
Simulating linear covariant gauges on the lattice: a new approach

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Abstract

We present a **new** implementation of the **linear covariant gauge** on the lattice. In particular, we discuss details of the **numerical procedure** for fixing the gauge.

We also present **preliminary results** for the **transverse and longitudinal gluon propagators** for the $SU(2)$ gauge group in four space-time dimensions.

Why study the linear covariant gauge?

- Study **Green's functions** in the **infrared limit** of Yang-Mills theories in order to understand low-energy properties of the theory.
- Since they depend on the gauge, consider **different gauges** (Landau gauge, Coulomb gauge, λ -gauge, maximally Abelian gauge).
- **Linear covariant gauge**, very popular in **continuum studies**, proved quite hostile to the **lattice approach**.

Some analytic results

- **Coupled Dyson-Schwinger equations** for gluon and ghost propagators (R. Alkofer et al., 2003): **scaling solution** (the gluon propagator is infrared suppressed whereas the ghost propagator is infrared enhanced).
- **Infrared-finite ghost propagator** (A.C. Aguilar and J. Papavassiliou, 2008) using **Schwinger-Dyson equation** of the ghost propagator.
- **Gribov's analysis** for small values of the gauge parameter ξ (R.F. Sobreiro and S.P. Sorella, 2005): the **transverse gluon propagator** is infrared suppressed, the **longitudinal part** is unchanged and the “**ghost propagator**” $[-\partial_\mu D_\mu(A_t)]^{-1}$ is infrared enhanced.

Linear covariant gauge on the lattice (I)

We want to impose the condition

$$\partial_\mu A_\mu^b(x) = \Lambda^b(x),$$

for real-valued functions $\Lambda^b(x)$.

Landau gauge [$\Lambda^b(x) = 0$] is obtained on the lattice by **minimizing** the functional

$$\mathcal{E}_{LG}[U^g] = -\text{Tr} \sum_{\mu, x} g(x) U_\mu(x) g^\dagger(x + e_\mu).$$

Linear covariant gauge on the lattice (II)

Problem: a no-go theorem (L. Giusti, 1996).

There is no minimizing functional $\mathcal{E}_{LCG}[U^g, \Lambda]$ for the linear covariant gauge!

Proof: Suppose $\mathcal{E}_{LCG}[U^g, \Lambda]$ exists. Then, it should be given by $\mathcal{E}_{LG}[U^g] + \mathcal{F}[U^g]$, for some $\mathcal{F}[U^g]$. The gauge-fixing condition is obtained from the stationarity condition $\frac{\partial \mathcal{E}_{LCG}}{\partial w^b(x)} = 0$, when $g(x) = e^{iw(x)}$. Also, the second variation should satisfy

$$\frac{\partial^2 \mathcal{E}_{LCG}}{\partial w^b(x) \partial w^c(y)} = \frac{\partial^2 \mathcal{E}_{LCG}}{\partial w^c(y) \partial w^b(x)}.$$

However, one can show that this equality is not realized since the two terms are, respectively, proportional to f^{acb} and f^{abc} .

First Solution

Consider a different gauge (L. Giusti, 1996):

$$F[\partial_\mu A_\mu^a(x) - \Lambda^a(x)] = 0$$

with $F[s] = 0$ when $s = 0$, for which the minimizing functional $\mathcal{E}[U^{(g)}, \Lambda]$ exists.

Problems:

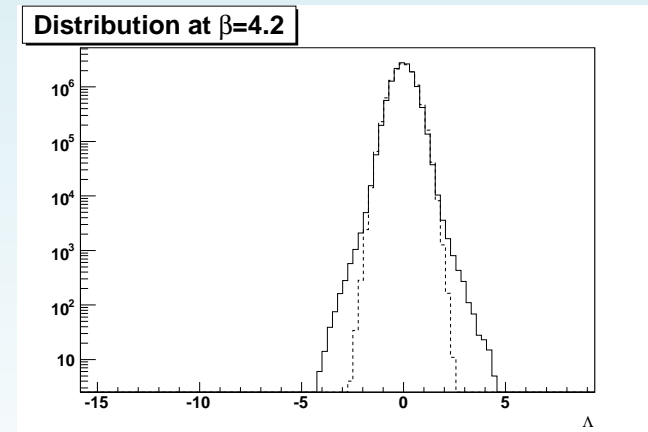
- Possible **spurious** solutions, $F[s] = 0$ when $s \neq 0$.
- With the choice $D_\nu \partial_\nu [\partial_\mu A_\mu^a(x) - \Lambda^a(x)] = 0$ the gauge fixing is **numerically** difficult.
- The **Faddeev-Popov matrix** is also different from that of the linear covariant gauge.

Second solution

Do not minimize! (A. C., A. Maas, T. Mendes, 2008)

1. Fix $U_\mu(x)$ to Landau gauge.
2. Solve $(\partial_\mu D_\mu^{ab} \phi^b)(x) = \Lambda^a(x)$. For small $\phi^b(x)$ one has $\partial_\mu A'_\mu{}^a(x) = \partial_\mu (A_\mu^a + D_\mu^{ab} \phi^b)(x) = \Lambda^a(x)$. **Problem:** only correct for infinitesimal gauge transformations.

If one considers functions $\Lambda^a(x)$ with a Gaussian distribution of width $\sqrt{\xi}$, then we should check if $\partial_\mu A'_\mu{}^a(x)$ also has a Gaussian distribution of width $\sqrt{\xi}$ and if $p^2 D_L(p^2) = \text{constant}$.



Third solution (2009)

Some details of the no-go theorem: for the minimizing functional $\mathcal{E}_{LCG}[U^g, \Lambda]$ the gauge condition is given by the **first variation**, i.e.

$$\delta\mathcal{E}_{LCG}[U^g, \Lambda] = \frac{\partial\mathcal{E}_{LCG}}{\partial U} \cdot \frac{\partial U}{\partial g} \delta g = 0 .$$

Solution: remove an **implicit hypothesis** of the no-go theorem, i.e. that the gauge transformation appears in the minimizing functional in the “**canonical**” way $g(x)U_\mu(x)g(x + e_\mu)$. Thus, we can look for a minimizing functional of the type $\mathcal{E}_{LCG}[U^g, g, \Lambda]$!

Simple hint: if you want to solve $B\phi = c$ just minimize $\frac{1}{2}\phi B\phi - \phi c$! In our case

$$\mathcal{E}_{LCG}[U^g, g, \Lambda] \sim \mathcal{E}_{LG}[U^g] - g\Lambda .$$

The minimizing functional

The lattice linear covariant gauge condition can be obtained by **minimizing** the functional

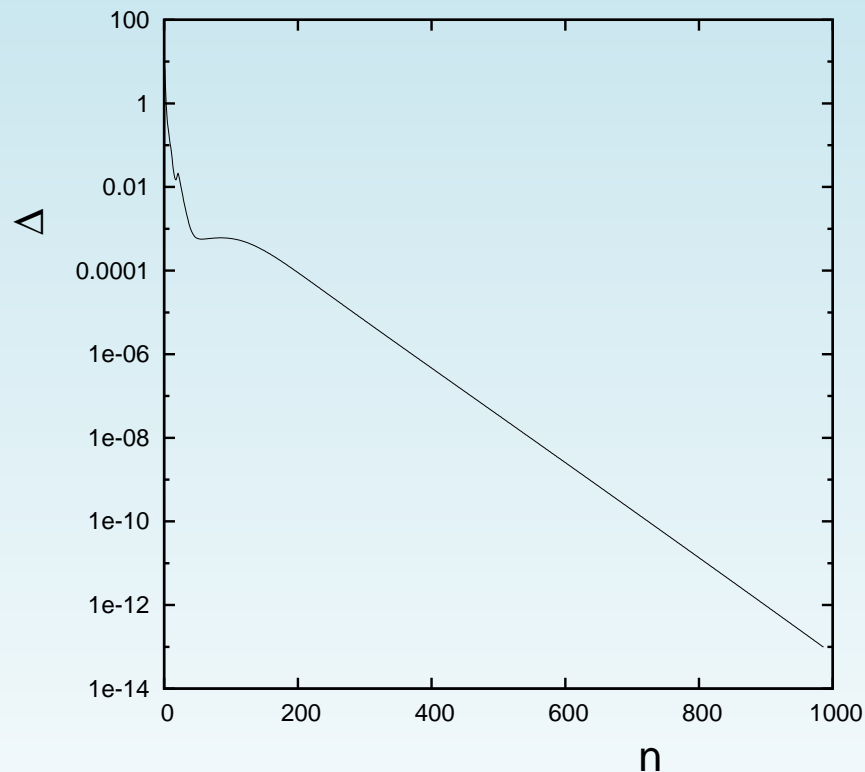
$$\mathcal{E}_{LCG}[U^g, g, \Lambda] = \mathcal{E}_{LG}[U^g] + \Re \text{Tr} \sum_x i g(x) \Lambda(x)$$

where

$$\mathcal{E}_{LG}[U^g] = -\text{Tr} \sum_{\mu, x} g(x) U_{\mu}(x) g^{\dagger}(x + e_{\mu}).$$

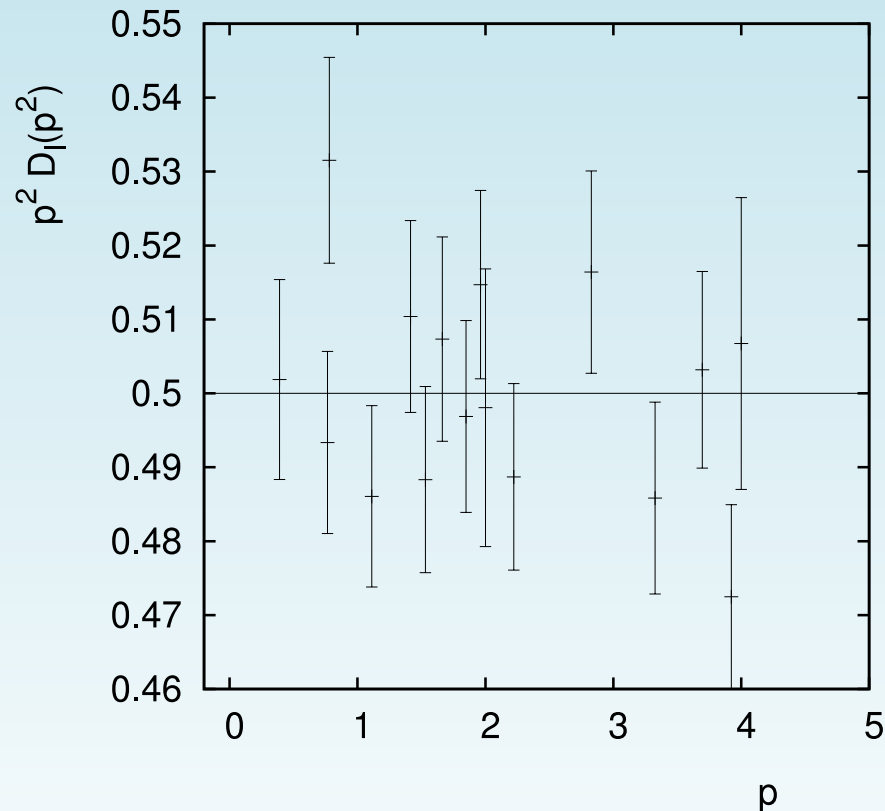
One can interpret the Landau-gauge functional $\mathcal{E}_{LG}[U^g]$ as a **spin-glass Hamiltonian** for the **spin variables** $g(x)$ with a **random interaction** given by $U_{\mu}(x)$. Then, our new functional corresponds to the same spin-glass Hamiltonian when a random external **magnetic field** $\Lambda(x)$ is applied.

Numerical tests of the gauge fixing



We have performed some numerical tests using the so-called stochastic-overrelaxation algorithm for the 4d $SU(2)$ case at $\beta = 4.0$, with $V = 8^4$ and 16^4 , and $\xi = 0.001, 0.1, 0.5$. We show the value of $\Delta = \sum_{x,b} [\nabla \cdot A^b(x) - \Lambda^b(x)]^2$ as a function of the number of iterations n for $\beta = 4$, $\xi = 0.5$ and $V = 8^4$.

Longitudinal gluon propagator



We checked that

$$p^2 D_l(p^2) = \xi$$

for $V = 16^4$ and $\beta = 4$ in the $SU(2)$ case for $\xi = 0.01, 0.1$ and 0.5 . In the case $\xi = 0.5$ a fit a/p^b for $D_l(p^2)$ gives $a = 0.502(5)$ and $b = 2.01(1)$.

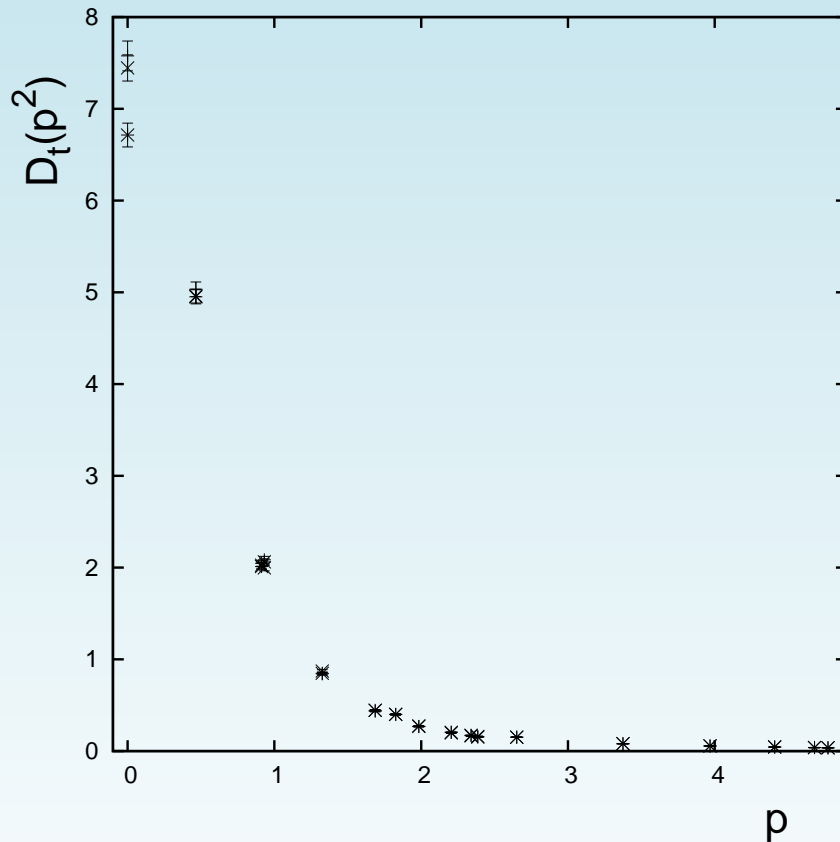
Discretization effects

- The gluon field $A_\mu^a(x)$ is (usually) **bounded** (compact formulation).
- The functions $\Lambda^b(x)$ satisfy a Gaussian distribution, i.e. they are **unbounded**.
- **Convergence problems** (the problem gets more severe when ξ is larger).

First observation: On the lattice, the function $\Lambda^b(x)$ is actually generated from a **Gaussian distribution** with width $\sigma = \sqrt{2N_c\xi/\beta}$ in the $SU(N_c)$ case. The situation is better for SU(3) than for SU(2).

Second observation: One can try to use **different discretizations** of the gluon fields. The situation gets better with the recently introduced **stereographic projection** (or modified lattice Landau gauge) (L. von Smekal et al., 2007).

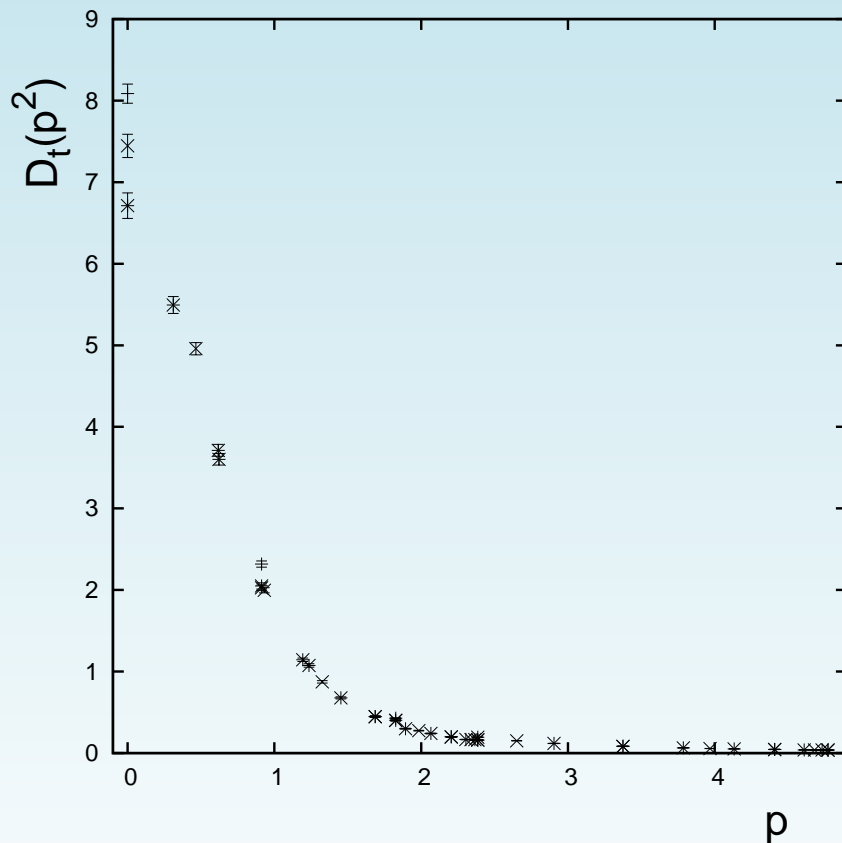
Transverse gluon propagator (I)



Transverse gluon propagator $D_t(p^2)$ as a function of the momentum p (both in physical units) for the lattice volume $V = 16^4$, $\beta = 2.3$ and $\xi = 0$ (+), 0.05 (\times), 0.1 (*).

$D_t(0)$ decreases as ξ increases (in agreement with L. Giusti et al., 2001).

Transverse gluon propagator (II)



Transverse gluon propagator $D_t(p^2)$ as a function of the momentum p (both in physical units) for the gauge coupling $\xi = 0.05$, $\beta = 2.3$, with the lattice volumes $V = 8^4$ (+), 16^4 (x) and 24^4 (*).

$D_t(0)$ decreases as V increases (as in Landau gauge).

Linear covariant gauge: conclusions

- We have found a **minimizing functional** for the **linear covariant gauge**.
- **Open problem**: simulations for large lattice volumes when the gauge parameter ξ is also large [it should be less severe for the SU(3) group].
- **Next**: simulations for the **SU(3)** case and also in **3d**.
- Understanding how the geometrical picture of the **(Landau) Gribov-Zwanziger confinement scenario** should be modified in the **general covariant gauge** (small value of ξ).
- Study **BRST breaking** in Gribov-Zwanziger approach.

**SECOND PART:
SOME INSIGHTS
FROM THE LATTICE**

Upper and Lower Bounds for $D(0)$

By using the inequality

$$\left(\frac{1}{n} \sum_{i=1}^n X_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^n X_i^2 ,$$

we have proven that

$$\langle \overline{M} \rangle^2 \leq D(0)/V \leq d(N_c^2 - 1) \langle \overline{M}^2 \rangle$$

were $D(0)$ is the gluon propagator at zero momentum and

$$\overline{M} = \frac{1}{d(N_c^2 - 1)} \sum_{b,\mu} |\tilde{A}_\mu^b(0)| = \frac{1}{d(N_c^2 - 1)V} \sum_{b,\mu} \left| \sum_x A_\mu^b(x) \right| .$$

Bounds for $D(0)$: Results

If \overline{M} goes to zero as $V^{-\alpha}$ we find that

$$D(0) \rightarrow 0, \quad 0 < D(0) < +\infty \quad \text{or} \quad D(0) \rightarrow +\infty$$

respectively if α is larger than, equal to or smaller than $1/2$.

- In 3d and in 4d Landau gauge we find $\alpha = 1/2$.
- This is consistent with a paramagnetic system with domains. On each domain the magnetization M is nonzero and the value in different domains is Gaussian distributed.

A Theoretical Experiment

In the paper D. Zwanziger, NPB 412 (1994) 657, the infinite-volume limit is taken in two steps:

1. first, considering the $V \rightarrow \infty$ limit for the gauge transformation $g(x)$;
2. then, taking the same limit for the gluon field [i.e. the link variables $U_\mu(x)$].

How can one do that?

The Two Limits

1. Consider a d -dimensional link configuration $U_\mu(x)$ with **periodicity** N , i.e. $V = N^d$ and $U_\mu(x + Ne_\nu) = U_\mu(x)$;
2. **replicate** this configuration m^d times, so that you get a configuration of volume $V = (mN)^d$;
3. do a **gauge fixing** of this (**large**) configuration using a gauge transformation $g(x)$ with **periodicity** mN , i.e. $g(x + mNe_\nu) = g(x)$.

The Extended Gauge Transformation (I)

One can prove that $g(x)$ can be written as

$$g(x) = \exp(t^a \theta_\mu^a x^\mu) h(x),$$

where

- $h(x)$ has **periodicity** N , i.e. $h(x + Ne_\nu) = h(x)$;
- θ_μ^a are elements of a Cartan subalgebra, i.e. they **commute**;
- $t^a \theta_\mu^a$ have **eigenvalues** $2\pi i n_\mu / (mN)$, with n_μ integer.

The Extended Gauge Transformation (II)

In the $SU(2)$ case (one-dimensional Cartan subalgebra) we can write

$$g(x) = \exp [2\pi i (u\sigma_3 u^\dagger) n_\mu x^\mu / (mN)] h(x) ,$$

with $u \in SU(2)$.

Note that $\exp (t^a \theta_\mu^a x^\mu)$ can be a “large” gauge transformation, maybe related to **Gribov copies** (as in the Abelian case).

What is the new minimizing functional?

The Extended Minimizing Functional

Instead of considering

$$I[U, h] = - \sum_{x, \mu} \text{Re Tr } U_{\mu}^h(x)$$

with $U_{\mu}^h(x) = h(x)U_{\mu}(x)h^{\dagger}(x + e_{\mu})$ we have, because of the cyclicity of the trace,

$$J[U, h, \theta] = - \sum_{x, \mu} \text{Re Tr } [U_{\mu}^h(x) \exp(-t^a \theta_{\mu}^a)] .$$

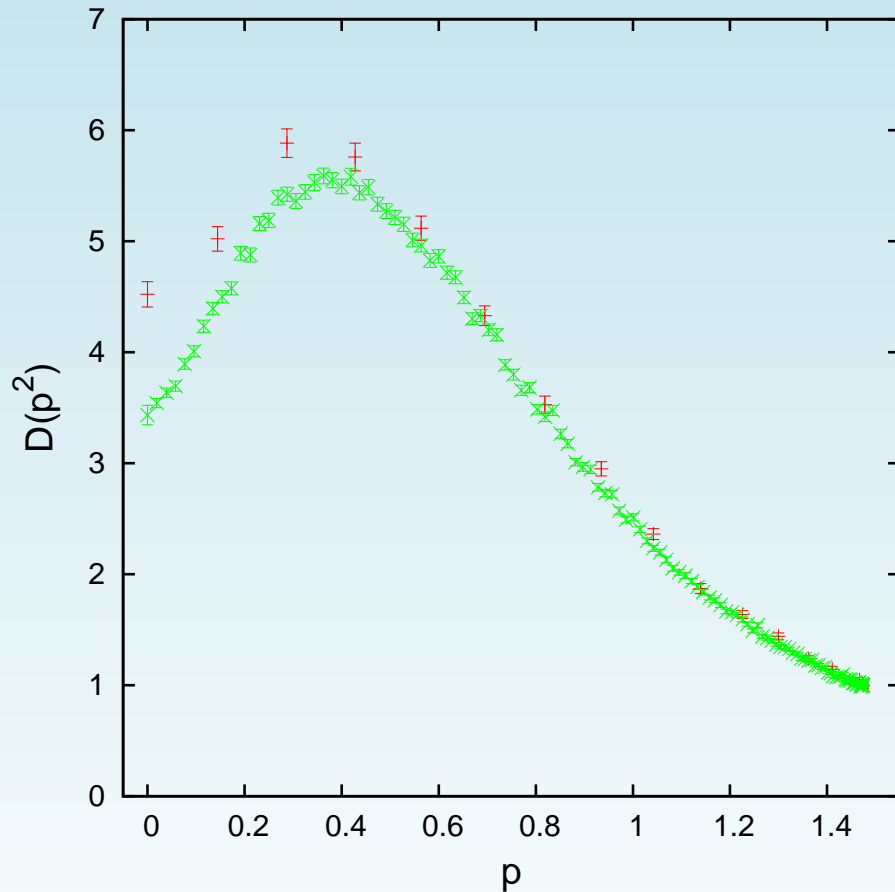
This still has **periodicity** N (so we work with a lattice of size $N^d!$), but it includes **large gauge transformations**! The gauge-fixed configuration is (of course) **transverse**.

Numerical Simulations

In the $SU(2)$ case:

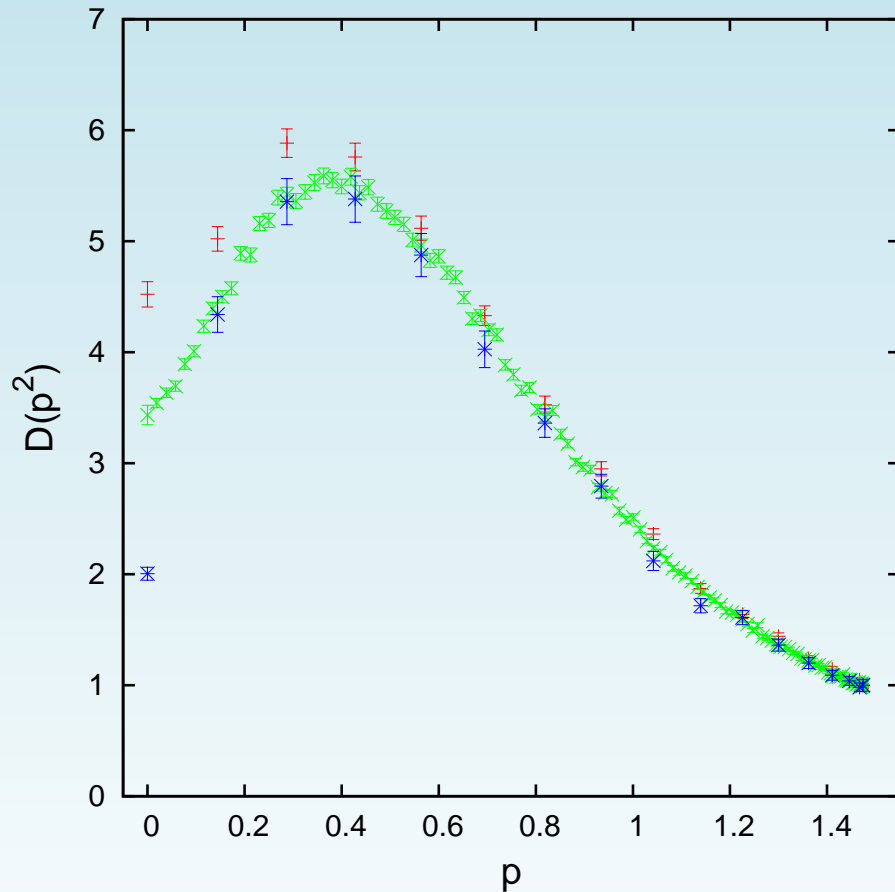
1. **generate** a thermalized d -dimensional link configuration $U_\mu(x)$ with **periodicity** N , i.e. $V = N^d$;
2. **minimize** $J[U, h, \theta]$ with respect to $h(x)$ and to $\exp(-u\sigma_3 u^\dagger \theta_\mu)$ with $\theta_\mu = 2\pi i n_\mu / (mN)$;
3. **evaluate the propagators** using the **large** configuration, obtained by **replicating** $U_\mu^h(x)$ **plus** the large gauge fixing.

Gluon Propagator in 3d (I)



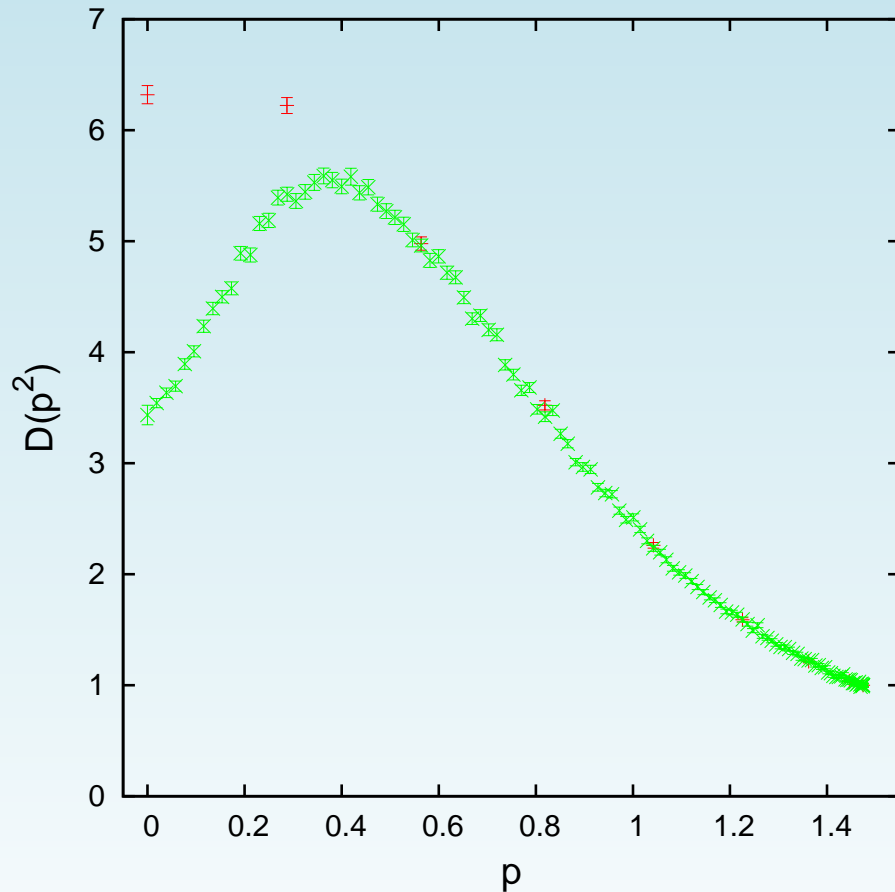
Gluon propagator $D(p^2)$ as a function of the lattice momentum p for a 32^3 lattice and a 240^3 lattice at $\beta = 3.0$. Here we do **NOT** use the extended gauge fixing.

Gluon Propagator in 3d (II)



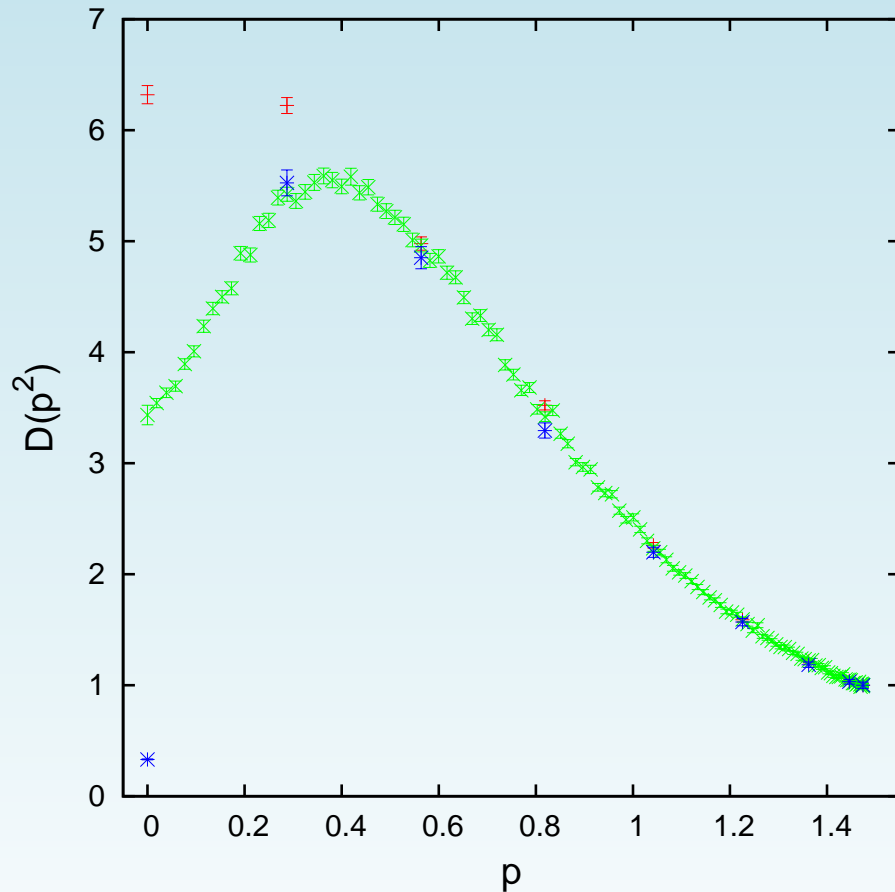
Gluon propagator $D(p^2)$ as a function of the lattice momentum p for a 32^3 lattice, a 240^3 lattice and a $32^3 \times 8^3 = 256^3$ lattice at $\beta = 3.0$.

Gluon Propagator in 3d (III)



Gluon propagator $D(p^2)$ as a function of the lattice momentum p for a 16^3 lattice and a 240^3 lattice at $\beta = 3.0$. Here we do **NOT** use the extended gauge fixing.

Gluon Propagator in 3d (IV)



Gluon propagator $D(p^2)$ as a function of the lattice momentum p for a 16^3 lattice, a 240^3 lattice and a $16^3 \times 16^3 = 256^3$ lattice at $\beta = 3.0$.

The Limit $m \rightarrow \infty$

Recall that

$$D(p^2) \propto V \sum_{b,\mu} \langle |\tilde{A}_\mu^b(p)|^2 \rangle$$

with

$$\tilde{A}_\mu^b(p) = V^{-1} \sum_x e^{-2\pi i k \cdot x} A_\mu^b(x)$$

and $p_\mu = 2 \sin(\pi k_\mu/N)$.

If one uses the “large” gauge configuration **without** the extended gauge fixing, the gluon propagator gets a factor m^d [this factor **cancels** in the evaluation of $\tilde{A}_\mu^b(p)$ but not in that of $D(p^2)$].

The Limit $m \rightarrow \infty$ (II)

Using the extended gauge fixing, $D(p^2)$ for $p \neq 0$ still gains a large factor (smaller than m^d). This is not the case for $D(0)$.

Since we (re)normalized using $D(p_{max}^2) = 1$, $D(0)$ is strongly suppressed, but there is probably a discontinuity $D(p^2 \rightarrow 0) \neq D(0)$ in the limit $m \rightarrow \infty$.

Indeed, note that when we used a 16^3 volume to simulate an extended volume of 256^3 the value of $D(0)$ was more suppressed than when considering the 32^3 volume. This is because the factor m^d is larger in the first case.

The Zero-Momentum Propagator

Question: why is $D(0)$ so suppressed?

In the $m \rightarrow \infty$ limit the extended gauge fixing is given by

$$\exp [2\pi i(u\sigma_3 u^\dagger)(w_\mu x^\mu)] = \cos(w_\mu x^\mu)\mathbf{1} + iu\sigma_3 u^\dagger \sin(w_\mu x^\mu)$$

in the $SU(2)$ case. By minimizing $J[U, h, w, u]$ with respect to w_μ and u we try to find a **global rotation** u that makes the gauge configuration close to an **“Abelian” configuration** and then we remove the **“Abelian zero mode”**.

The extended gauge-fixing is removing the **zero modes!**

Question: do we find $D(0) = 0$ at **finite** volume? Only if the **relevant** Landau-gauge configurations are **“Abelian”** (modulo a global rotation).

The Extended Gauge Transformation

- The infinite-volume limit in **two steps** seems to give, for the **gluon propagator in Landau gauge**, the same results obtained using the standard one-step limit, with **much less computational effort**.
- **Questions**: what is this result telling us about the **relevant Landau-gauge configurations**? Is this confirming the **paramagnetic** picture? Connection with the **Maximally Abelian gauge**?
- **Problem** to be solved: find a way of obtaining other values of p for which $D(p^2) \neq 0$. If successful, apply this method to the **ghost propagator** and to the **4d case**. Also do the **2d case**.