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# Ghost and gluon propagators, Gribov-Zwanziger horizon condition, and Kugo-Ojima confinement criterion

Kei-Ichi Kondo*
(Univ. of Tokyo/Chiba Univ., Japan)

Based on

- K.-I. Kondo, Kugo-Ojima color confinement criterion and Gribov-Zwanziger horizon condition, arXiv:0904.4897[hep-th]. Phys.Lett.B. 678, 322-330 (2009).

- K.-I. Kondo, Infrared behavior of the ghost propagator in the Landau gauge Yang-Mills theory, arXiv:0907.3249 [hep-th].

- K.-I. Kondo, A nilpotent “BRST” symmetry for the Gribov-Zwanziger theory arXiv:0905.1899[hep-th].

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* On sabbatical leave of absence from Chiba University

# § Introduction

We consider the quantum Yang-Mills theory in the Landau gauge $\partial\mathscr{A} = 0$

$$Z_{\rm YM} := \int [d\mathscr{A}]\delta(\partial\mathscr{A})\det(-\partial D[\mathscr{A}]))\exp\{-S_{YM}[\mathscr{A}]\}. \tag{1}$$

However, the gauge fixing condition $\partial\mathscr{A} = 0$ can not fix the gauge uniquely. This is because each gauge orbit intersects the gauge fixing hypersurface $\Gamma := \{\mathscr{A}; \partial\mathscr{A} = 0\}$ many times. The unique representative can not be chosen. There are Gribov copies.

In order to avoid the Gribov copies, Gribov (1978) proposed to restrict the functional integral to the (1st) Gribov region $\Omega$

$$Z_{\rm YM} := \int_\Omega [d\mathscr{A}]\delta(\partial\mathscr{A})\det(-\partial D[\mathscr{A}]))\exp\{-S_{YM}[\mathscr{A}]\} \tag{2}$$

where

$$\Omega := \{\mathscr{A}; \partial\mathscr{A} = 0 \ \& \ -\partial D[\mathscr{A}] > 0\} \subset \Gamma \tag{3}$$

Note that $-\partial D[\mathscr{A} = 0] = -\partial\partial > 0$, i.e., $\{\mathscr{A} = 0\} \in \Omega$.
The boundary of $\Omega$ is called the Gribov horizon:

$$\partial\Omega := \{\mathscr{A}; \partial\mathscr{A} = 0 \ \& \ -\partial D[\mathscr{A}] = 0\} \tag{4}$$

He predicted that the resulting Green functions exhibit unexpected behavior in the deep infrared (IR) region and that they play the essential role in confinement.

Define the gluon 2-point function (full or complete propagator)

$$D_{\mu\nu}^{AB}(k) := \delta^{AB}\left[\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{F(k^2)}{k^2} + \frac{\alpha}{k^2}\frac{k_\mu k_\nu}{k^2}\right] \quad (\alpha = 0) \tag{5}$$

and the ghost propagator

$$G^{AB}(k) := -\delta^{AB}\frac{G(k^2)}{k^2}. \tag{6}$$

In constrast to the free case

$$F(k^2) = 1, \quad G(k^2) = 1. \tag{7}$$

Gribov predicted their IR behaviors in the deep IR region $k^2 \ll 1$

$$\frac{F(k^2)}{k^2} \cong \frac{k^2}{(k^2)^2 + M^4} \downarrow 0, \quad \frac{G(k^2)}{k^2} \cong \frac{M^2}{(k^2)^2} \uparrow \infty \quad (k^2 \downarrow 0). \tag{8}$$

The gluon propagator vanishes in the IR limit $k^2 \downarrow 0$, while the ghost propagator becomes more singular than the free case in the IR region. This power like behavior should be compared with the UV behavior with the logarithmic corrections. ...

For the review [R. Alkofer and L. von Smekal, Phys.Rept.**353**, 281 (2001)]

$$F(k^2) = A \times (k^2)^{\alpha}, \quad G(k^2) = B \times (k^2)^{\beta}, \quad \alpha + 2\beta = 0, \quad 0 < A, B < \infty, \tag{9}$$

$$\alpha = 2\kappa > 1, \quad \beta = -\kappa < 0, \quad 1/2 < \kappa < 1 \text{ (Gribov } \kappa = 1).$$

Running coupling constant:

$$g^2(k) := g^2 F(k^2) G^2(k^2) \to 0 < g^2 A B^2 < \infty \ (k^2 \to 0) \text{ IR fixed point}$$

This IR behavior was considered to be reasonable from the viewpoint of color confinement. Due to Kugo-Ojima (1977-1978), all color non-singlet objects can not be observed or confined, in other words, only color singlet objects are observed, if $u(0) = -1$ in the Lorentz covariant gauge (a sufficient condition for color confinement). In the Landau gauge, Kugo-Ojima criterion for color confinement $u(0) = -1$ is equivalent to the divergent ghost dressing function $G(0) = \infty$, since in the Landau gauge

$$G(k) = [1 + u(k^2)]^{-1}$$

Until 2006, it seemed that this prediction has been confirmed by the Schwinger-Dyson equation (the scaling solution), the functional renormalization group equation and numerical simulations on lattice. $\Longrightarrow$ ghost dominance picture for confinement

So far so good.

However, ....

By careful analyses of the Schwinger-Dyson equation, socalled the decoupling solution was discovered:

$$F(k^2) = A' \times (k^2)^{\alpha}, \quad G(k^2) = B' \times (k^2)^{\beta}, \quad \alpha = 1, \quad \beta = 0, \quad 0 < A', B' < \infty,$$

$$g^2(k) := g^2 F(k^2) G^2(k^2) \cong g^2 A' B'^2 k^2 \to 0 \ (k^2 \to 0)$$

The ghost dressing function must be finite. See [Boucaud, Leroy, Yaouanc, Micheli, Pene and Rodriguez-Quintero, hep-ph/0803.2161, JHEP **06**, 099 (2008).]

Moreover, reexaminations of numerical simulations on lattices, functional renormalization group equation seem to converge the result:

The gluon propagator goes to the non-zero and finite constant in the IR limit, while the ghost propagator behaves like free (i.e., the ghost dressing function G(0) is non-zero and finite in the IR limit).

[Note that the Kugo-Ojima theory is based on the usual BRST formulation and does not take into account the Gribov problem where the exact color symmetry and the well-defined BRST charge are assumed.]

# § Main results of this talk

I discuss how the restriction of the integration region to the (1st) Gribov region constrains the possible value for the ghost dressing function and the Kugo-Ojima parameter for color confinement.

Within the Gribov-Zwanziger theory for the D-dimensional SU(N) Yang-Mills theory in the Landau gauge,

(1) I prove that the ghost dressing function $G(k^2)$ is non-zero finite in the limit $k \to 0$ and hence the ghost propagator behaves like free in the deep infrared regime.

(2) The Kugo-Ojima color confinement criterion u(0)=-1 is not satisfied in an original form. Rather, I find u(0)=-2/3 for $D = 4$ irrespective of $N$.

(3) However, it is possible to find a nilpotent “BRST” like symmetry in the Gribov-Zwanziger theory (restricted to the 1st Gribov region). $\boldsymbol{\delta} S_{\rm GZ} = \boldsymbol{\delta} \tilde{S}_{\gamma} \neq 0$
This is important to look for a modified color confinement criterion a la Kugo-Ojima.

These results are in harmony with decoupling solution of the Schwinger-Dyson equation, recent numerical simulation results on huge lattices.

However, they depend on the choice of the (non-local) horizon term.

# § Gribov-Zwanziger theory and horizon condition

- Gribov-Zwanziger theory [D.Zwanziger, Nucl. Phys. B**323**, 513–544 (1989)]

$$Z_\gamma := \int \mathcal{D}\mathscr{A}\,\delta(\partial^\mu \mathscr{A}_\mu)\det M \exp\{-S_{YM} - \gamma \int d^D x h(x)\}, \tag{1}$$

where $S_{YM}$ is the Yang-Mills action, $K$ is the Faddeev-Popov operator $K := -\partial_\mu D_\mu = -\partial_\mu(\partial_\mu + g\mathscr{A}_\mu \times)$ and $h(x) = h[\mathscr{A}](x)$ is the Zwanziger horizon function given by

$$h(x) := \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y). \tag{2}$$

Here the parameter $\gamma$ called the Gribov parameter is determined by solving a gap equation, commonly called the horizon condition:

$$\langle h(x) \rangle^\gamma = (N^2 - 1)D. \tag{3}$$

The action corresponding to the partition function (1) contains the *non-local* horizon term:

$$\int d^D x h(x) := \int d^D x \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y). \tag{4}$$

# § A localized Gribov-Zwanziger theory

[D. Zwanziger, Nucl. Phys. B**399**, 477–513 (1993).]

$$e^{-\gamma \int d^D x h(x)} = \int [d\xi][d\bar{\xi}][d\omega][d\bar{\omega}] \exp\left\{ -\tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}] \right\}, \tag{1}$$

where

$$\begin{aligned} \tilde{S}_\gamma =: \int d^D x [ & \bar{\xi}^{CA}_\mu K^{AB} \xi^{CB}_\mu - \bar{\omega}^{CA}_\mu K^{AB} \omega^{CB}_\mu \\ & + i\gamma^{1/2} g f^{ABC} \mathscr{A}^B_\mu \xi^{AC}_\mu + i\gamma^{1/2} g f^{ABC} \mathscr{A}^B_\mu \bar{\xi}^{AC}_\mu ]. \end{aligned} \tag{2}$$

The localized action $S_{\rm GZ}$ for the Gribov-Zwanziger theory is obtained

$$\begin{aligned} S_{\rm GZ} = & S^{\rm tot}_{\rm YM}[\mathscr{A}, \mathscr{C}, \bar{\mathscr{C}}, \mathscr{B}] + \tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}] \\ = & S_{\rm YM}[\mathscr{A}] + S_{\rm GF+FP}[\mathscr{A}, \mathscr{C}, \bar{\mathscr{C}}, \mathscr{B}] + \tilde{S}_\gamma[\mathscr{A}, \xi, \bar{\xi}, \omega, \bar{\omega}], \end{aligned} \tag{3}$$

where

$$\mathscr{L}_{\rm GF+FP} := \int d^D x \left\{ \mathscr{B} \cdot \partial_\mu \mathscr{A}_\mu + \frac{\alpha}{2} \mathscr{B} \cdot \mathscr{B} + i \bar{\mathscr{C}} \cdot \partial_\mu D_\mu \mathscr{C} \right\}. \tag{4}$$

The localized GZ theory is known to be multiplicatively renormalizable to all orders.

$$\begin{aligned}
\mathscr{A}_\mu =& Z_A^{1/2}\mathscr{A}_\mu^R, \quad \mathscr{B} = Z_B^{1/2}\mathscr{B}^R, \quad Z_B = Z_A^{-1}, \\
\mathscr{C} =& Z_C^{1/2}\mathscr{C}^R, \quad \bar{\mathscr{C}} = Z_C^{1/2}\bar{\mathscr{C}}^R, \\
g =& Z_g g_R, \quad Z_g = \tilde{Z}_1 Z_A^{-1/2} Z_C^{-1},
\end{aligned} \tag{5}$$

$$\begin{aligned}
\xi_\mu =& Z_\xi^{1/2}\xi_\mu^R, \quad \bar{\xi}_\mu = Z_{\bar{\xi}}^{1/2}\bar{\xi}_\mu^R, \quad Z_\xi = Z_{\bar{\xi}} = Z_C, \\
\omega_\mu =& Z_\omega^{1/2}\omega_\mu^R, \quad \bar{\omega}_\mu = Z_{\bar{\omega}}^{1/2}\bar{\omega}_\mu^R, \quad Z_\omega = Z_{\bar{\omega}} = Z_C, \\
\gamma =& Z_\gamma \gamma_R, \quad Z_\gamma = Z_A^{-1} Z_C^{-1},
\end{aligned} \tag{6}$$

# § Horizon condition and ghost dressing function

[K.-I. K., arXiv:0904.4897[hep-th], arXiv:0907.3249 [hep-th]]:
The average of the horizon function is exactly rewritten as

$$\langle h(0) \rangle = -\left(N^2-1\right)\left\{Du(0)+w(0)-G(0)[u(0)+w(0)]^2\right\}, \tag{1}$$

where

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI} = \left[ g_{\mu\nu} u(k^2) + \frac{k_\mu k_\nu}{k^2} w(k^2) \right] \delta^{AB}. \tag{2}$$

Here $u(k^2)$ is the Kugo-Ojima function defined by

$$\langle (D_\mu \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k := \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \delta^{AB} u(k^2). \tag{3}$$

In the Landau gauge, the ghost dressing function $G(k^2)\delta^{AB} := -k^2 \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k$ satisfies

$$G(k^2) = [1 + u(k^2) + w(k^2)]^{-1}. \tag{4}$$

The last equality was derived by [Kugo, hep-th/9511033] and also in [P.A. Grassi, T. Hurth, A. Quadri, e-Print: hep-th/0405104, Phys.Rev. D**70**, 105014 (2004)].

The IR limit of the ghost dressing function satisfies ($G(0) \geq 0$)

$$G(0) = [1 + u(0) + w(0)]^{-1}. \tag{5}$$

The horizon condition is exactly rewritten as

$$\langle h(0) \rangle = -(N^2 - 1) \left\{ Du(0) + w(0) - G(0)[u(0) + w(0)]^2 \right\} = (N^2 - 1)D, \tag{6}$$

1) $G(0) = 0 \iff G(0)^{-1} = \infty \iff u(0) = \infty$ or $w(0) = \infty$ from (5)
$\implies \langle h(0) \rangle = \infty.$ The horizon condition (6) is not satisfied.

2) $G(0) = \infty \iff G(0)^{-1} = 0 \iff u(0) + w(0) = -1$ from (5)
$\implies \langle h(0) \rangle = \infty.$ The horizon condition (6) is not satisfied.

The horizon condition is satisfied only when $0 < G(0) < \infty.$ The ghost propagator behaves like free in the deep IR regime.

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The Kugo-Ojima criterion $u(0) = -1 \iff G(0) = w(0)^{-1} \implies$

$$\langle h(0) \rangle = -(N^2 - 1) \left\{ -D + w(0) - [1/w(0)][-1 + w(0)]^2 \right\} = (N^2 - 1)D$$

$\implies w(0) = 1/2$ and $G(0) = 2$ for any $D$.
In other words, $w(0) \neq 1/2 \implies u(0) \neq -1$ for any $D$.

# § Proof of the main result

Step 1: Rewriting the horizon condition

$$\langle h(x)\rangle = (\mathrm{dim}G)D \tag{1}$$

The average of the horizon function reads

$$\begin{aligned}
\langle h(x)\rangle &= \int d^D y \langle g f^{ABC}\mathscr{A}_\mu^B(x)(K^{-1})^{CE}(x,y) g f^{AFE}\mathscr{A}_\mu^F(y)\rangle \\
&= -\int d^D y \langle g f^{ABC}\mathscr{A}_\mu^B(x)\mathscr{C}^C(x)\bar{\mathscr{C}}^E(y) g f^{AFE}\mathscr{A}_\mu^F(y)\rangle \\
&= -\int d^D y \langle (g\mathscr{A}_\mu \times \mathscr{C})^A(x)(g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A(y)\rangle \\
&= -\lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A(g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k .
\end{aligned} \tag{2}$$

In what follows, we define the Fourier transform of the two-point function for composite operators by

$$\langle \phi_1^A \phi_2^B\rangle_k := \int d^D x e^{ik(x-y)}\langle 0|T[\phi_1^A(x)\phi_2^B(y)]|0\rangle . \tag{3}$$

Step 2: We consider the non-selfcontracted form

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k = \lambda_{\mu\nu}^{AB}(k) + \Delta_{\mu\nu}^{AB}(k), \tag{4}$$

where

$$\begin{aligned}
\lambda_{\mu\nu}^{AB}(k) :=& \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{m1PI}}, \\
\Delta_{\mu\nu}^{AB}(k) :=& \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^C \rangle_k^{\mathrm{1PI}} \langle \mathscr{C}^C \bar{\mathscr{C}}^D \rangle_k \langle \mathscr{C}^D (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{1PI}}.
\end{aligned} \tag{5}$$

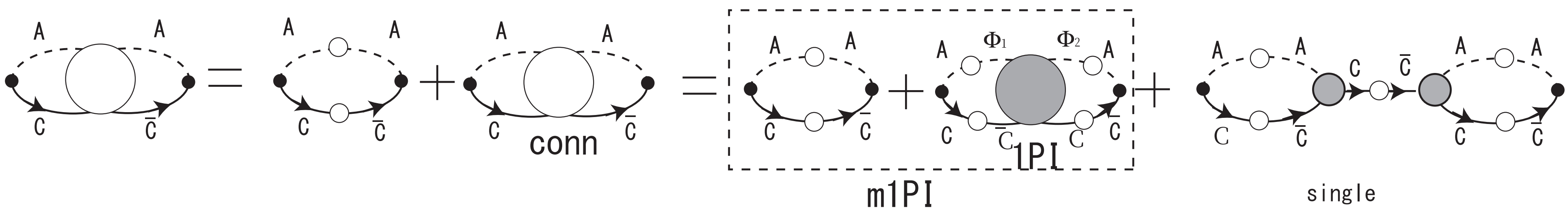



Figure 1: Diagrammatic representation of $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{conn}}$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{1PI}}$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\mathrm{m1PI}}$.

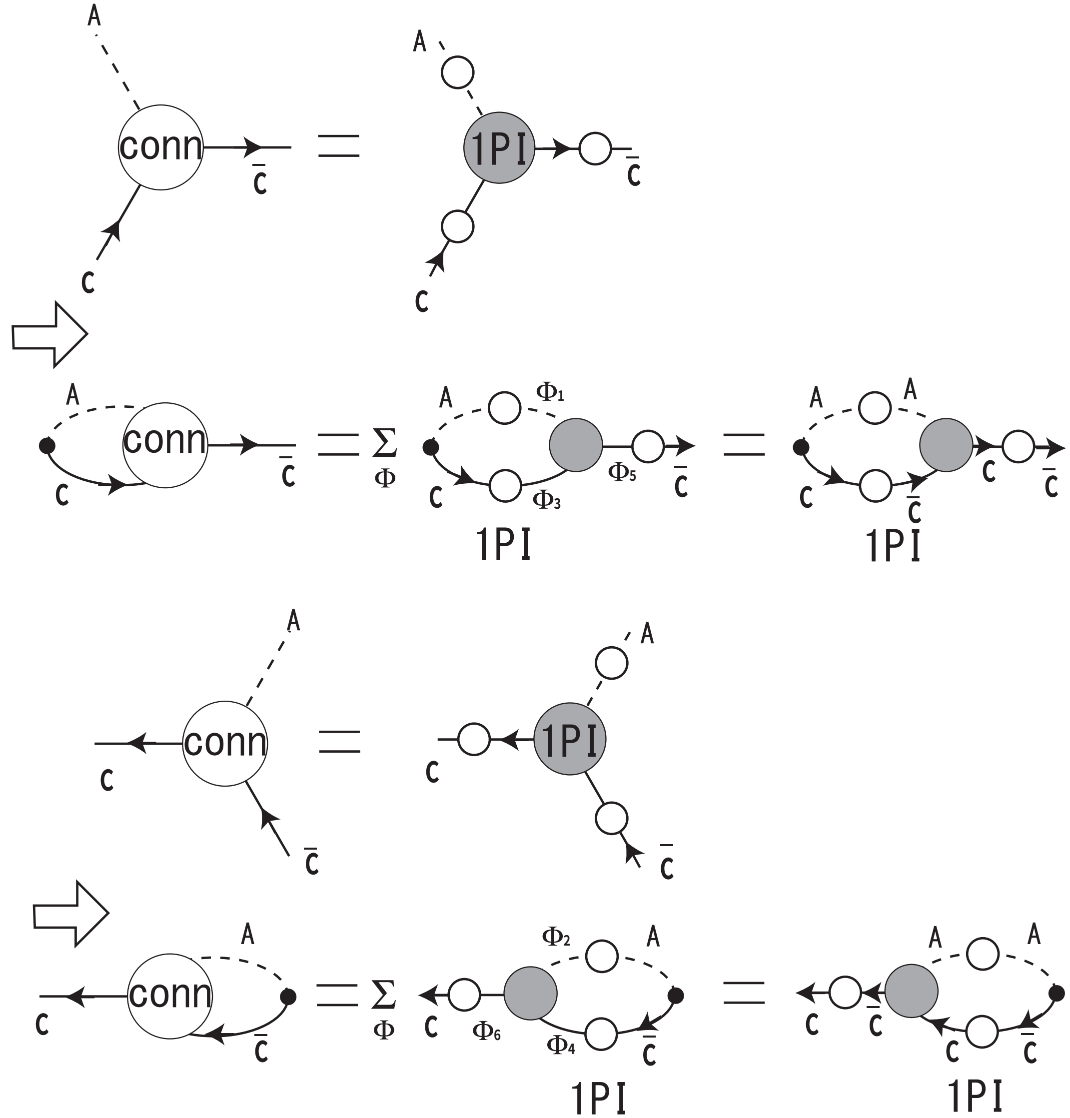



Figure 2: Diagrammatic representation of (a) $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{\rm 1PI}$, (b) $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$ and $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\rm 1PI}$.

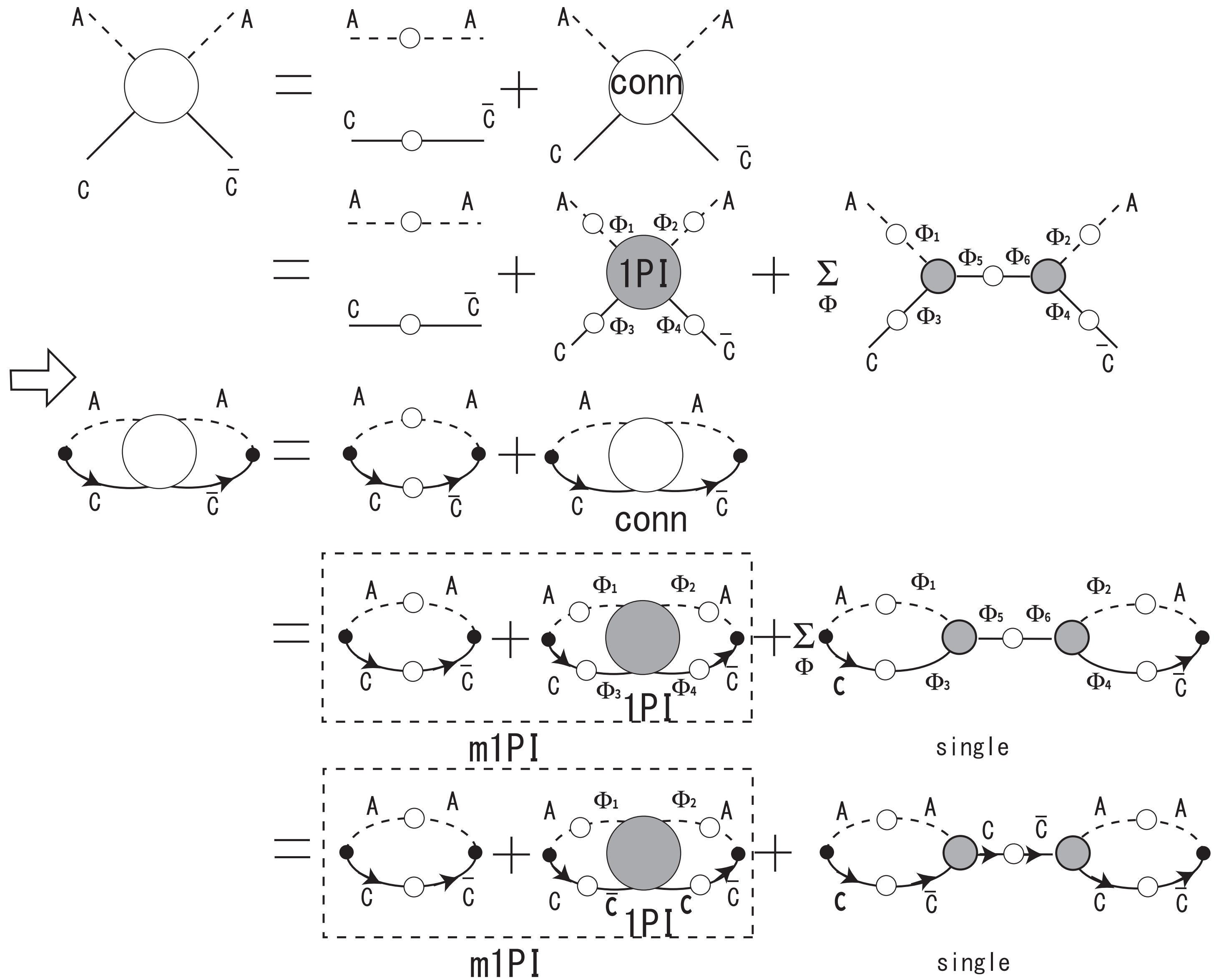



Figure 3: Diagrammatic representation of $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\rm conn}$, $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\rm 1PI}$ and $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{\rm m1PI}$.

Step 3: mutual relationships $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{m1PI}$, $\langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{1PI}$ (or $\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}$) and $\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k$.

(a) In the manifestly covariant gauge of the Lorenz type,

$$ik_\mu \lambda_{\mu\nu}^{AB}(k) = \langle \mathscr{C}^A (g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B \rangle_k^{1PI}, \tag{6}$$

(b) In the Landau gauge, the FP conjugation invariance leads to

$$-ik_\nu \lambda_{\mu\nu}^{AB}(k) = \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}, \tag{7}$$

(c)

$$\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = -ik_\mu \left( -\delta^{AB} + \frac{-1}{k^2} \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} \right), \tag{8}$$

which is stronger than the resulting ghost propagator Schwinger-Dyson equation:

$$\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} = -k^2 \delta^{AB} - ik^\mu \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}. \tag{9}$$

Thus, $\lambda_{\mu\nu}^{AB}$ is related to the ghost propagator:

$$ik_\mu \lambda_{\mu\nu}^{AB}(k)(-ik_\nu) = ik^\mu \langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = -\delta^{AB} k^2 - \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1}. \tag{10}$$

Step 4: The general form is (assuming unbroken color symmetry)

$$\lambda_{\mu\nu}^{AB}(k) = \left[g_{\mu\nu}u(k^2) + \frac{k_\mu k_\nu}{k^2}w(k^2)\right]\delta^{AB}, \tag{11}$$

where $u$ is the Kugo-Ojima function (usually defined by)

$$\langle(D_\mu \mathscr{C})^A(g\mathscr{A}_\nu \times \bar{\mathscr{C}})^B\rangle_k := \delta^{AB}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)u(k^2), \tag{12}$$

and $w$ is an unfamiliar function.

In the Landau gauge, we obtain a relationship between the Kugo-Ojima function and the ghost dressing function defined by $G(k^2)\delta^{AB} := -k^2\langle \mathscr{C}^A \bar{\mathscr{C}}^B\rangle_k$

$$G(k^2)^{-1} = 1 + u(k^2) + w(k^2), \tag{13}$$

since

$$ik_\mu \lambda_{\mu\nu}^{AB}(k)(-ik_\nu) = -\delta^{AB}k^2 - \langle \mathscr{C}^A \bar{\mathscr{C}}^B\rangle_k^{-1}. \tag{14}$$

In the Landau gauge, therefore, we find

$$\lambda^{AA}_{\mu\mu}(k) =(\mathrm{dim}G)[Du(k^2)+w(k^2)], \tag{15}$$

$$\begin{aligned}\Delta^{AA}_{\mu\mu}(k) =& -i\lambda^{AC}_{\mu\sigma}(k)k_\sigma \frac{-G(k^2)}{k^2}\delta^{CD} ik_\rho \lambda^{DA}_{\rho\mu}(k) \\ =& -(\mathrm{dim}G)G(k^2)[u(k^2)+w(k^2)]^2 \\ =& -(\mathrm{dim}G)\frac{[u(k^2)+w(k^2)]^2}{1+u(k^2)+w(k^2)}.\end{aligned} \tag{16}$$

The average of the horizon function reads

$$\begin{aligned}\langle h(0)\rangle =& -\lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k \\ =& -\lambda^{AA}_{\mu\mu}(0) - \Delta^{AA}_{\mu\mu}(0) \\ =& -(\mathrm{dim}G)\left\{Du(0)+w(0)-G(0)[u(0)+w(0)]^2\right\} \\ =& -(\mathrm{dim}G)\left\{Du(0)+w(0)-\frac{[u(0)+w(0)]^2}{1+u(0)+w(0)}\right\}.\end{aligned} \tag{17}$$

The existence of the last term $\Delta^{AA}_{\mu\mu}(0)$ is crucial to obtain a finite ghost dressing function at $k=0$.

# § Plugging into the Schwinger-Dyson equation

The Schwinger-Dyson (SD) equation for the ghost propagator is

$$\langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k^{-1} = -\delta^{AB} k^2 - i \frac{k^\mu}{k^2} \langle (g \mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI}. \tag{1}$$

By using $-i\frac{k^\mu}{k^2}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A \bar{\mathscr{C}}^B \rangle_k^{1PI} = \frac{k^\mu k^\nu}{k^2}\frac{\lambda_{\mu\nu}^{AA}(k)}{(N^2-1)} = u(k^2) + w(k^2)$, the SD equation is rewritten as

$$G^{-1}(k^2) = 1 + u(k^2) + w(k^2). \tag{2}$$

Now we incorporate the horizon condition into the SD equation. Following the idea of Gribov, we substitute the horizon condition of the form

$$1 = \frac{\langle h(0) \rangle}{(N^2-1)D} = -u(0) - \frac{w(0)}{D} + \frac{G^{-1}(0) - 2 + G(0)}{D}. \tag{3}$$

Then, it is observed that the first term $-u(0)$ in the horizon condition cancels the term $u(k^2)$ at $k = 0$ in the right-hand side of the SD equation,

$$G^{-1}(0) = \frac{G^{-1}(0) - 2 + G(0)}{D} + \left( -\frac{1}{D} + 1 \right) w(0). \tag{4}$$

By solving

$$G^2(0) - [2 + (1-D)w(0)]G(0) + 1 - D = 0. \tag{5}$$

we have

$$G(0) = 1 + (1-D)w(0)/2 + \sqrt{[1 + (1-D)w(0)/2]^2 - 1 + D} > 0, \tag{6}$$

Using $w(0) = 0$ by an independent argument [Binosi's talk], we find that $G(0)$ is determined selfconsistently by solving the SD equation for the ghost propagator:

$$G(0) = 1 + \sqrt{D} > 0, \quad u(0) = (-D \pm \sqrt{D})/(D-1) \tag{7}$$

$$G(0) = 3 > 0, \quad u(0) = -2/3 \quad (D = 4). \tag{8}$$

In other words, the horizon condition determines the boundary condition for $G(0)$ in the SD equation. Consequently, we have the decoupling solution $0 < G(0) < \infty$.

The scaling solution $G(0) = \infty$ is obtained only when $\Delta_{\mu\mu}^{AA}(0)$ is vanishing. To obtain the scaling solution, the constant terms must cancel exactly or disappear at the $k = 0$ limit on the right-hand side of the SD equation. This is what implicitly assumed, but not stated explicitly, as pointed out by [Boucaud, Leroy, Yaouanc, Micheli, Pene and Rodriguez-Quintero, arXiv:0801.2721[hep-ph], JHEP **06**, 012 (2008).]

# § Another choice of the horizon term up to the total derivative term

We point out that the result crucially depends on the explicit form of the non-local horizon term adopted.

If the total derivative was neglected in the Gribov-Zwanziger theory, the horizon term could be rewritten as

$$\begin{aligned} \int d^D x h(x) &:= \int d^D x \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y) \\ ? &= \int d^D x \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y) \\ ? &= \int d^D x \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) D_\mu[\mathscr{A}]^{AE}(y), \end{aligned} \tag{1}$$

The last horizon term yielded the average of the horizon function:

$$\langle h(0) \rangle = - \lim_{k \to 0} \langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k, \tag{2}$$

Thus, the horizon condition for this horizon function defined from this form, i.e.,

$$\langle h(0)\rangle = -\lim_{k\to 0}\langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A\rangle_k = -(N^2-1)\left\{(D-1)u(0)-1\right\} = (N^2-1)D, \tag{3}$$

led to the Kugo-Ojima criterion:

$$u(0) = -1, \tag{4}$$

and the divergent ghost dressing function with an imput $w(0)=0$:

$$G(0) = [1+u(0)+w(0)]^{-1} = w(0)^{-1} = \infty. \tag{5}$$

Thus, if one starts from the horizon term in the last form of (1) by neglecting the total derivative term in the non-local horizon function, then one is led to the opposite conclusion to ours.

Two horizon functions cannot be the same. If the average of two horizon functions agree to each other, then $u(0)=-1/2$ for any $D$ assuming $w(0)=0$.

If we adopt another horizon function,

$$1 = \frac{\langle h(0) \rangle}{(N^2-1)D} = -\frac{\lim_{k\to 0}\langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k}{(N^2-1)D} = \frac{1}{D}[1+u(0)] - u(0). \tag{6}$$

the SD equation $G^{-1}(k^2) = 1 + u(k^2) + w(k^2)$ is rewritten as

$$G^{-1}(k^2) = \frac{1}{D}[1+u(0)] - u(0) + u(k^2) + w(k^2). \tag{7}$$

In the deep IR limit, such a cancellation occurs for $u(0)$:

$$G^{-1}(0) = \frac{1}{D}[1+u(0)] + w(0), \tag{8}$$

$$G^{-1}(0) = \frac{1}{D}G^{-1}(0) - \frac{1}{D}w(0) + w(0). \tag{9}$$

This is solved to give for $D \neq 1$

$$G^{-1}(0) = w(0). \tag{10}$$

If $w(0) = 0$, then we obtain the scaling solution $G^{-1}(0) = 0$.

# § Lattice data for the Kugo-Ojima parameter

- H. Nakajima and S. Furui, hep-lat/9909008,
- H. Nakajima and S. Furui, hep-lat/0006002,

- A. Sternbeck, section 5.2 in hep-lat/0609016. [Ph.D. thesis]
- A. Sternbeck, E.-M. Ilgenfritz, M. Müller-Preussker, A. Schiller and I.L. Bogolubsky, hep-lat/0610053,

In pure Yang-Mills theory,

$$u(0) = -0.6 \sim -0.8 > -1$$

See Figure 5.3 in section 5.2.2 on page 104

$$\tilde{u}(k^2, \mu^2) := Z^{-1}(k^2, \mu^2) - 1 = u(k^2, \mu^2) + w(k^2)$$

See Figure 5.4 on page 106 [Figure 6 of hep-lat/0610053],

$$u(k^2, \mu^2), \quad \tilde{u}(k^2, \mu^2) - u(k^2, \mu^2) = w(k^2)$$

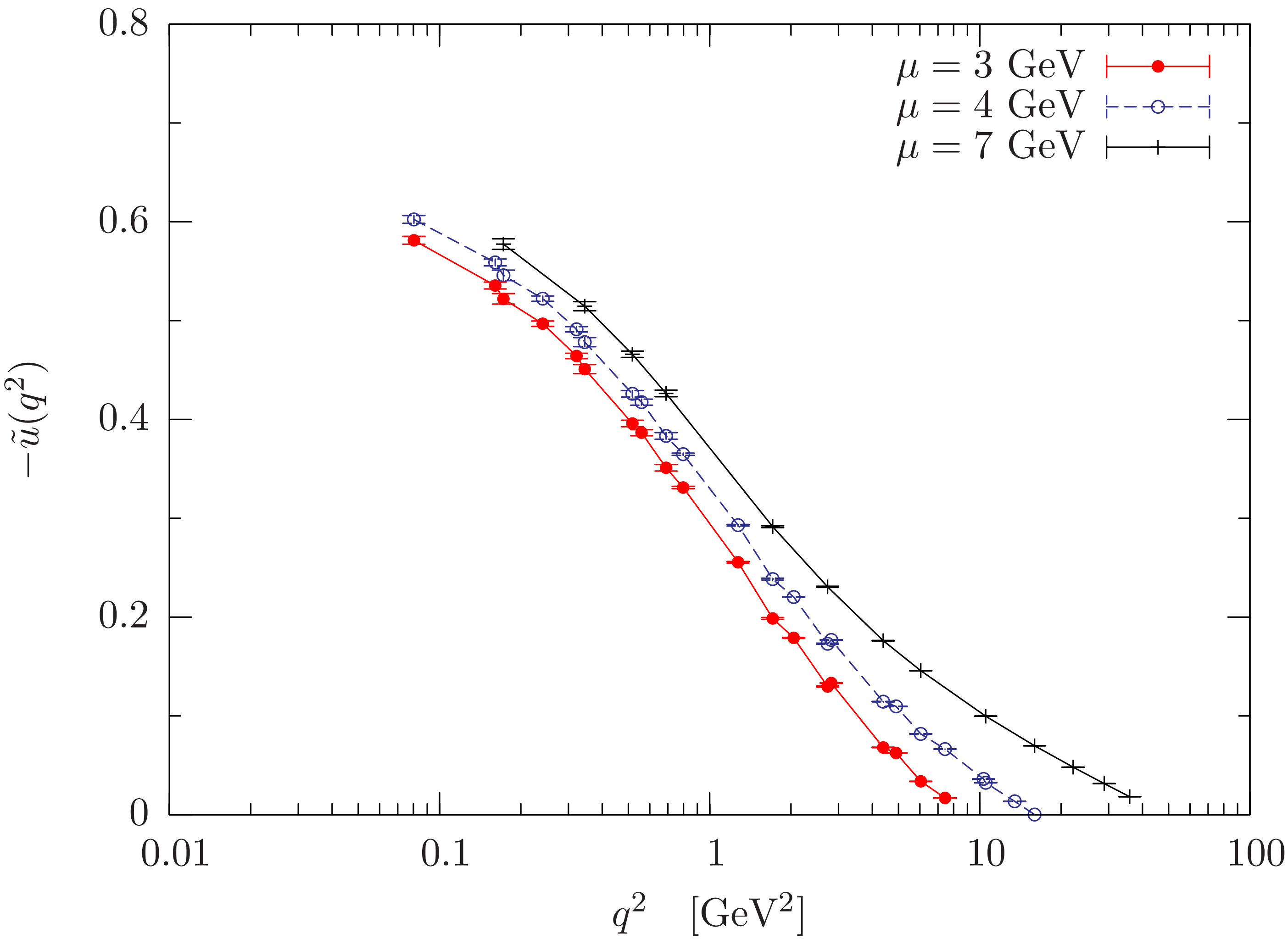
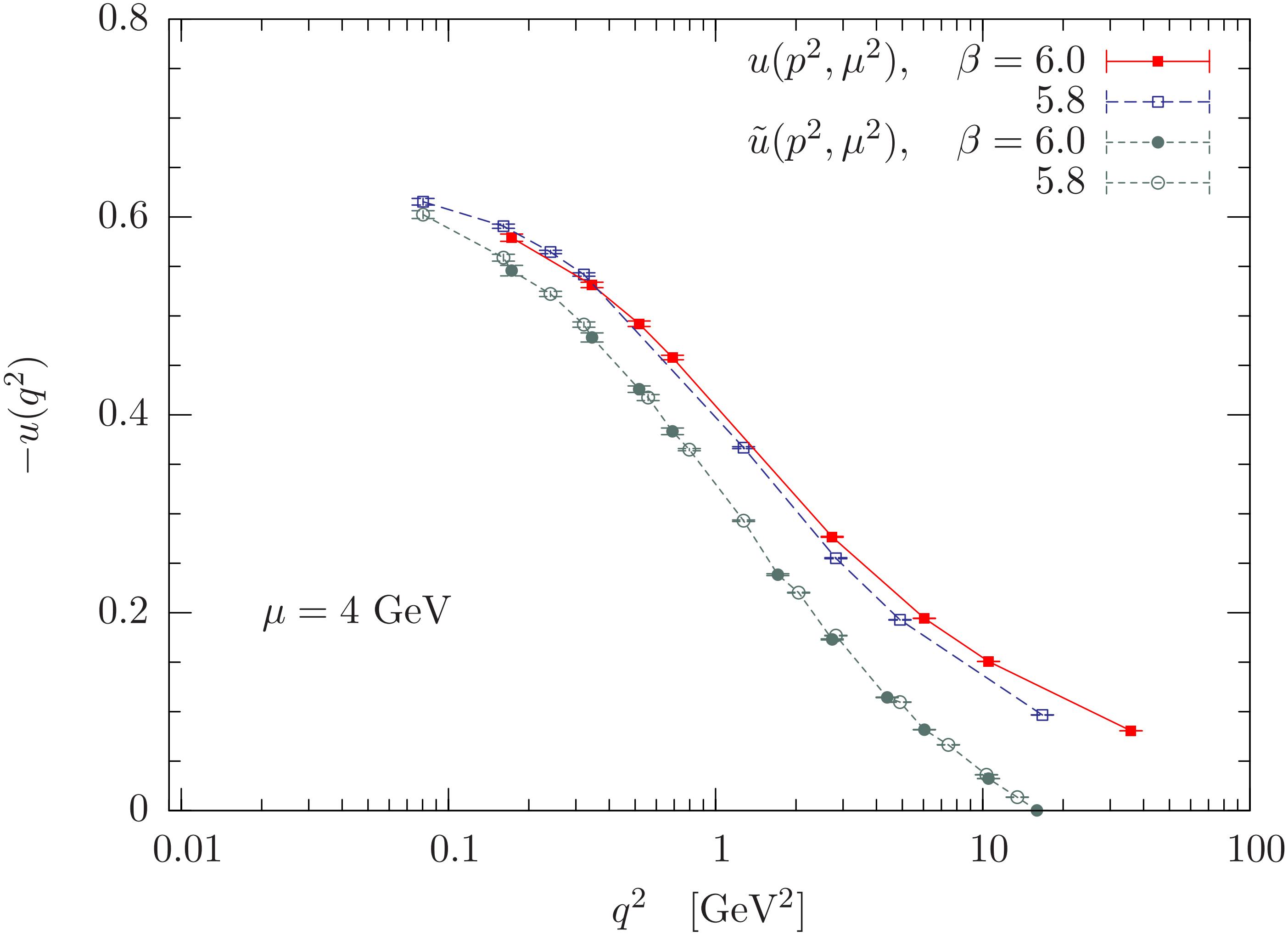



Figure 4: The asymptote $-\tilde{u}(q^2, \mu^2)$ as defined in ( ) is shown as a function of momentum $q^2$. For the ghost dressing function we used our data at $\beta = 5.8$ and $6.0$ renormalized either at $\mu = 3$, $4$ or $7$ $GeV$. The lattice size is $32^4$. Lines are drawn to guide the eye.




Data for the function $u(q^2, \mu^2)$ at $\beta = 5.8$ and 6.0 are shown using full and open squares. Additionally, data of the asymptote $\tilde{u}(q^2, \mu^2)$ are shown at the same $\beta$ values (circles). All data refer to the same quenched configurations on a $32^4$ lattice and are renormalized at $\mu = 4\ GeV$ as described in the text. Lines are drawn to guide the eye.

# § Lattice data for ghost and gluon dressing function

[Bogolubsky-Ilgenfritz-Muller-Preussker-Sternbeck-2009-PLB676-69-73, 0901.0736]
SU(3)

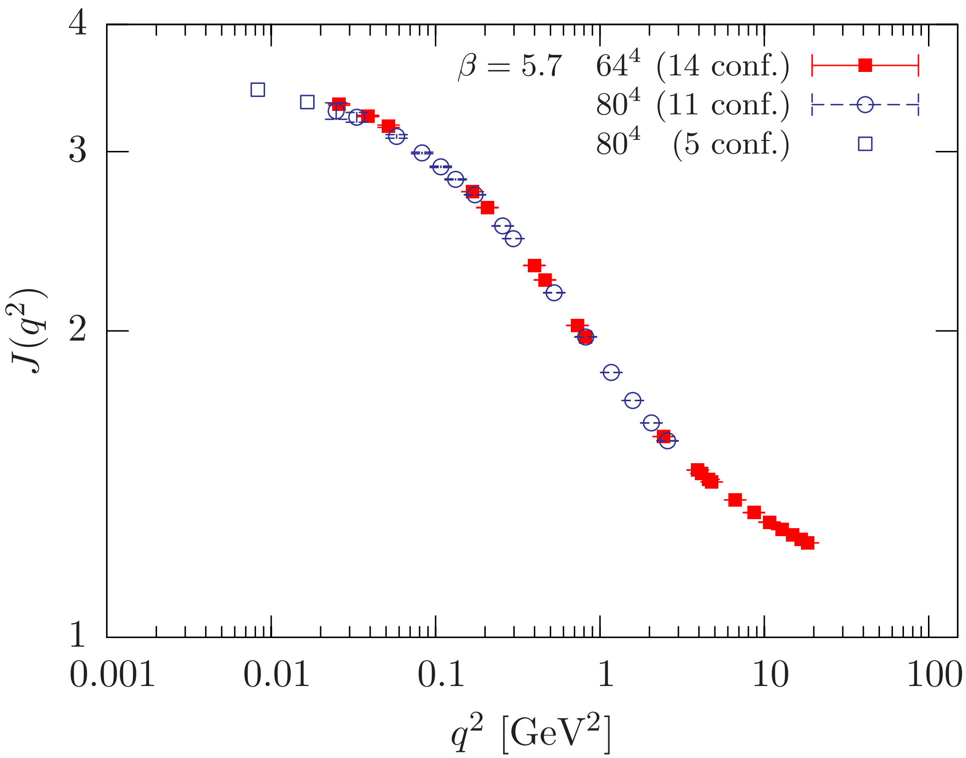



Figure 5: [SU(3)] Bare ghost dressing function $J(q^2)$ versus $q^2$ for $L = 64, 80$ at $\beta = 5.70$. Errors are not shown at the two lowest $q^2$ (squares).

[Bogolubsky-Ilgenfritz-Muller-Preussker-Sternbeck, 0812.2761] SU(2)

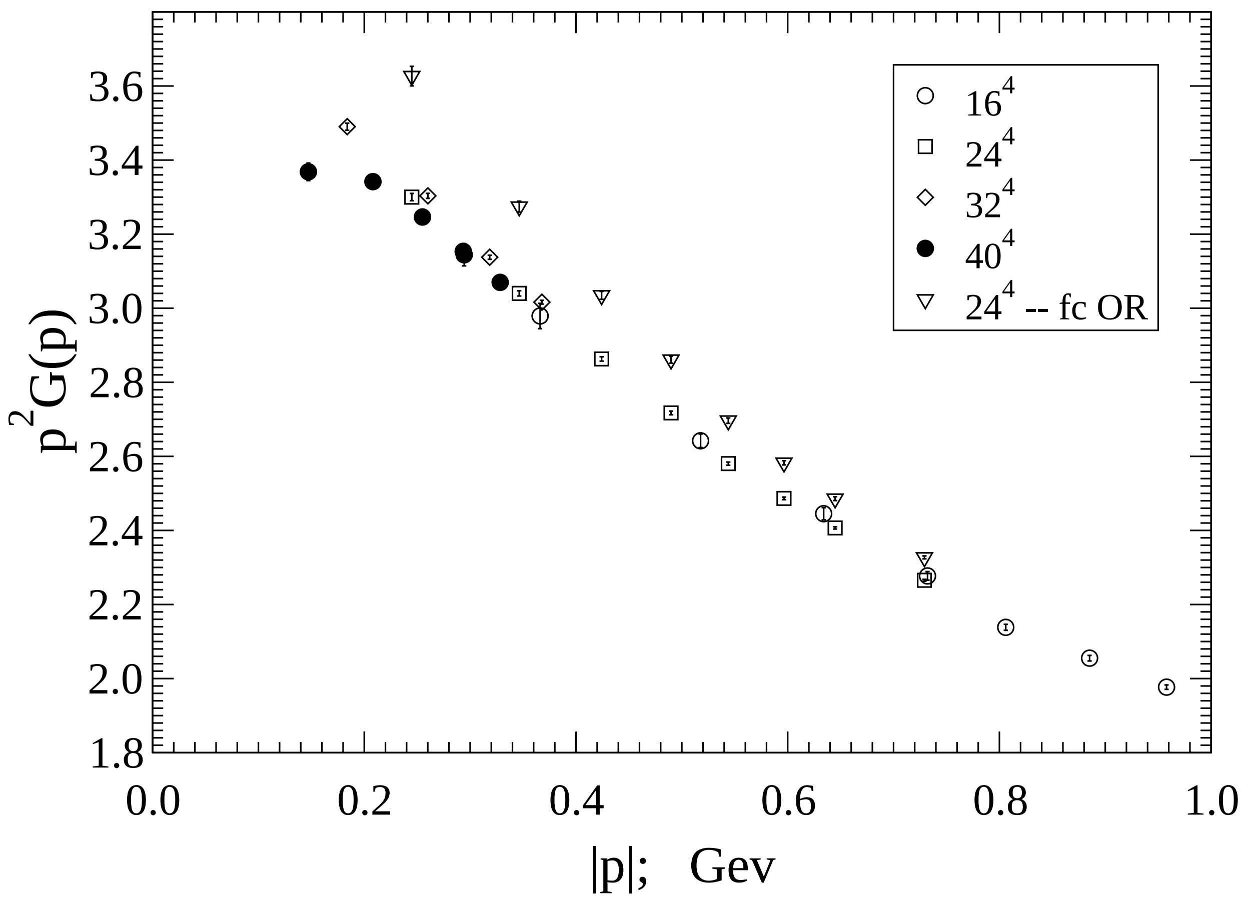



Figure 6: [SU(2)] The momentum dependence of the ghost dressing function $p^2 \cdot G(p)$ on the various lattices. For comparison results obtained with OR algorithm on $24^2$ lattices are also shown.

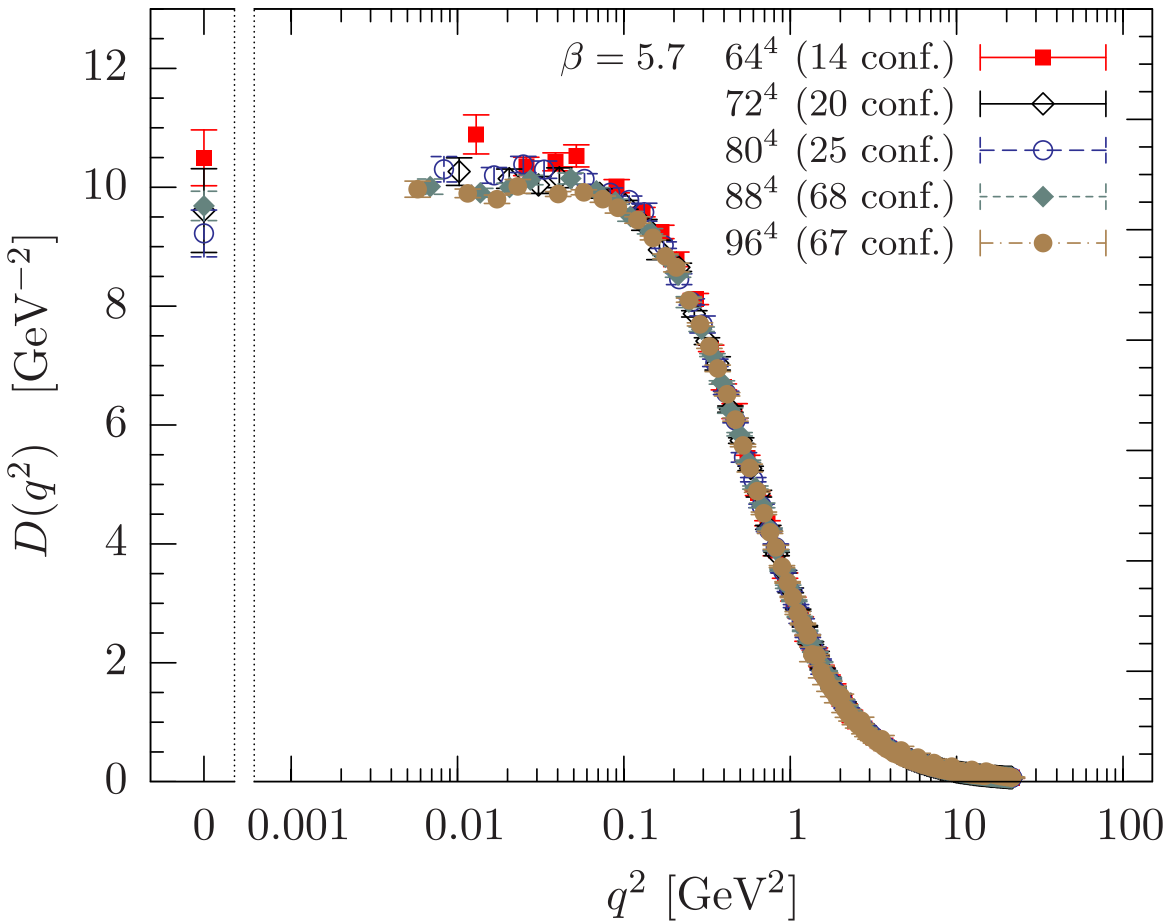



Figure 7: [SU(3)] The bare lattice gluon propagator $D(q^2)$ versus $q^2$ for $\beta = 5.70$ and various lattice sizes. We also show data on $D(0)$ (left).

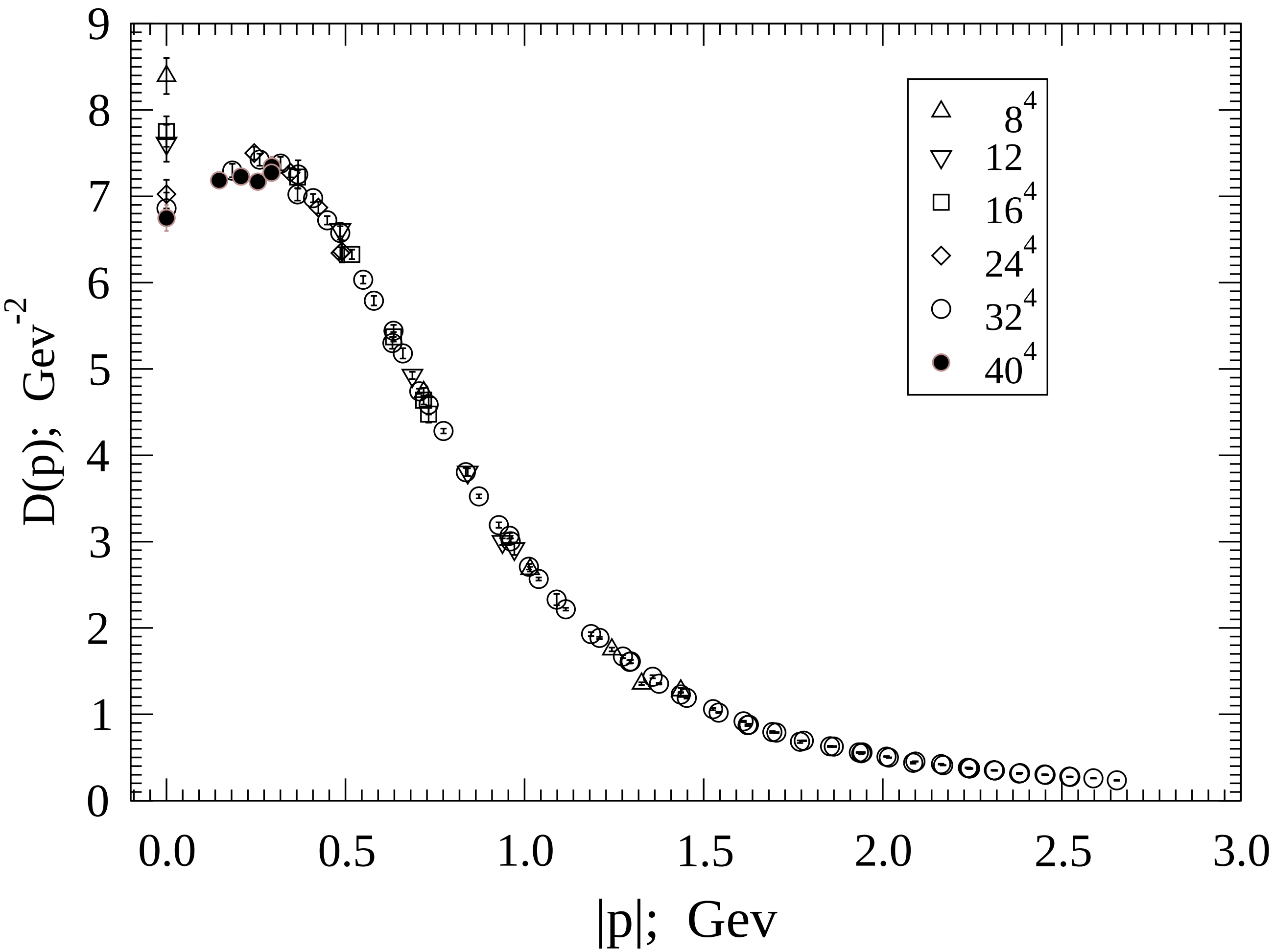



Figure 8: [SU(2)] The momentum dependence of the gluon propagator $D(p)$ on various lattice size. bc results are shown throughout.

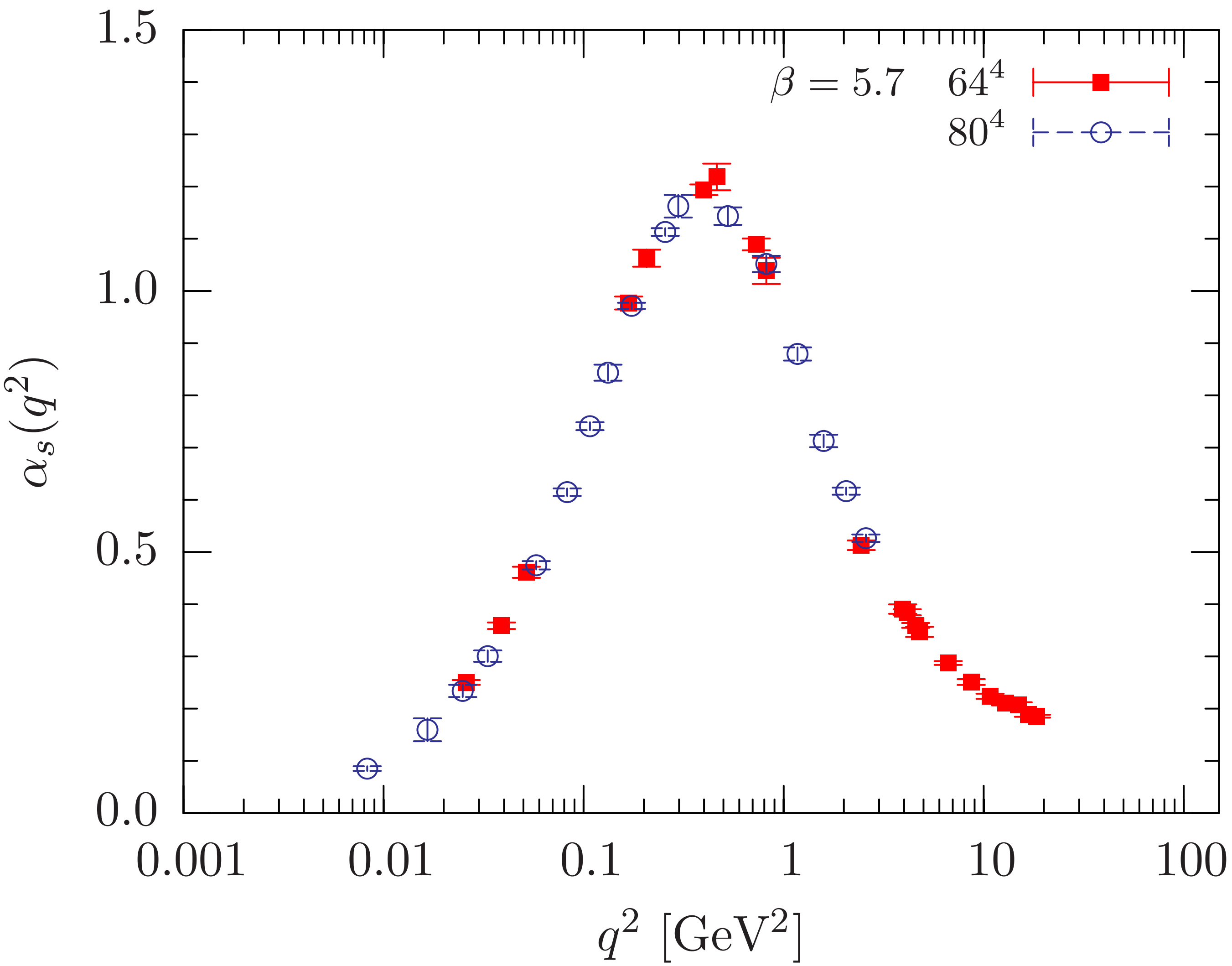



Figure 9: [SU(3)] Running coupling $\alpha_s(q^2)$ versus $q^2$ for lattice sizes $64^4$ and $80^4$ at $\beta = 5.70$.

## § Remark: A diagram connected by a single ghost line

Comparison with the paper: D. Zwanziger, Some exact infrared properties of gluon and ghost propagators and long-range force in QCD, arXiv:0904.2380[hep-th].

The essential difference between the Zwanziger result and ours comes from the 2nd term $\Delta_{sing}(k) := \Delta_{\mu\mu}^{AA}(k)$.

In fact, if $\Delta_{sing}(k) := \Delta_{\mu\mu}^{AA}(k)$ vanishes in the limit $k = 0$ as claimed in the paper, the average of the horizon function became instead equal to

$$\langle h(0) \rangle = -\lambda_{\mu\mu}^{AA}(0) = -(N^2 - 1)Du(0), \tag{1}$$

and the horizon condition $\langle h(0) \rangle = (N^2 - 1)D$ led to the Kugo-Ojima criterion and the divergent ghost dressing function:

$$u(0) = -1, \quad G(0) = [1 + u(0) + w(0)]^{-1} = w(0)^{-1} = \infty. \tag{2}$$

However, $\Delta_{sing}(k)$ does not vanish and remains non-zero even after taking the $k = 0$ limit, as we have examined.

# § Renormalization

We have calculated the horizon condition using

$$\langle h(x) \rangle = -\lim_{k\to 0} \langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A \rangle_k \tag{1}$$

However, the composite operator $(g\mathscr{A}_\mu \times \mathscr{C})$ is not multiplicative renormalizable due to operator mixing:

$$g\mathscr{A}_\mu \times \mathscr{C} = Z_C^{-1/2} (g\mathscr{A}_\mu \times \mathscr{C})_R + Z_C^{-1/2} (1 - Z_C) \partial_\mu \mathscr{C}_R. \tag{2}$$

On the other hand, the composite operator$D_\mu[\mathscr{A}]\mathscr{C}$ is multiplicative renormalizable

$$D_\mu[\mathscr{A}]\mathscr{C} = Z_C^{-1/2} (D_\mu[\mathscr{A}]\mathscr{C})_R, \tag{3}$$

In this sense, the horzon condition is good:

$$\langle h(x) \rangle = -\lim_{k\to 0} \langle (D_\mu[\mathscr{A}]\mathscr{C})^A (D_\mu[\mathscr{A}]\bar{\mathscr{C}})^A \rangle_k, \tag{4}$$

The SD equation for the ghost propagator is form-invariant under the renormalization:

$$\langle \mathscr{C}_R^A \bar{\mathscr{C}}_R^B \rangle_k = -\delta^{AB} \frac{1}{k^2} - i \frac{k^\mu}{k^2} \langle (g\mathscr{A}_\mu \times \mathscr{C})_R^A \bar{\mathscr{C}}_R^B \rangle_k. \tag{5}$$

An unrenormalized horizon condition $0 = (N^2-1)D - \langle h(0) \rangle$ is

$$\begin{aligned} 0 =& (N^2-1)D - \langle h(0) \rangle = (N^2-1)D + \lim_{k \to 0} \langle (D_\mu \mathscr{C})^A (D_\mu \bar{\mathscr{C}})^A \rangle_k \\ =& (N^2-1)(D-1)[1+u(0)], \end{aligned} \tag{6}$$

The multiplicatively renormalized horizon condition is

$$\begin{aligned} 0 =& Z_C[(N^2-1)D - \langle h(0) \rangle] = (N^2-1)(D-1) Z_C [1+u(0)] \\ =& (N^2-1)(D-1)[1+u_R(0)], \end{aligned} \tag{7}$$

if we adopt the renormalization

$$1 + u(0) = Z_C^{-1}[1+u_R(0)]. \tag{8}$$

The horizon condition is satisfield only when $u_R(0) = -1$ and $G_R^{-1}(0) = w_R(0)$ also after normalization. We have a definite result.

We return to the first horizon condition:

$$\langle h(x)\rangle = -\lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k.$$

The method (1): We must use the localized GZ theory which is multiplicative renormalizable. The Slavnov-Taylor identity means

$$\langle i\gamma^{-1/2} g^2 f^{ABC} \mathscr{A}_\mu^B(x) \bar{\xi}_\mu^{CA}(x)\rangle = \langle h(x)\rangle = (N^2-1)D. \tag{9}$$

Then the horizon condition is multiplicatively renormalized:

$$\begin{aligned} &\langle i\gamma_R^{-1/2} g_R^2 f^{ABC} \mathscr{A}_{\mu,R}^B(x) \bar{\xi}_{\mu,R}^{CA}(x)\rangle = Z_C (N^2-1)D \\ \Longleftrightarrow\ & Z_C g_R^{-2} = i\langle \gamma_R^{-1/2} f^{ABC} \mathscr{A}_{\mu,R}^B(x) \bar{\xi}_{\mu,R}^{CA}(x)\rangle / [(N^2-1)D]. \end{aligned} \tag{10}$$

An overall renormalization constant $Z_C$ is enough to renormalize this horizon condition.

If the horizon condition is incorporated into the SD equation, a partial cancellation at $k=0$ occurs between the horizon condition and the ghost self-energy. This cancellation occurs for the always multiplicative renormalizable part coming from $\lambda_{\mu\mu}(0)$.

$$\begin{aligned} 0 =&(N^2-1)D - \langle h(0)\rangle \\ =&(N^2-1)D + \lim_{k\to 0}\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k \\ =&(N^2-1)\left\{D[1+u(0)] + w(0) - G(0)[u(0)+w(0)]^2\right\}, \end{aligned} \tag{11}$$

This horizon condition is not multiplicatively renormalizable!

$$\begin{aligned} 0 =&Z_C[(N^2-1)D - \langle h(0)\rangle] \\ =&(N^2-1)DZ_C + \lim_{k\to 0} Z_C\langle (g\mathscr{A}_\mu \times \mathscr{C})^A (g\mathscr{A}_\mu \times \bar{\mathscr{C}})^A\rangle_k \\ =&(N^2-1)\left\{DZ_C[1+u(0)] + Z_C w(0) - Z_C G(0)[u(0)+w(0)]^2\right\} \\ \neq&(N^2-1)\left\{D[1+u_R(0)] + w_R(0) - G_R(0)[u_R(0)+w_R(0)]^2\right\}?, \end{aligned} \tag{12}$$

This is reasonable, since we did not use the localized renormalizable GZ theory. Without introducing the Zwanziger ghost field $\xi, \bar{\xi}, \omega, \bar{\omega}$, the usual framework of the multiplicative renormalization can not be applied to this horizon condition.

The method (2): We recall the SD equation with the horizon condition:

$$G^{-1}(k^2) = -u(0) - \frac{w(0)}{D} + \frac{G^{-1}(0) - 2 + G(0)}{D} + u(k^2) + w(k^2). \qquad (13)$$

This is unrenormalized version. The UV cutoff $\Lambda$ must be introduced to make the self-energy part $u(k^2)$ finite. [$w(k^2)$ is finite from the begining by some reason. See Binosi et al.] So, $G$ depends on $\Lambda$. $G(k^2, \Lambda)$.

A novel situation occurs by introducing the horizon condition. By the resulting subtraction $u(k^2) - u(0)$, the ultraviolet divergence cancels. Then this is regarded as a self-consistent equation to give a finite ghost function $G(k^2) = \lim_{\Lambda\to\infty} G(k^2, \Lambda) < \infty$.

In other words, the SD equation is self-organized (in a non-perturbative way) to give a finite result for any $k$. $\Longrightarrow$ no need for UV renormalization!

We have the contribution from the remainig term $\Delta_{\mu\mu}(0)$ which is non-zero. Therefore, we have the decoupling solution, $G^{-1}(0) \neq 0$.

This situation does not occur for another horizon condition.

$$G^{-1}(k^2) = \frac{1}{D}[1 + u(0)] - u(0) + u(k^2) + w(k^2). \qquad (14)$$

We need the UV renormalization.

The renormalization point dependence: From

$$G_R(k^2, \mu^2) = Z_C^{-1}(\mu^2, \Lambda^2) G(k^2, \Lambda^2), \tag{15}$$

we have

$$\frac{G_R(k^2, \mu^2)}{G_R(\mu^2, \mu^2)} = \frac{G(k^2, \Lambda^2)}{G(\mu^2, \Lambda^2)} \tag{16}$$

In particular, at $k^2 = 0$

$$G_R(0, \mu^2) = G_R(\mu^2, \mu^2) \frac{G(0, \Lambda^2)}{G(\mu^2, \Lambda^2)} = G_R(\mu^2, \mu^2) \frac{1 + \sqrt{D}}{G(\mu^2, \Lambda^2)} \tag{17}$$

For the decoupling solution, if we take the renormalization condition, e.g.,

$$G_R(\mu^2, \mu^2) = 1 \Longrightarrow G_R(0, \mu^2) = \frac{G(0)}{G(\mu^2)} = \frac{1 + \sqrt{D}}{G(\mu^2)} \tag{18}$$

For instance, if one chooses $\mu = 1.5$GeV and $D = 4$, then $G(\mu^2) = 1.2$ for a given b.c. $G(0) = 3$. So, one gets $G_R(0, \mu^2) = 3/1.2 = 2.5$.

[C.S. Fischer, A. Maas and J.M. Pawlowski, arXiv:0810.1987 [hep-ph], Ann. of Phys]
It is only a matter of infrared boundary conditions $G(0)$ whether scaling or decoupling occurs.

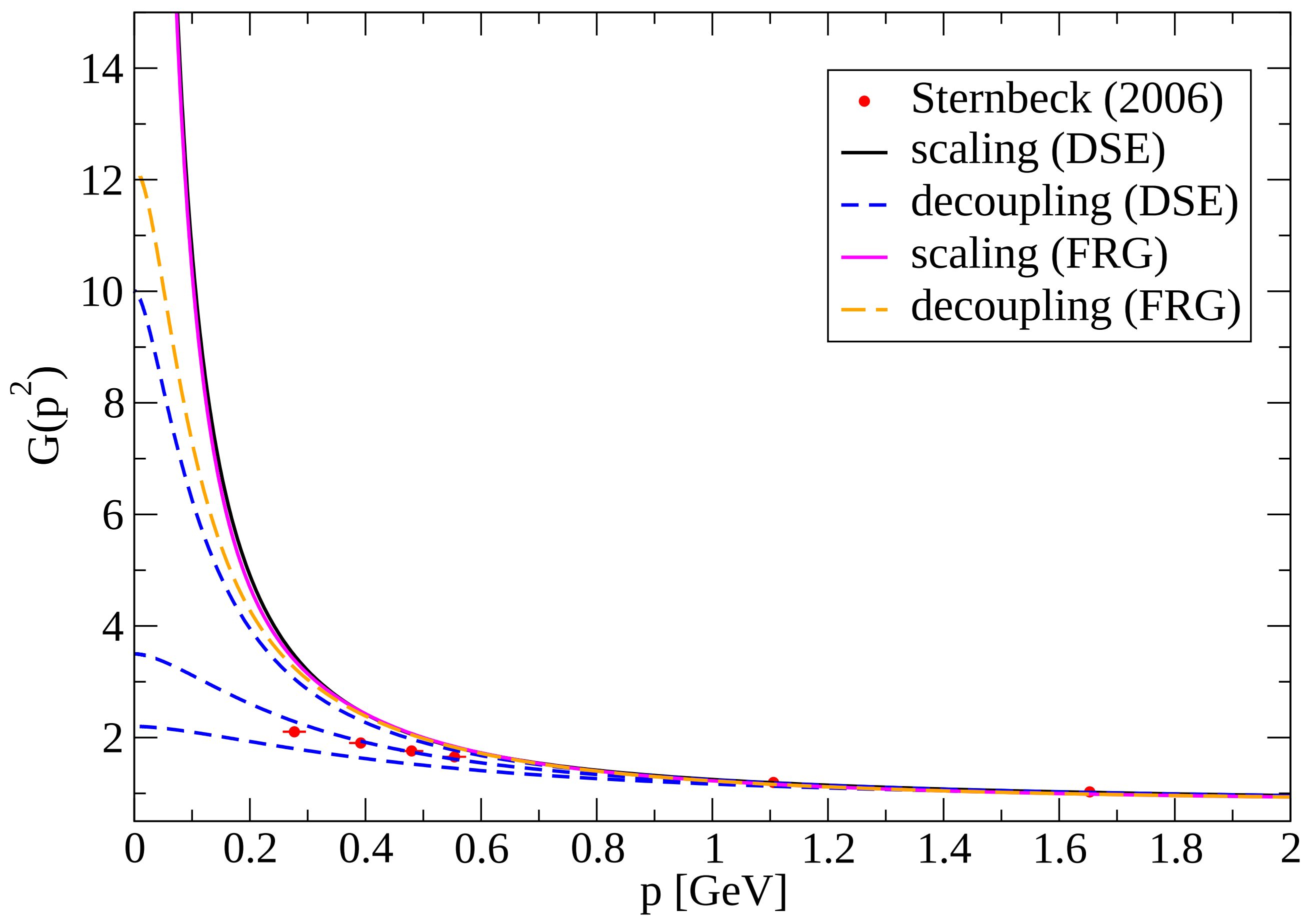



Figure 10: Ghost dressing functions obtained from the Schwinger-Dyson equation and functional renormalization group compared to lattice results in minimal Landau gauge from [Sternbeck, 2006][Bowman et al, 2004].

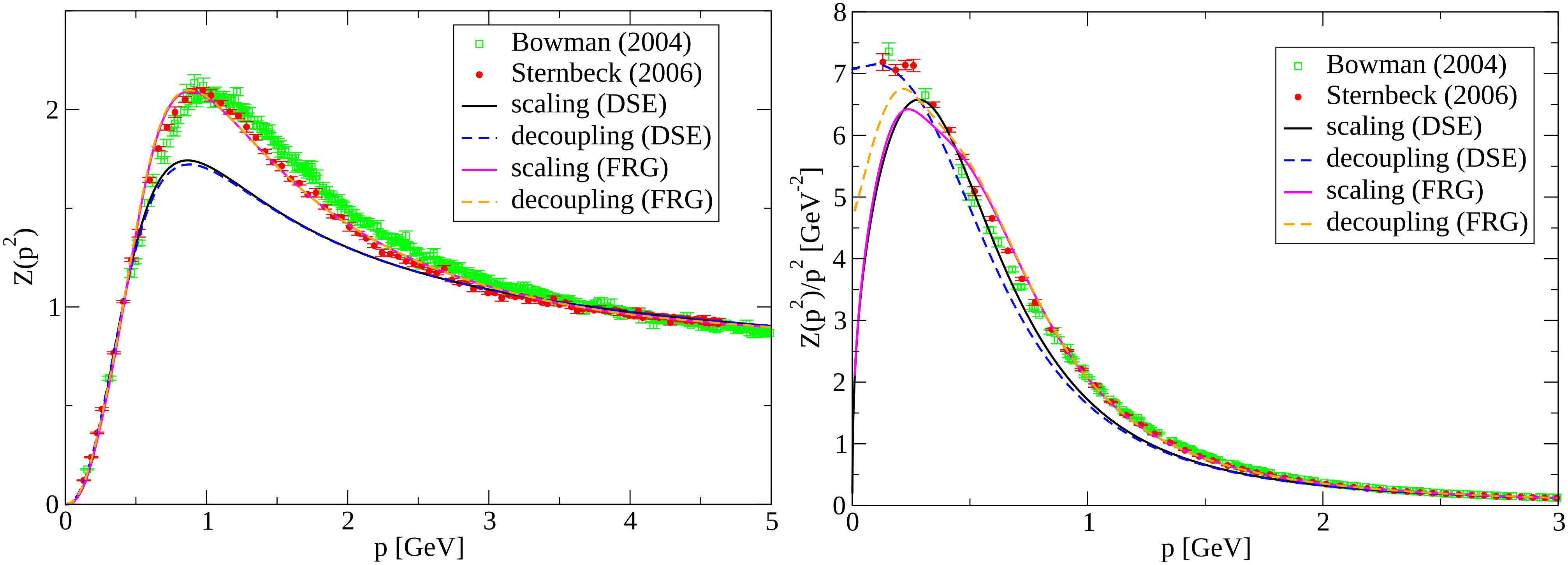



Figure 11: Gluon dressing functions and gluon propagators obtained from the Schwinger-Dyson equation and functional renormalization group compared to lattice results in minimal Landau gauge from [sternbeck06][Bowman:2004jm]. [C.S. Fischer, A. Maas and J.M. Pawlowski, arXiv:0810.1987 [hep-ph]]

# § Conclusion and discussion

We have discussed how the existence of the Gribov horizon modifies the deep infrared behavior of the Landau gauge SU(N) Yang-Mills theory, in the Gribov-Zwanziger framework with the horizon condition $\langle h(x) \rangle = (\mathrm{dim}G)D$.
The result crucially depends on the choice of the non-local horizon term adopted.
⊙ **decoupling solution** For the original horizon term, [D.Zwanziger,(1989)]

$$h(x) = \int d^D y g f^{ABC} \mathscr{A}_\mu^B(x) (K^{-1})^{CE}(x,y) g f^{AFE} \mathscr{A}_\mu^F(y), \tag{1}$$

the decoupling solution, i.e., finite ghost dressing function $G(k^2)$ even in the limit $k \to 0$: $G(0) < \infty$ is obtained. The Kugo-Ojima criterion $u(0) = -1$ is not satisfied. $G(0) = 1 + \sqrt{D}$, $u(0) = (-D + \sqrt{D})/(D-1)$ up to renormalization point dependence.
⊙ **scaling solution** For another horizon term (which differs by the total derivatives)

$$h(x) = \int d^D y D_\mu[\mathscr{A}]^{AC}(x) (K^{-1})^{CE}(x,y) D_\mu[\mathscr{A}]^{AE}(y), \tag{2}$$

the Kugo-Ojima criterion $u(0) = -1$ is satisfied. The scaling solution, i.e., infinite ghost dressing function in the limit $k \to 0$: $G(0) = w(0)^{-1} = \infty$ with an input $w(0) = 0$, even after renormalization, as already known. [D. Dudal et al, arXiv:0904.0641[hep-th]]

⊙ **Schwinger-Dyson equation** We have shown how to incorporate the horizon condition into the Schwinger-Dyson equation for the ghost propagator to discriminate between scaling and decoupling.

⊙ **Renormalization** A possible renormalization scheme and the renormalization point dependence of the decoupling solution has been discussed. This should be compared with [A.C. Aguilar, D. Binosi and J. Papavassiliou, arXiv:0907.0153 [hep-ph]. ]

⊙ **BRST symmetry and color confinement** How the existence of the horizon is relevant for color confinement. In the Gribov-Zwanziger theory (restricted to the 1st Gribov region), the BRST symmetry is broken by the existence of the horizon. $\boldsymbol{\delta} S_{\mathrm{GZ}} = \boldsymbol{\delta} \tilde{S}_{\gamma} \neq 0$

Nevertheless, there exists a "BRST" like symmetry (without nilpotency [Sorella,0905.1010[hep-th]] or with nilpotency [K.-I. K., 0905.1899[hep-th]]) which leaves the Gribov-Zwanziger action invariant. Then we could apply the Kugo-Ojima idea to the Gribov-Zwanziger theory, which opens the path to searching for the modified color confinement criterion $a\ la$ Kugo and Ojima.

⊙ **Quark confinement** [J. Braun, H. Gies and J.M. Pawlowski, Quark Confinement from Color Confinement. e-Print: arXiv:0708.2413 [hep-th]] It is shown that all solutions (decoupling as well as scaling) lead to quark confinement by proving the vanishing of the order parameter of quark confinement, the Polyakov loop.

⊙ **Coulomb gauge** In the Coulomb gauge, there exists the same problem as in the Landau gauge. Be careful!

**Thank you for your attention!**

# § A modified BRST transformation

Our main motivation is to find out a modified BRST transformation $\boldsymbol{\delta}'$ such that $\boldsymbol{\delta}'$ leaves the action $S_{\rm YM}^{\rm tot}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}]$ invariant, i.e.,

$$\boldsymbol{\delta}'(S_{\rm YM}^{\rm tot}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}])=0, \tag{1}$$

and $\boldsymbol{\delta}'$ obeys the nilpotency, i.e.,

$$\boldsymbol{\delta}'\boldsymbol{\delta}'=0. \tag{2}$$

Such a transformation could be non-local.

Suppose that $\tilde{S}_\gamma$ is written in the BRST-exact form

$$\begin{aligned} & S_{\rm GF+FP}[\mathscr{A},\mathscr{C},\bar{\mathscr{C}},\mathscr{B}]+\tilde{S}_\gamma[\mathscr{A},\xi,\bar{\xi},\omega,\bar{\omega}] \\ =& \int d^Dx\{\mathscr{B}^A\partial_\mu\mathscr{A}_\mu^A-i\bar{\mathscr{C}}^AK^{AB}\mathscr{C}^B+\bar{\xi}_\mu^{CA}K^{AB}\xi_\mu^{CB}-\bar{\omega}_\mu^{CA}K^{AB}\omega_\mu^{CB} \\ & +i\gamma^{1/2}gf^{ABC}\mathscr{A}_\mu^B\xi_\mu^{AC}+i\gamma^{1/2}gf^{ABC}\mathscr{A}_\mu^B\bar{\xi}_\mu^{AC}\} \qquad (3) \\ =& \int d^Dx\{-i\boldsymbol{\delta}'\left[\bar{\mathscr{C}}^A(\partial_\mu\mathscr{A}_\mu^A)\right]+\boldsymbol{\delta}'[\bar{\omega}_\mu^{CA}(-\partial_\rho D_\rho^{AB}[\mathscr{A}])\xi_\mu^{CB}]\}, \qquad (4) \end{aligned}$$

The BRST invariance of $S_{\rm GF+FP}+\tilde{S}_\gamma$, i.e., $\boldsymbol{\delta}'(S_{\rm GF+FP}+\tilde{S}_\gamma)=0$, is guaranteed by the nilpotency ($\boldsymbol{\delta}'\boldsymbol{\delta}'=0$) of the modified BRST transformation.

A modified BRST transformation

$$
\begin{aligned}
\boldsymbol{\delta}' \mathscr{A}_\mu^A(x) =& (D_\mu[\mathscr{A}]\mathscr{C}(x))^A, \\
\boldsymbol{\delta}' \mathscr{C}^A(x) =& -\frac{g}{2}(\mathscr{C}(x) \times \mathscr{C}(x))^A, \\
\boldsymbol{\delta}' \bar{\mathscr{C}}^A(x) =& i\mathscr{B}^A(x) + F^A(x), \\
\boldsymbol{\delta}' \mathscr{B}^A(x) =& i\boldsymbol{\delta}' F^A(x), \\
\boldsymbol{\delta}' \xi_\mu^{AB}(x) =& \omega_\mu^{AB}(x) + G_\mu^{AB}(x), \\
\boldsymbol{\delta}' \omega_\mu^{AB}(x) =& -\boldsymbol{\delta}' G_\mu^{AB}(x), \\
\boldsymbol{\delta}' \bar{\omega}_\mu^{AB}(x) =& \bar{\xi}_\mu^{AB}(x) + H_\mu^{AB}(x), \\
\boldsymbol{\delta}' \bar{\xi}_\mu^{AB}(x) =& -\boldsymbol{\delta}' H_\mu^{AB}(x),
\end{aligned} \tag{5}
$$

where

$$
F^A(x) = \gamma^{1/2} \int d^D y \Delta^{-1}(x,y) g f^{ABC} \partial_\mu \bar{\xi}_\mu^{BC}(y), \tag{6}
$$

$$
G_\mu^{AB}(x) = \int d^D y (K^{-1})^{AC}(x,y) \partial_\rho [g f^{CFE} (D_\rho \mathscr{C})^F(y) \xi_\mu^{EB}(y)], \tag{7}
$$

$$
H_\mu^{AB}(x) = \int d^D y i \gamma^{1/2} g f^{BCE} \mathscr{A}_\mu^C(y) (K^{-1})^{EA}(y,x). \tag{8}
$$

Note that $G$ does not vanish even in the limit $\gamma \to 0$ and the modified BRST transformation $\boldsymbol{\delta}'\xi_\mu^{AB}(x)$ has the part $G$ involving the Yang-Mills field and the ghost field. Even in the limit, therefore, the horizon term is not decoupled from the usual Yang-Mills-Faddeev-Popov theory. This issue is cured by redefining the auxiliary field $\omega_\mu^{AB}(x)$, i.e., shifting it by $G_\mu^{AB}(x)$

$$\omega'^{AB}_\mu(x) := \omega_\mu^{AB}(x) + G_\mu^{AB}(x). \tag{9}$$

Then the modified BRST transformation is simplified

$$\boldsymbol{\delta}'\mathscr{A}_\mu^A(x) =(D_\mu[\mathscr{A}]\mathscr{C}(x))^A, \tag{10a}$$

$$\boldsymbol{\delta}'\mathscr{C}^A(x) =-\frac{g}{2}(\mathscr{C}(x)\times\mathscr{C}(x))^A, \tag{10b}$$

$$\boldsymbol{\delta}'\bar{\mathscr{C}}^A(x) =i\mathscr{B}^A(x)+F^A(x), \tag{10c}$$

$$\boldsymbol{\delta}'\mathscr{B}^A(x) =i\boldsymbol{\delta}'F^A(x), \tag{10d}$$

$$\boldsymbol{\delta}'\xi_\mu^{AB}(x) =\omega'^{AB}_\mu(x), \tag{10e}$$

$$\boldsymbol{\delta}'\omega'^{AB}_\mu(x) =0, \tag{10f}$$

$$\boldsymbol{\delta}'\bar{\omega}_\mu^{AB}(x) =\bar{\xi}_\mu^{AB}(x)+H_\mu^{AB}(x), \tag{10g}$$

$$\boldsymbol{\delta}'\bar{\xi}_\mu^{AB}(x) =-\boldsymbol{\delta}'H_\mu^{AB}(x), \tag{10h}$$

This result should be compared with the result of Sorella,0905.1010[hep-th] , another modified BRST transformation without nilpotency.

Since we have found a modified BRST transformation which leaves the Gribov-Zwanziger action invariant, then we could apply the Kugo-Ojima idea to the Gribov-Zwanziger theory, which opens the path to searching for the modified color confinement criterion $a\ la$ Kugo and Ojima. ...

# § On the gluon dressing function

The purpose is to search the solution which is consistent with the general principles of quantized gauge field theory:

- Non-perturbative multiplicative renormalizability
- Analyticity
- Spectral condition
- Poincaré group structure

Then the gluon dressing function vanishes in the IR limit $p^2 \to 0$.

The gluon propagator can be finite and non-zero.

[K.-I. K., hep-th/0303251]

Short version [K.-I. K., hep-lat/0309142]

First, we consider the case of $w(0) = 0$ [supported by numerical calculations].

- a new relationship $\langle h(0) \rangle = (N^2 - 1) \left\{ -Du(0) + \frac{u(0)^2}{1 + u(0)} \right\},$ (1)
- the horizon condition $\langle h(0) \rangle = (N^2 - 1)D,$ (2)

$$\to (D-1)u(0)^2 + 2Du(0) + D = 0, \quad u(0) = (-D \pm \sqrt{D})/(D-1) \tag{3}$$

$$\to G(0)^2 - 2G(0) + 1 - D = 0, \quad G(0) = 1 \pm \sqrt{D} \tag{4}$$

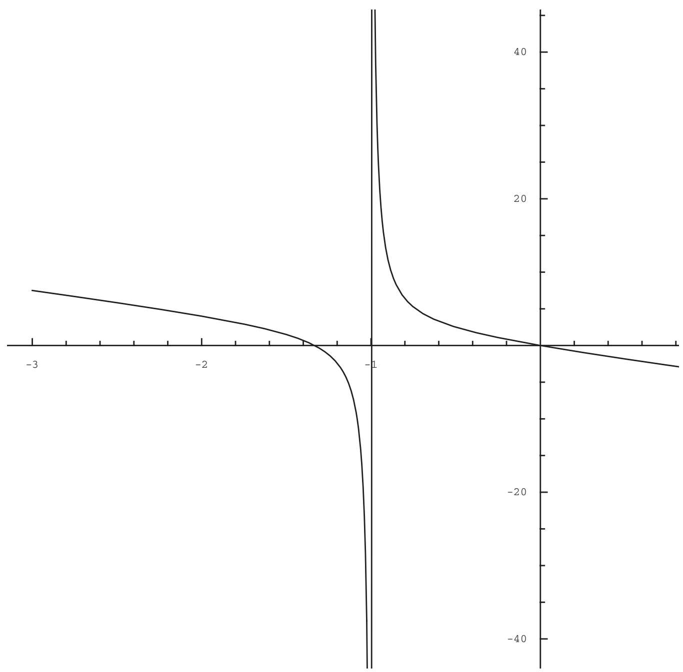



Figure 12: The plot of $\langle h(0) \rangle$ versus $u(0)$ for $D = 4$.

For $D = 4$, $3u(0)^2 + 8u(0) + 4 = 0$ has solutions $u(0) = -2/3, -2$. We obtain irrespective of the number of color N

$$u(0) = -\frac{2}{3}, \quad G(0) = [1 + u(0)]^{-1} = 3. \tag{5}$$

The ghost dressing function $G(k^2)$ is finite even in the deep infrared limit $k^2 \to 0$.

This differs from the Gribov result. Is it possible to reconcile this result with the old Gribov result? Yes. See the following.

Recall that the Gribov result was obtained by taking into account the $O(g^2)$ terms.

The formal power series expansion in $u(0)$ yields the horizon condition

$$\langle h(0) \rangle = (N^2 - 1) \left\{ -Du(0) + u(0)^2 - u(0)^3 + \cdots \right\} = (N^2 - 1)D. \tag{6}$$

If we took into account only a linear term in $u(0) = O(g^2)$ on the left-hand side, then the horizon condition would lead to the Kugo-Ojima criterion $u(0) = -1$ and the ghost dressing function $G(0) = [1 + u(0)]^{-1}$ would diverge.

In this way we can reproduce the Gribov approximate (wrong) result.

• Second, we consider the (unlikely) case of $w(0) \neq 0$. [See numerical results]

The horizon condition alone is not sufficient to determine both $u(0)$ and $w(0)$. Suppose the Kugo-Ojima confinement criterion is satisfied $u(0) = -1$. Then the horizon condition is

$$\langle h(0) \rangle = -(N^2 - 1) \left\{ -D + 1 + \frac{-1 + w(0)}{w(0)} \right\} \cong (N^2 - 1) D. \tag{7}$$

This leads to the value of $w(0)$ irrespective of the spacetime dimension $D$ and the number of color $N$:

$$w(0) = 1/2 \quad \text{for any} \quad D. \tag{8}$$

Even if $u(0) = -1$, therefore, the ghost propagator behaves like free $1/k^2$ at $k = 0$, no more singular than $1/k^2$: irrespective of the spacetime dimension $D$ and the number of color $N$

$$\lim_{k^2 \to 0} [-k^2 \langle \mathscr{C}^A \bar{\mathscr{C}}^B \rangle_k]^{-1} = \delta^{AB} w(0) = \frac{1}{2} \delta^{AB} \neq 0, \quad G(0) = 2. \tag{9}$$

Thus, the ghost propagator behaves like free at low momenta, while the gluon propagator is non-vanishing at low momenta The original Gribov prediction is wrong?