p = 0$ pion field is excluded from the path integral

$$\frac{1}{V} \sum_{p \neq 0} \frac{1}{p^2} \, ,$$

How to take exactly into account the global mode:
Insert into the partition function

$$1 = \int d\mathbf{m}\, \delta\left(\mathbf{m} - \frac{1}{V} \sum_x \mathbf{S}(x) \right)$$

$$\mathbf{m} = m\mathbf{e}, \quad d\mathbf{m} = m^{N-1} dm\, d\mathbf{e}, \quad \mathbf{e} = \Omega^{\mathrm{T}}\mathbf{n}, \quad \mathbf{n} = (1, 0, \ldots, 0)$$
$$d\mathbf{e} \to d\Omega, \quad \mathbf{S}(x) = \Omega^{\mathrm{T}}\mathbf{R}(x)$$

$$Z = \int d\Omega \int [d\mathbf{R}(x)] \prod_{i=1}^{N-1} \delta\left(\frac{1}{V} \sum_x R_i(x) \right)$$
$$\exp\left[-\mathcal{A}_{\mathrm{eff}}(\mathbf{R}, \Omega\mathbf{H}) + (N-1) \log\left(\frac{1}{V} \sum_x R_0(x) \right) \right]$$

Parametrization

$$\mathbf{R}(x) = \left(\sqrt{1-\pi^2(x)/F^2}\,, \pi_i(x)/F\right)\,, \quad i = 1,2,3$$

$$[d\mathbf{R}(x)] = \frac{1}{2\sqrt{1-\pi^2(x)}} \prod_{i=1}^{N-1} d\pi_i(x)$$

The $p = 0$ mode is missing because of the presence of

$$\delta\left(\frac{1}{V}\sum_x \pi_i(x)\right)$$

It is replaced by $\Omega \in \mathrm{O}(N)$.

ϵ vs. p regime

In practice one is often between ϵ and p regimes, $ML \approx 1$.

Can one have an expression which works for all ML values?
The answer is yes.

Take the p-regime calculation but treat the global mode exactly, as in the ϵ regime (even if it is not needed!)

Damgaard, Fukaya, arXiv:0812.2792
“The chiral condensate in a finite volume”

Interpolate between the ϵ and p regimes.

One would think that at $m_q = 0$ ($M = 0$) and finite V one is always in the ϵ-regime since $m\Sigma V = 0$.
However, for cylindrical geometry, $L_t \gg L_s$ there is no global constant mode, only $\mathbf{S}(t, \mathbf{x}) \sim \mathbf{S}(t)$

- δ-regime (ChPT):
  $FL \gg 1$, $ML_s < 1$, $L_t \gg L_s$ (cylindric geometry)
  small spatial volume, the infrared regulator is L_s.
  Time dependent constant mode (the direction of $\sum_{\mathbf{x}} \mathbf{S}(t, \mathbf{x})$) is treated exactly

For the δ-regime

$$\mathcal{A}_{\text{eff}} = \frac{1}{2}F^2 \int d^4x\, (\partial_\mu \mathbf{S}(x))^2 \approx \frac{1}{2}F^2 L_s^3 \int dt\, (\partial_t \mathbf{S}(t))^2$$

QM rotator with moment of inertia $\Theta = F^2 L_s^3$

(Leutwyler, 1987)

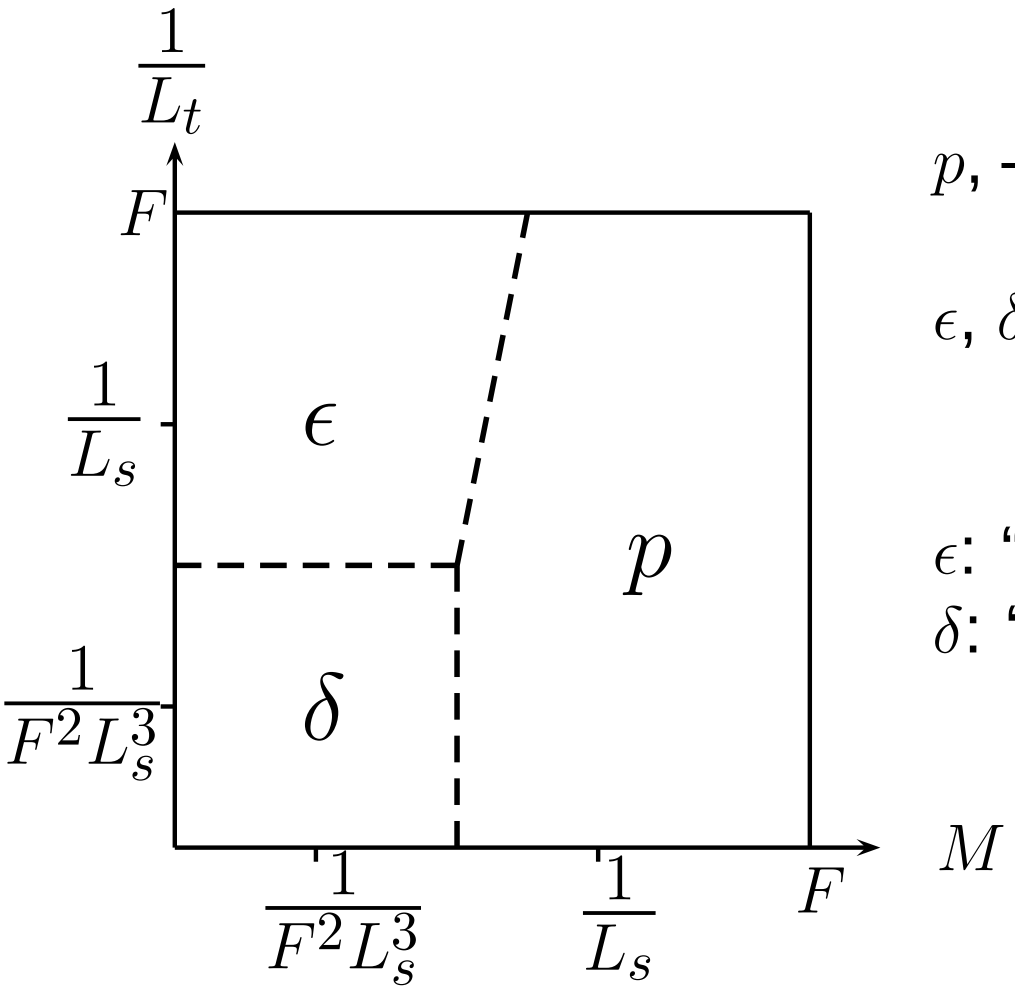
Rotator spectrum

$$E_l = \frac{1}{2\Theta}\mathcal{C}_l\,, \qquad \mathcal{C}_l = l(l+N-2) \quad \text{(Casimir)}$$

Mass gap

$$\Delta E = E_1 - E_0 = \frac{N-1}{2\Theta} = \frac{3}{2F^2 L_s^3} \quad \text{(for O(4))}$$

Regimes of QCD treated by ChPT



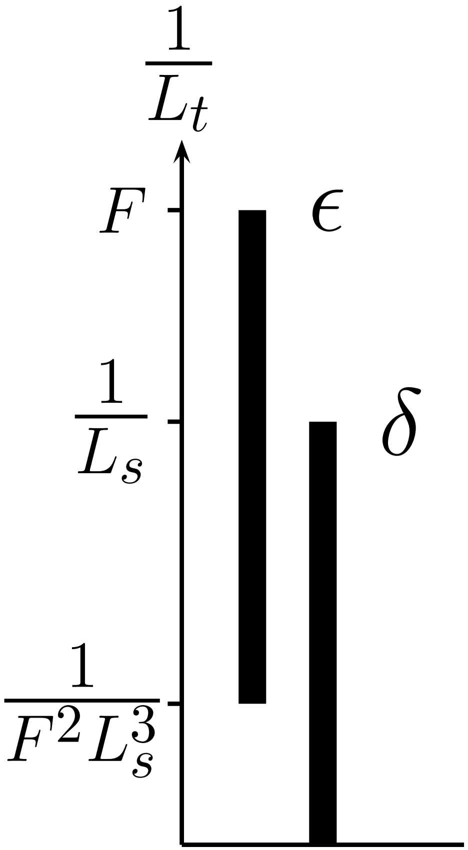
p, – symmetry broken, pions,

ϵ, δ – symmetry restored,
rotator modes dominate

ϵ: “high temp.” (cubic geom.)
δ: “low temp.” (cyl. geom.)

ϵ vs. δ regime

Actually, the ϵ and δ regimes overlap. At $M = 0$ one has in δ regime $\tau = 1/\Delta E \approx F^2 L_s^3$



ϵ: globally constant mode, $\mathbf{S}(t, \mathbf{x}) \approx \mathbf{e}$

δ: spatially constant modes, $\mathbf{S}(t, \mathbf{x}) \approx \mathbf{e}(t)$

Overlap of the two regimes
for $L_s < L_t < F^2 L_s^3$

For $L_s < L_t < F^2 L_s^3$: $\quad \mathbf{S}(\mathbf{t},\mathbf{x}) \sim \mathbf{e}$

ϵ regime: as usual

δ regime: In the transfer matrix language, only the $\mathbf{p}=0$ (spatially constant, rotator modes) enter, since

$$\exp\left(-\frac{2\pi}{L_s}L_t\right) \ll 1$$

However, higher values of l are important:

$$\exp\left(-E_l L_t\right) = \exp\left(-\frac{3\,\mathcal{C}_l}{2F^2 L_s^3}L_t\right) \sim \mathrm{O}(1)$$

For $L_t \gg F^2 L_s^3$: $\quad \mathbf{S}(\mathbf{t}, \mathbf{x}) \sim \mathbf{e}(t)$

purely δ regime: physically the simplest case
Only the rotator mass gap $\Delta E = E_1 - E_0$ is important.

However, it is not easy to calculate the NNL correction.

Relation between the ϵ and δ regimes

Once the rotator spectrum is known for arbitrary l (δ regime) one can calculate the $\langle \mathbf{S}(0)\mathbf{S}(t)\rangle$ correlator in the overlapping regime.

However the matching to the ϵ regime formulae requires that the rotator spectrum is not quite the same as we know from QM – it is a more general function of the O(N) Casimir operator:

$$E_l = \frac{1}{2\Theta}\mathcal{C}_l + \ldots \mathcal{C}_l^2 + \ldots$$

where

$$\mathcal{C}_l = l(l+N-2)\,.$$

It would be interesting to measure the masses of states with $l > 1$ (tensor correlators).

NL and NLL corrections

- ϵ regime:

The pseudoscalar correlator has been calclated to the NNL order

(Hasenfratz, Leutwyler, 1989)

At $L_s = L_t = L \approx 2\,\mathrm{fm}$:

$$\langle P(0)P(t)\rangle|_{t=L/2} \approx \Sigma^2 \left(1 - \frac{0.07}{F^2L^2} + \frac{0.02}{F^4L^4}\right)$$

$$= \Sigma^2\,(1 - 0.095 + 0.036)$$

Note: non-leading LEC's (4-derivatives) enter here only at the NNL order, in contrast to the p regime, where they enter in the NL order.

- δ regime:

The mass gap and correlators have been calculated only up to the NL order

(Hasenfratz, F.N, 1993)

Rotator with mass gap (for massless quarks)

$$\Delta E_{\mathrm{rot}} = \frac{3}{2}\frac{1}{F^2 L_s^3}\left(1 - \frac{0.45}{F^2 L_s^2} + \ldots\right)$$

At $L_s = 2\,\mathrm{fm}$

$$\Delta E_{\mathrm{rot}} \approx 200\,\mathrm{MeV} \times (1 - 0.61 + \ldots)\ ,$$

NNL correction is needed!

Mass gap in δ regime I.

(Hasenfratz, Weingart)

Consider the average direction in a time slice and the fluctuations arund it

$\mathbf{S}(t,\mathbf{x}) = \mathbf{e}(t) + \pi(t,\mathbf{x})$, $\quad \sum_{\mathbf{x}} \pi(t,\mathbf{x}) = 0$

Integrate out $\pi(t,\mathbf{x})$ in a fixed backround $\mathbf{e}(t)$ to get

$$\hat{\mathcal{A}}_{\mathrm{eff}}(\mathbf{e}) = \frac{1}{2}\Theta(L_s)\dot{\mathbf{e}}^2(t) + \dots$$

and find

$$\Theta(L_s) = F^2L^3\left(1 + \frac{c_1}{F^2L^2} + \frac{c_2}{F^4L^4} + \frac{c_3}{F^4L^4}\log(L\Lambda) + \dots\right)$$

Mass gap in δ regime II.

(F.N., Weiermann)

Use the Lüscher, Weisz, Wolff trick invented for the calculation of the mass gap in 2d O(N) non-linear sigma model:

$$\begin{aligned} C(t;g) = \langle \mathbf{S}(0)\mathbf{S}(t)\rangle \propto \mathrm{e}^{-m(g)t} &= 1 - m(g)t + \frac{1}{2}m(g)^2t^2 \\ &= 1 + (A_0 + A_1 t)g^2 + (B_0 + B_1 t + B_2 t^2)g^4 + \ldots \end{aligned}$$

where

$$m(g) = c_1 g^2 + c_2 g^4 + \ldots$$

By calculating PT expansion and separating the t^0, t terms one gets the expansion for $m(g)$

Summary

- Different regimes of QCD can be useful to obtain different physical quantities: masses, scattering lengths, LEC's, etc.
- The ϵ and δ regimes are ideal to determine the leading LEC's F and Σ since the others enter only in NNL corrections.
- NNL order is badly needed for the δ regime mass gap, and large L_s in simulations.
- The QM rotator of the δ regime is expected to be modified (in higher orders): it will contain $\mathcal{C}_l^2$.