Effective Action for Low-Energy Quantum Field Fluctuations

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QCD Green's functions, confinement, and phenomenology
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$$Z_{YM} = \int \mathcal{D}A_\mu \, exp\left\{-\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}\right\}$$

Ab initio framework: Lattice Qcd

Powerful tool for obtaining quantitative information on Non-Perturbative QCD: hadronic spectrum, matrix elements, etc..

Q.: Are there field configurations (instantons, monopoles, centre-vortices, etc..) which dominate a given matrix element?

We present a method for building an "Effective Theory" for a chosen set of vacuum field configurations $\tilde{A}_\mu[\{\gamma_i\}]$, starting from configurations obtained via Lattice simulation

$$Z_{\text{YM}} = \int d\gamma_1 \ \ldots \int d\gamma_k \ \exp\left\{ - F(\gamma_1, \ldots, \gamma_k) \right\}.$$

Ex: Instanton-AntiInstanton interaction, etc..

Suppose we want to evaluate the expectation value of $\mathcal{O}[\Psi(x)]$

$$\langle \mathcal{O}[A_\mu] \rangle = Z^{-1} \int \mathcal{D}A_\mu(x) \mathcal{O}[A_\mu(x)] e^{-S_{YM}(x)}$$

Consider a family of configurations $\tilde{A}_\mu(x, \{\gamma_i\})$ s.t.

$$\langle \mathcal{O}[A_\mu] \rangle \simeq \langle \mathcal{O}[\tilde{A}_\mu(\{\gamma_i\}) + \cancel{B}_\mu] \rangle$$

$$A_\mu(x) = \tilde{A}_\mu(x, \{\gamma_i\}) + B_\mu(x); \qquad (B \cdot g^i_\gamma(\{\bar{\gamma}_j\})) = 0$$

$$\langle \mathcal{O}[A_\mu(x)] \rangle \simeq Z^{-1} \int \prod_i d\gamma_i \mathcal{O}[\tilde{A}_\mu(\gamma)] \int \mathcal{D}B \left(\prod_i \delta(B \cdot g^i_\gamma(\{\bar{\gamma}_j\})) \right) \times$$

Jacobian, Gauge Fixing,...

$$\times \det \left[D_\mu(\tilde{A}) D_\mu(\tilde{A} + B) \right] \times \ldots \times e^{-S[\tilde{A}+B] + \frac{1}{\alpha^2} \int \mathrm{d}^4x (D_\mu(\tilde{A}) B_\mu)^2}$$

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By evaluating $F(\gamma_1, \ldots, \gamma_k)$ for different $\tilde{A}_\mu(\{\gamma\})$ we obtain important information on Non-Perturbative dynamics.

From Lattice Simulation to "Effective Theory"

1. Identify a set of vacuum configurations $\tilde{A}_\mu(\{\gamma_i\})$ i.e. a sub-manifold $M_{\tilde{\Psi}}$ of the full functional space.
2. Decompose the partition function,

$$Z_{YM} = \int \mathcal{D}A_\mu e^{-S\left[A_\mu = \tilde{A}_\mu(\{\gamma_i\}) + B_\mu\right)} = \int \mathrm{d}\gamma_1 \ldots \mathrm{d}\gamma_m \, e^{-F(\{\gamma_j\})}$$

$$\boxed{A_\mu(x) = \tilde{A}_\mu(x, \{\gamma_i\}) + B_\mu(x); \qquad (B \cdot g^i_\gamma(\{\bar{\gamma_j}\})) = 0}$$

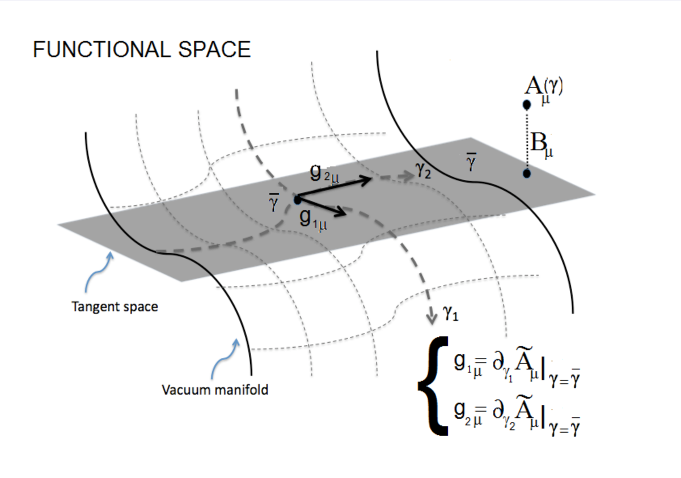
$$Z_{YM} = \int \left(\prod_{l=1}^{m} d\gamma_l\right) \int \mathcal{D}B \left(\prod_i \delta(B \cdot g^i_\gamma(\{\bar{\gamma}_j\}))\right) \times$$

Jacobian, Gauge Fixing,...

$$\times \det\left[D_\mu(\tilde{A}) D_\mu(\tilde{A} + B)\right] \times \ldots \times e^{-S[\tilde{A}+B] + \frac{1}{\alpha^2} \int \mathrm{d}^4x \left(D_\mu(\tilde{A}) B_\mu\right)^2}$$

From Lattice Simulation to "Effective Theory"

Step 1 and 2



From Lattice Simulation to "Effective Theory"

3) Generate a statistically representative set of independent configurations with Lattice simulations

$$\{A_{\mu,1}(x), A_{\mu,2}(x), A_{\mu,3}(x), \ldots, A_{\mu,N}(x)\}$$

4) For each configuration $A_\mu(x)$, project onto the tangent space of $M_{\tilde{A}_\mu}$ generated by the m "vectors"

$$g^i_{\gamma,\mu}(x, \{\bar{\gamma_j}\}); \text{ s.t. } \quad (B(x) \cdot g^i_\gamma(\{\bar{\gamma_j}\})) = 0$$

$$\Gamma_1[A_{sim}] = (A_{sim} \cdot g^1_\gamma) = \left(\tilde{A}(\gamma_1, \gamma_2, \ldots, \gamma_m) \cdot g^1_\gamma\right) = \Theta_1[\gamma_1, \gamma_2, \ldots, \gamma_m]$$

$$\vdots$$

$$\Gamma_m[A_{sim}] = (A_{sim} \cdot g^m_\gamma) = \left(\tilde{A}(\gamma_1, \gamma_2, \ldots, \gamma_m) \cdot g^m_\gamma\right) = \Theta_m[\gamma_1, \gamma_2, \ldots, \gamma_m]$$

$$(\Theta_1[A_1], \Theta_2[A_1], \ldots)\,;\; (\Theta_1[A_2], \Theta_2[A_2], \ldots)\,;\; \ldots\; (\Theta_1[A_N], \Theta_2[A_N], \ldots)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$(\gamma_1[A_1], \gamma_2[A_1], \ldots); \quad (\gamma_1[A_1], \gamma_2[A_2], \ldots); \ldots \quad (\gamma_1[A_N], \gamma_2[A_N], \ldots)$$

TEST: Double Well

This is the simplest system with non-trivial vacuum structure

$$V(x[t]) = m\alpha\left(x^2[t] - \beta^2\right)^2 ,$$

There are 4 solutions to the (Im. time) Equation of Motion

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x[t] - 4\alpha x[t]\left(x[t]^2 - \beta^2\right) = 0$$

$$\begin{array}{lll} 2 & \text{trivial vacua:} & x_0 = \pm\beta \\ 2 & \text{Instanton-like solutions:} & x_{I,\bar{I}} = \pm\tanh[\sqrt{2\alpha}\beta(t-\bar{t})] \end{array} \quad (1)$$

Instantons are good Degrees of Freedom, but in our method is not necessary for the family of configurations $\tilde{\Psi}[\gamma]$ to be solutions of EoM.

TEST: Double Well

We rewrite Z in terms of a sum of N instantons and N antiinstantons

$$Z[T;-\beta,-\beta] = \int \left(\prod_{i=1}^{N} \mathrm{d}t_i \mathrm{d}\bar{t}_i \right) \; e^{-\frac{1}{\hbar} \; F_N(t_1,\bar{t}_1,\ldots,t_N,\bar{t}_N)}$$

In the limit of very high barrier $\tanh[\sqrt{(2\alpha)}\beta t] \rightarrow (1-2\theta[t])$

$$Z[T;-\beta,-\beta] \simeq \left(\frac{2\langle\mathcal{N}\rangle}{T}\right)^{2N} \int \left(\prod_{i=1}^{N} \int \mathrm{d}t_i \mathrm{d}\bar{t}_i \right) \theta(\bar{t}_i - t_i)\theta(t_{i+1} - \bar{t}_i)$$

If the barrier is lowered we can consider only a single couple

$$Z[T;-\beta,-\beta] \simeq \int \mathrm{d}t_1 \mathrm{d}\bar{t}_1 \Big(g_{2,IA}(\bar{t}_1 - t_1) + g_{2,AI}(t_1 - \bar{t}_1) \Big)$$

We will calculate $g_{2,IA}(\bar{t}_1 - t_1)$.

Double Well: from Lattice simulation to F_2

We choose a set of configuration given by a superposition of Instantons:

$$x(t) = \tilde{x}^{IA}(t-\chi;\xi) + y(t); \quad (y \cdot g_{\chi/\xi}) = 0$$

with

$$\tilde{x} = -\beta\left\{1 - \tanh\left[\sqrt{2\alpha}\,\beta\left(t-\chi+\frac{\xi}{2}\right)\right] + \tanh\left[\sqrt{2\alpha}\,\beta\left(t-\chi-\frac{\xi}{2}\right)\right]\right\}$$

The manifold $M_{\tilde{x}}$ is identified by the "Instanton coordinates" $\{\chi = \frac{1}{2}(t_1+t_2), \xi = t_2 - t_1\}$ and its tangent space, whose vectors are

$$\begin{aligned} g_\chi(t;\bar{\chi},\bar{\xi}) &= \partial_\chi\, \tilde{x}\left(t;\chi - \frac{1}{2}\xi, \chi + \frac{1}{2}\xi\right)\Big|_{\chi=\bar{\chi},\xi=\bar{\xi}} \\ g_\xi(t;\bar{\chi},\bar{\xi}) &= \partial_\xi\, \tilde{x}\left(t;\chi - \frac{1}{2}\xi, \chi + \frac{1}{2}\xi\right)\Big|_{\chi=\bar{\chi},\xi=\bar{\xi}} \end{aligned}$$

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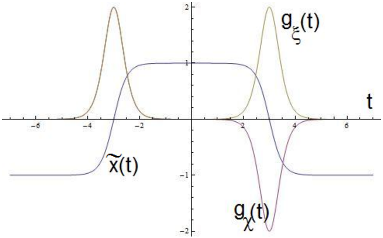


Double Well: from simulation to F_2

We choose a set of configuration given by a superposition of Instantons:

$$x(t) = \tilde{x}^{IA}(t-\chi;\xi) + y(t); \quad (y \cdot g_{\chi/\xi}) = 0$$

with

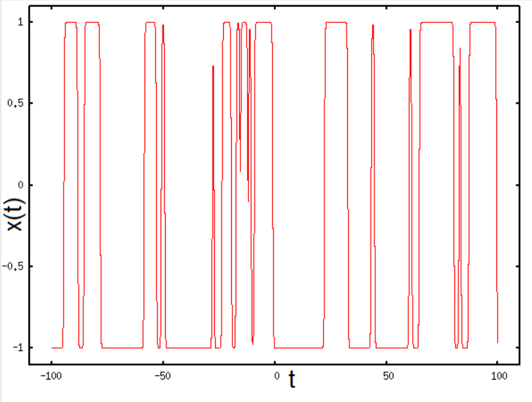
$$\tilde{x} = -\beta \left\{ 1 - \tanh\left[\sqrt{2\alpha}\,\beta \left(t - \chi + \frac{\xi}{2} \right) \right] + \tanh\left[\sqrt{2\alpha}\,\beta \left(t - \chi - \frac{\xi}{2} \right) \right] \right\}$$

$$\begin{aligned} \Gamma_\chi[x_{sim}] = (x_{sim} \cdot g_\chi) &= (\tilde{x}(t-\chi,\xi) \cdot g_\chi) = \Theta_\chi[\chi,\xi] \\ \Gamma_\xi[x_{sim}] = (x_{sim} \cdot g_\xi) &= (\tilde{x}(t-\chi,\xi) \cdot g_\xi) = \Theta_\xi[\chi,\xi] \end{aligned}$$

$$\begin{array}{ccc} (\Theta_\chi[x_1], \Theta_\xi[x_1]) ; & (\Theta_\chi[x_2], \Theta_\xi[x_2]) ; \ \ldots & (\Theta_\chi[x_N], \Theta_\xi[x_N]) \\ \downarrow & \downarrow & \downarrow \\ (\chi_1, \xi_1); & (\chi_2, \xi_2); \ldots & (\chi_N, \xi_N) \end{array}$$

TEST: Dilute Instanton Gas Approximation

$T = 200;\ m = 1,\ \beta = 1,\ \alpha = 7,\ \langle \mathcal{N} \rangle \simeq 31,\ \tau_I = 0.13$



Example of configuration generated with DIGA distribution.

TEST: Dilute Instanton Gas Approximation

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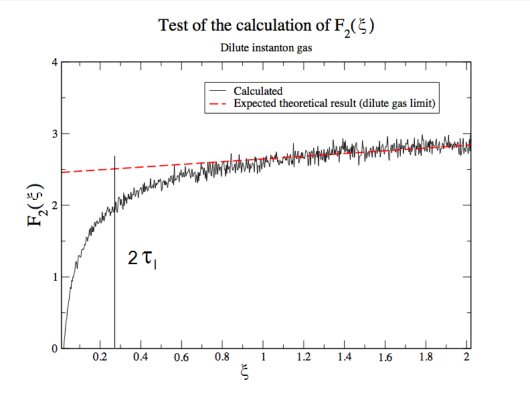
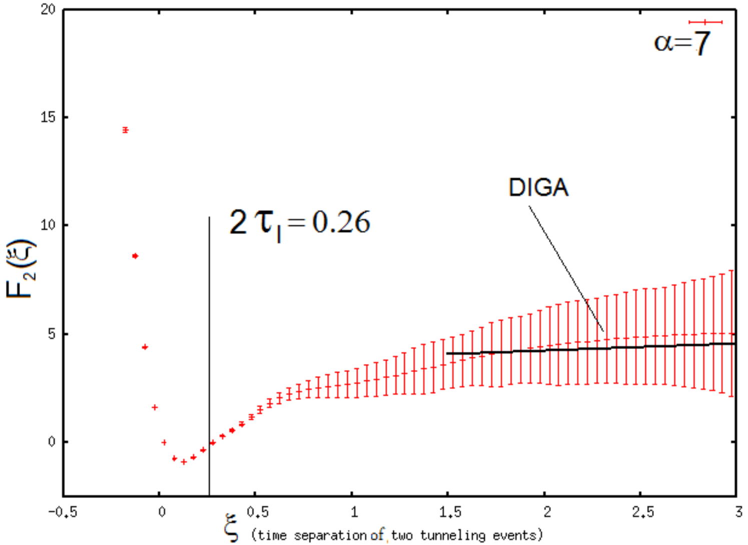
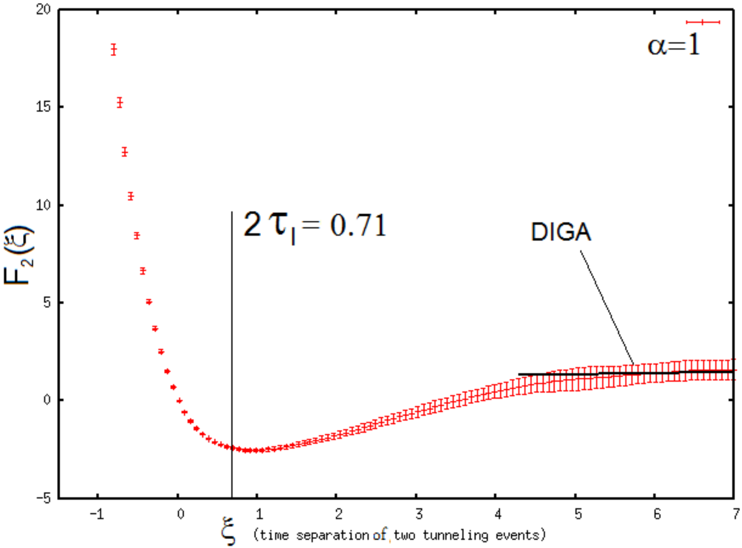


$$F_{2,DIGA} = -(2\langle \mathcal{N} \rangle - 1)\text{Log}[T - \xi]$$

"Effective Potential" for $\alpha = 7$: high well.



"Effective Potential" for $\alpha = 1$: low well.



Stochastic Quantization: Perturbative Analysis

Introduce a new coordinate: stochastic time.

$$\frac{d}{d\tau}\, x(t,\tau) = -k \frac{\delta\, S[x]}{\delta\, x(t,\tau)} + \sqrt{\hbar}\, \eta(t,\tau); \quad P[\eta] \propto exp\left\{-\frac{1}{4k}(\eta \cdot \eta)\right\}$$

Parisi and Wu (1981) have shown that

$$\mathcal{P}[x] \stackrel{\tau\to\infty}{\longrightarrow} \exp\left\{ - S[x(t)]\right\}$$

In the double well case we have

$$x(t) \to x[t,\tau] = \tilde{x}[t-\chi(\tau), \xi(\tau)] + y(t,\tau)$$

We expand the fields in $\varepsilon = \sqrt{\hbar}$

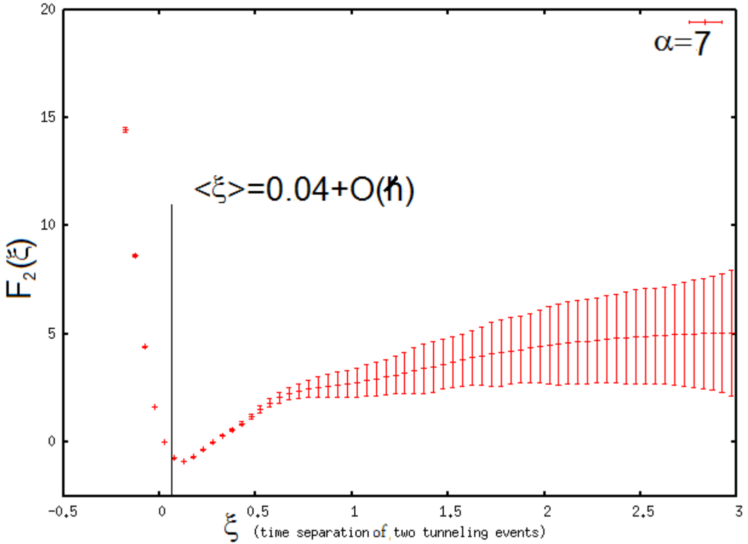
$$x(t,\tau) = \sum_{i=0}^{\infty} \varepsilon^i x_i(t,\tau)$$

$$\xi(\tau) = \sum_{i=0}^{\infty} \varepsilon^i \xi_i(\tau); \quad \chi(\tau) = \sum_{i=0}^{\infty} \varepsilon^i \chi_i(\tau); \quad y(t,\tau) = \sum_{i=1}^{\infty} \varepsilon^i y_i(t,\tau)$$

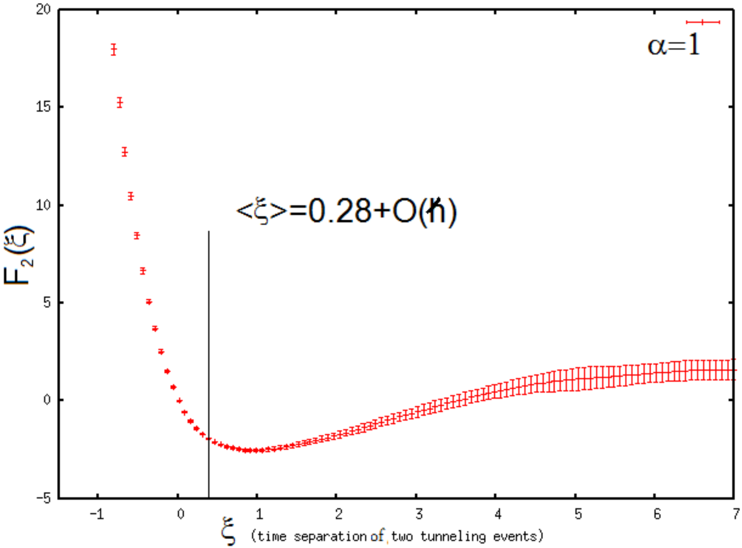
We can evaluate the mean distance up to $\hbar$.

$$\begin{aligned} \langle \chi \rangle &= 0 + \mathcal{O}(\hbar); \text{ Trivial...} \\ \langle \xi \rangle &= \frac{9}{32} \frac{\hbar}{m\alpha\beta^4} + \mathcal{O}(\hbar) > 0! \\ \langle \xi^2 \rangle &= \frac{9}{32} \frac{\hbar}{m\alpha\beta^4} \left(\frac{\pi^2 - 9}{3\sqrt{2\alpha}\beta} \right) + \mathcal{O}(\hbar) \end{aligned}$$

"Effective Potential" for $\alpha = 7$: high well.



"Effective Potential" for $\alpha = 1$: low well.



QCD: Instantons.

We now briefly present the status of the work.

WORK IN PROGRESS

$$A^a_\mu = \bar{A}^a_\mu(\gamma) + B^a_\mu; \quad (g^i_\gamma \cdot B) = 0$$

$$Z_{YM} = \prod_i \int \mathrm{d}\gamma_i \int \mathcal{D}B \left(\prod_i \delta(B \cdot g^i_\gamma(\{\bar{\gamma}_j\})) \right) \left| \det_{ij}[f_{ij}(g_\gamma, \bar{A}, B) \right| \times$$
$$\times \det\left[D_\mu(\bar{A}) D_\mu(\bar{A} + B)\right] e^{-S[\bar{A}+B] + \frac{1}{\alpha^2}\int \mathrm{d}^4x \left(D_\mu(\bar{A})B_\mu\right)^2}$$

- Instanton size distribution;
- Instanton-AntiInstanton interaction.

Single Instanton

Instanton in Singular Gauge

$$\bar{A}^a_{I,\mu}(x_\mu; x_{0,\nu}, \Delta^a, \rho) = 2\bar{\eta}^b_{\mu\nu} T^{ab}[\Delta] \frac{\rho^2 (x - x_0)_\nu}{(x - x_0)^2 \left[(x - x_0)^2 - \rho^2\right]}$$

Vectors of tangent space, associated to the manifold M_{ρ,Δ,x_0}.

$$\zeta^{\rho,a}_{I,\mu} = \frac{\partial}{\partial \rho} \bar{A}^a_{I,\mu}\Big|_{(\rho=\bar{\rho},\Delta=\bar{\Delta},x_0=\bar{x}_0)} \longrightarrow (\bar{A}_I \cdot \zeta^\rho) = \Theta_\rho[\rho, \Delta, x_0]$$

$$\zeta^{\Delta,a,c}_{I,\mu} = \frac{\partial}{\partial \Delta^c} \bar{A}^a_{I,\mu}\Big|_{(\rho=\bar{\rho},\Delta=\bar{\Delta},x_0=\bar{x}_0)} \longrightarrow (\bar{A}_I \cdot \zeta^{\Delta,c}) = \Theta^c_\Delta[\rho, \Delta, x_0]$$

$$\zeta^{x_0,a}_{I,\mu,\nu} = \frac{\partial}{\partial x_{0,\nu}} \bar{A}^a_{I,\mu}\Big|_{(\rho=\bar{\rho},\Delta=\bar{\Delta},x_0=\bar{x}_0)} \longrightarrow (\bar{A}_I \cdot \zeta^{x_0}_\nu) = \Theta_{\nu,x_0}[\rho, \Delta, x_0]$$

CONCLUSIONS

- We described a method which enables to to obtain an "Effective Theory" for a given family of vacuum configurations, using Lattice simulations;
- the method has been tested with the Double Well "toy model": the simplest system with a non trivial vacuum structure;
- we presented the current status of the work to implement the method for QCD: single instanton and double instanton.