QCD as basic field theory :
difficulties in building new bridges from perturbative regions to simple properties of hadrons

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Topics

1 The two central anomalies : scale and chiral U1

2 The perturbatively accessible region and the long way back to hadronic scales

0.1 **fm** $\leq D \leq (\sim) \, 2$ **fm**

3 Outlook

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List of contents

List of figures

1-1 QCD – the two central anomalies : scale and chiral U1

Premises

We face the theoretical abstraction of QCD with $N_{fl} = 6$, representing strong interactions – adaptable to two or three light flavors (u , d , s) of quarks and antiquarks. $\leftrightarrow$

quarks : color is counted in $\pi^0 \to \gamma\gamma$

spin and flavor are clearly seen in $q\overline{q}$ and $3q$, $3\overline{q}$ spectroscopy.

(1)
$$\mathcal{L} = \left[\overline{q}^{\dot{c}'}_{\dot{\mathcal{B}}\dot{f}} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \delta_{c'c} \\ - v^A_\mu \left(\frac{1}{2}\lambda^A \right)_{c'\dot{c}} \end{array} \right\} \gamma^\mu_{\dot{\mathcal{B}}\mathcal{A}} q^c_{\mathcal{A} f} - m_f \overline{q}^{\dot{c}}_{\dot{\mathcal{A}}\dot{f}} q^c_{\mathcal{A} f} \right] - \frac{1}{4 g^2} F^{\mu\nu A} F^A_{\mu\nu} + \Delta\mathcal{L}$$

quarks : c' , $c = 1, 2, 3$ **color** , $f = 1, \cdots, 6$ **flavor**

$\mathcal{B}, \mathcal{A} = 1, \cdots, 4$ **spin** , m_f **mass**

$\longrightarrow$

gauge bosons :

$$F_{\mu\nu}^{A} = \partial_{\nu} v_{\mu}^{A} - \partial_{\mu} v_{\nu}^{A} - f_{ABC}\, v_{\nu}^{B}\, v_{\mu}^{C}$$

(2)

$$A, B, C = 1, \cdots, dim\,(G = SU3_{c}) = 8$$

Lie algebra labels, $\left[\frac{1}{2}\lambda^{A}, \frac{1}{2}\lambda^{B}\right] = i f_{ABC}\,\frac{1}{2}\lambda^{C}$

perturbative rescaling :

$$v_{\mu}^{A} = g\, v_{\mu\, pert}^{A}\, , \; F_{\mu\nu}^{A} = g\, F_{\mu\nu\, pert}^{A}$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.

Completing $\Delta\mathcal{L}$ in Fermi gauges

$$\Delta\mathcal{L} = \left\{ \begin{array}{c} -\frac{1}{2\,\eta\, g^{2}} \left(\partial_{\mu} v^{\mu\, A}\right)^{2} \\ + \partial^{\mu}\, \overline{c}^{A} \left(D_{\mu}\, c\right)^{A} \end{array} \right\} \; ; \; \eta \; : \textbf{ gauge parameter}$$

(3)

ghost fermion fields : $c\, ,\, \overline{c}\; ;\; \left(D_{\mu}\, c\right)^{A} = \partial_{\mu}\, c^{A} - f_{ABD}\, v_{\mu}^{B}\, c^{D}$

gauge fixing constraint : $C^{A} = \partial_{\mu}\, v^{\mu\, A}$

→

Gauge boson binary bilocal and adjoint string operators

One goal is, to identify – not just some candidate resonance – gluonic mesons, binary and higher modes, and to relate them to the base quantities within QCD.

(4)
$$
\begin{aligned}
& B_{[\mu_1 \nu_1], [\mu_2 \nu_2]}(x_1, x_2) = \\
& = F_{[\mu_1 \nu_1]}(x_1; A) \, U(x_1, A; x_2, B) \, F_{[\mu_2 \nu_2]}(x_2; B) \\
& A, B, \cdots = 1, \cdots, 8 \quad ; \quad \text{no flavor but spin}
\end{aligned}
$$

$F_{[\mu\nu]}(x; A)$ **denote the color octet of field strengths.**
The quantity $U(x, A; y, B)$ **in eq. (4) denotes the octet string operator, i. e. the path ordered exponential over a straight line path** $\mathcal{C}$ **from y to x**

(5)
$$
\begin{aligned}
& U(x, A; y, B) = P \exp \left(\int_y^x \Big|_{\mathcal{C}} dz^{\mu} \, \tfrac{1}{i} \, v_{\mu}(z, D) \, \mathcal{F}_D \right)_{AB} \\
& (\mathcal{F}_D)_{AB} = i f_{ADB}
\end{aligned}
$$

with the local limit $\longrightarrow$

(6) $$B_{[\mu_1 \nu_1],[\mu_2 \nu_2]}(x_1 = x_2 = x) = (:) F^{A}_{[\mu_1 \nu_1]}(x) F^{A}_{[\mu_2 \nu_2]}(x)(:)$$

no flavor but spin

The same procedure involving a triplet string applies to $\overline{q}\, q$ bilinears

$$B^{q}_{[\dot{\mathcal{A}} \dot{f}_1 , \mathcal{B} f_2]}(x_1 , x_2) = \overline{q}^{\,\dot{c}_1}_{\mathcal{B} \dot{f}_2}(x_1) U(x_1 , c_1 ; x_2 , \dot{c}_2) q^{c}_{\mathcal{A} f_1}(x_2)$$

(7) $$U(x_1 , c_1 ; x_2 , \dot{c}_2) = P \exp \left(\int_y^x \Big|_{\mathcal{C}} d z^{\mu} \tfrac{1}{i} v_{\mu}(z , D) \tfrac{1}{2} \lambda_D \right)_{c_1 \dot{c}_2}$$

flavor *and* spin

with the local limit

(8) $$B^{q}_{[\dot{\mathcal{A}} \dot{f}_1 , \mathcal{B} f_2]}(x_1 = x_2 = x) = (:) \overline{q}^{\,\dot{c}}_{\mathcal{B} \dot{f}_2}(x) q^{c}_{\mathcal{A} f_1}(x)(:)$$

The symbols $(:)$ in eqs. 6 and 8 should indicate that normal ordering of regulating the local limits is required and further that such normal ordering is *not* unique , and dependent on quark masses in the case of the $\overline{q}\, q$ bilinears . $\longrightarrow$

The U1-axial central anomaly involves the local chiral current projections from $B^{\,q}_{\,[\,\dot{\mathcal{A}}\,\dot{f}_1\,,\,\mathcal{B}\,f_2\,]}(x)$ **in eq. 8**

$$\begin{aligned} \left(j^{\,\pm}_{\,\mu} \right)_{\dot{f}_1\, f_2} (\,x\,) \quad &= \quad B^{\,q}_{\,[\,\dot{\mathcal{A}}\,\dot{f}_1\,,\,\mathcal{B}\,f_2\,]}(x)\,\left(\gamma_{\,\mu}\,\tfrac{1}{2}\,(\,\P \pm \gamma_{\,5}\,) \right)_{\mathcal{A}\,\dot{\mathcal{B}}} \\ &= \quad (:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}_2}\,\gamma^{\,\pm}_{\,\mu}\,q^{\,c}_{\,f_1}\,(\,x\,)\,(:) \end{aligned} \tag{9}$$

$$\gamma_{\,5} \ = \ \gamma_{\,5\,R} \ = \ \tfrac{1}{i}\,\gamma_{\,0}\,\gamma_{\,1}\,\gamma_{\,2}\,\gamma_{\,3} \quad ; \quad \gamma^{\,\pm}_{\,\mu} \ = \ \gamma_{\,\mu}\,\tfrac{1}{2}\,(\,\P \pm \gamma_{\,5}\,)$$

The equations of motion for the fermion fields are *and superficially imply*

$$\left. \begin{aligned} \partial\!\!\!/\; q^{\,c}_{\,f_2} \quad &= \quad \tfrac{1}{i}\,\left(v\!\!\!/^{\,c\,\dot{c}'} + \delta^{\,c\,\dot{c}'}\,m_{\,f_2} \right)\,q^{\,c'}_{\,f_2} \\ \overline{q}^{\,\dot{c}}_{\,\dot{f}_1}\;\overleftarrow{\partial\!\!\!/} \quad &= \quad \overline{q}^{\,\dot{c}'}_{\,\dot{f}_1}\,\tfrac{1}{i}\,\left(- v\!\!\!/^{\,c'\,\dot{c}} - \delta^{\,c'\,\dot{c}}\,m_{\,f_1} \right) \end{aligned} \right. \quad ; \quad \textbf{no sums over } \dot{f}_1\,,\; f_2 \ \rightarrow$$

$$\begin{aligned} \partial^{\,\mu}\,\left(j^{\,\pm}_{\,\mu} \right)_{\dot{f}_1\, f_2} &= \tfrac{1}{2\,i}\,\left((\,m_{\,f_2} - m_{\,f_1}\,)\,S_{\,\dot{f}_1\,f_2} \mp (\,m_{\,f_2} + m_{\,f_1}\,)\,P_{\,\dot{f}_1\,f_2} \right) \\ S_{\,\dot{f}_1\,f_2} = (:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}_1}\,q^{\,c}_{\,f_2}\,(:) \quad &, \quad P_{\,\dot{f}_1\,f_2} = (:)\,\overline{q}^{\,\dot{c}}_{\,\dot{f}_1}\,\gamma_{\,5}\;\,q^{\,c}_{\,f_2}\,(:) \end{aligned} \tag{10}$$

In eq. 10 $m_{\,f}$ **denotes the** *real* , *nonnegative* **quark mass for flavor f.** $\rightarrow$

From eq. 10 the relations for vector and axial vector currents ***superficially*** **follow**

(11)
$$\begin{aligned}
(j_\mu)_{\dot f_1 f_2} &= \left(j_\mu^{+}\right)_{\dot f_1 f_2} + \left(j_\mu^{-}\right)_{\dot f_1 f_2} \\
(j_\mu^5)_{\dot f_1 f_2} &= \left(j_\mu^{+}\right)_{\dot f_1 f_2} - \left(j_\mu^{-}\right)_{\dot f_1 f_2} \\
\partial^\mu (j_\mu)_{\dot f_1 f_2} &= \frac{1}{i}\left(m_{f_2} - m_{f_1}\right) S_{\dot f_1 f_2} \\
\partial^\mu (j_\mu^5)_{\dot f_1 f_2} &= \left(m_{f_2} + m_{f_1}\right) i P_{\dot f_1 f_2}
\end{aligned}$$

As it follows from the original derivation by Adler and Bell and Jackiw [1-1-] in QED, the vecor current Ward identities in eq. 11 can be implemented also in QCD , leaving the axial current ones reduced to the flavor non-singlet case , leaving the U1 axial current divergent anomalous

$$\partial^\mu (j_\mu)_{\dot f_1 f_2} = \frac{1}{i}\left(m_{f_2} - m_{f_1}\right) S_{\dot f_1 f_2} \quad \surd$$

(12)
$$\left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}^{NS}_{\dot f_1 f_2} = \left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}_{\dot f_1 f_2} - \frac{1}{N_{fl}}\, \delta_{\dot f_1 f_2} \sum_f \left\{ \begin{array}{c} j_\mu^5 \\ P \end{array} \right\}_{\dot f f}$$

$\longrightarrow$

and similarly

(13) $$\left\{ \begin{array}{c} j^{\,5}_{\,\mu} \\ P \end{array} \right\}^{S}_{\dot f_1 f_2} = \sum_f \left\{ \begin{array}{c} j^{\,5}_{\,\mu} \\ P \end{array} \right\}_{\dot f f}$$

Quark masses and splittings : m_f and $\Delta m_f = m_f - \langle m \rangle$

In the subtitle above $\langle m \rangle$ stands for the mean quark mass

(14) $$\langle m \rangle = \frac{1}{N_{fl}} \sum_f m_f$$

The identities for vector currents in eqs. 11 and 12 can be extended separating the conributions proportional to Δm_f and $\langle m \rangle$

(15) $$\begin{array}{l} \partial^{\,\mu} \left(j_{\,\mu} \right)_{\dot f_1 f_2} = \frac{1}{i} \left(\Delta m_{f_2} - \Delta m_{f_1} \right) S_{\dot f_1 f_2} \ \surd \\ \partial^{\,\mu} \left(j^{\,5}_{\,\mu} \right)^{NS}_{\dot f_1 f_2} = \quad \left(\Delta m_{f_2} + \Delta m_{f_1} \right) i\, P^{NS}_{\dot f_1 f_2} \ \surd \\ \partial^{\,\mu} \left(j^{\,5}_{\,\mu} \right)^{S}_{\dot f_1 f_2} = 2 \langle m \rangle \, i\, P^{S} \ \not\surd \ [\longrightarrow + \delta_5] \\ \delta_5 = \left(2 N_{fl} \right) \frac{1}{32 \pi^2} \, F^{\,A}_{\mu\,\nu} \widetilde{F}^{\,\mu\,\nu\,A} \Big|_{\rightarrow ren.gr.inv} \ ; \ \widetilde{F}^{\,A}_{\mu\,\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} F^{\,\mu\,\nu\,A} \end{array}$$

[a]

$\longrightarrow$

[a] δ_5 was – as far as I know – introduced by Murray Gell-Mann in lectures $\sim$ 1970 in Hawaii .

The singlet axial current anomaly

We shall return to the question of how the local operator $ch_2 (F) \equiv \frac{1}{32\pi^2} (:) F^{A}_{\mu\nu} \widetilde{F}^{\mu\nu A} (:)$ **is to be normalized and rendered renormalization group invariant [1-2-1991] . Here we just assume this to have been achieved and denote the U1-axial anomaly, the first of the central two, in its general form (eq. 15)**

(16)
$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \langle m \rangle \, i P^{S} + \delta_{5} \right\} (x)$$
$$\delta_{5} = \left(2 N_{fl} \right) \frac{1}{32\pi^2} \; (:) F^{A}_{\mu\nu} \widetilde{F}^{\mu\nu A} (:) \Big|_{\rightarrow ren.gr.inv}$$

From here it is conceptually clear how the scale- (or trace-) anomaly arises but strictly within QCD . The renormalizability of a field theory in the limit of uncurved space-time gives rise to a local , symmetric and *conserved* **energy momentum tensor , implying exact Poincaré invariance**

(17)
$$\{ \vartheta_{\mu\nu} = \vartheta_{\nu\mu} \} (x)$$
$$\partial^{\nu} \vartheta_{\mu\nu} = 0$$

In connection with the normal ordering questions it is important to admit in the precise form of the energy momentum tensor a nontrivial vacuum expected value , which $\longrightarrow$

in view of exact Poincaré invariance must be of the form

$$\langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle = \tfrac{1}{4} \eta_{\mu\nu} \tau$$

$$\left\{ \begin{array}{c} \eta_{\mu\nu} = diag(1, -1, -1, -1) \\ \tau \end{array} \right\} \text{ independent of x } \longrightarrow$$

(18)

$$\Delta \vartheta_{\mu\nu}(x) = \vartheta_{\mu\nu}(x) - \langle \Omega | \vartheta_{\mu\nu}(x) | \Omega \rangle \times \begin{cases} \widehat{\P} \\ \text{or } \ | \Omega \rangle \langle \Omega | \end{cases}$$

$$\text{with } \partial^{\nu} \Delta \vartheta_{\mu\nu}(x) = 0 \ ; \ \langle \Omega | \Delta \vartheta_{\mu\nu}(x) | \Omega \rangle = 0$$

In eq. 18 $\widehat{\P}$ **denotes the unit operator in the entire Hilbert space of states , while** $P_{\Omega} = | \Omega \rangle \langle \Omega |$ **stands for the projector on the ground state .**
Furthermore from the two local , *conserved* **tensors in eq. 18 only** $\Delta \vartheta_{\mu\nu}(x)$ **with vanishing vacuum expected value is acceptable as representing the conserved 4 momentum** *operators* **in the integral form**

(19)
$$\widehat{P}_{\mu} = \int_t d^3 x \, \Delta_{\mu\, 0}(t, \vec{x})$$

$\longrightarrow$

All these arguments *notwithstanding* **to subtract any eventual vacuum expected values of local operators , often put forward as mathematical prerequisites , it is wise** *not to do so* **in the presence of spontaneous parameters , the dynamical origin of spontaneous symmetry breaking, e.g. chiral symmetries in the limit ior neihbourhood of some** $m_f \rightarrow 0$ **.**

Using the (classical) equations of motion pertaining to the Lagrangean in eqs. 1 - 3

(20)

$$
\begin{aligned}
& (D_{\nu} F^{\mu\nu})^{A} = j^{\mu\ A} (\overline{q} , q) \ ; \ F \rightarrow F_{pert} \\
& (D_{\varrho} F^{\mu\ \nu})^{A} = \partial_{\varrho} F^{\mu\ \nu\ A} - f_{ABD}\, v_{\varrho}^{B} F^{\mu\ \nu\ D} \\
& j_{\mu}^{A} (\overline{q} , q) = g\, \overline{q}^{\dot{c}}_{\dot{\mathcal{A}}\ \dot{f}} (\gamma_{\mu})_{\dot{\mathcal{A}}\ \mathcal{B}} \tfrac{1}{2} \left(\lambda^{A} \right)_{c\dot{c}'} q^{c'}_{\mathcal{A}\ f} \\
& i\ (\gamma^{\mu} D_{\mu} q)^{c}_{\mathcal{A}\ f} = m_{f}\, q^{c}_{\mathcal{A}\ f} \quad \textbf{and} \quad q \rightarrow \overline{q} \\
& (D_{\mu} q)^{c}_{\mathcal{A}\ f} = \left[\partial_{\mu} \delta_{c\dot{c}'} + i\, g\, v_{\mu}^{D} \tfrac{1}{2} \left(\lambda^{D} \right)_{c\dot{c}'} \right] q^{c'}_{\mathcal{A}\ f}
\end{aligned}
$$

the associated form of the energy momentum becomes $\longrightarrow$

(21)
$$\vartheta^{(cl)}_{\mu\nu} = \begin{bmatrix} F^{A}_{\mu\varrho} F^{\varrho A}_{\nu} - \frac{1}{4}\eta_{\mu\nu} F^{A}_{\sigma\varrho} F^{\varrho\sigma A} + \\ + \frac{1}{2}\left[\overline{q}_{\dot{f}}\gamma_{\mu}\frac{i}{2}\overleftrightarrow{D}_{\nu} q_{f} + \mu \leftrightarrow \nu\right] \end{bmatrix}$$

and using once more the fermion part of the equations of motion the trace of the classical energy momentum tensor becomes

(22)
$$\vartheta^{\mu\ (cl)}_{\ \nu} = \sum_{f} m_{f} S_{\dot{f}f}$$
$$S_{\dot{f}_1 f_2} = (:)\,\overline{q}^{\dot{c}}_{\dot{f}_1} q^{c}_{f_2}\,(:)$$

The scale- or trace- anomaly

From the classical soft ferminic contribution to the trace of the energy momentum tensor there is a clear conjecture , also by Murray Gell-Mann , of the anomalous contribution , which subsequently became the scale- or trace- anomaly within QCD

(23)
$$\vartheta^{\mu}_{\ \mu} = \sum_{f} m_{f} S_{\dot{f}f} + \delta_{0}$$
$$\delta_{0} = \left(-2\beta(g)/g^{3}\right)\left[-\frac{1}{4}(:)F^{A}_{\mu\nu}F^{\mu\nu A}(:)\right]_{\to ren.gr.inv}$$

$\longrightarrow$

The two central anomalies alongside : scale- or trace- and U1-axial anomaly

We collect the two anomalous identities in eqs. 23 and 16

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_f m_f \, S_{\dot{f}\, f} + \delta_0 \right\} (x)$$
$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \, \langle m \rangle \, i \, P^{S} + \delta_5 \right\} (x)$$

(24)
$$\delta_0 = - \left(-2 \, \beta (g) / g^3 \right) \left[\tfrac{1}{4} \, (:) \, F^{A}_{\mu \nu} \, F^{\mu \nu A} \, (:) \right]_{\rightarrow ren.gr.inv}$$
$$\delta_5 = \left(2 \, N_{fl} \right) \frac{1}{8\pi^2} \left[\tfrac{1}{4} \, (:) \, F^{A}_{\mu \nu} \, \widetilde{F}^{\mu \nu A} \, (:) \right]_{\rightarrow ren.gr.inv}$$

$$- \beta / g^3 = \frac{1}{16\pi^2} \, b_0 + O(Y) \; ; \; Y = g^2 / (16 \, \pi^2$$

$\beta(g)$: **Callan-Symanzik rescaling function in QCD**

The qualification 'central' for the anomalies in eq. 24 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson ***renormalized*** **Lagrangean density x dependent**

(25)
$$\mathcal{L}_{g.b.} = - \frac{1}{g^2} \, \frac{1}{4} \, (:) \, F^{A}_{\mu \nu} \, F^{\mu \nu A} \, (:) + \vartheta \, \frac{1}{8\pi^2} \, \frac{1}{4} \, (:) \, F^{A}_{\mu \nu} \, \widetilde{F}^{\mu \nu A} \longrightarrow$$
$$g^2 \rightarrow g^2 (x) \; ; \; \vartheta \rightarrow \vartheta (x)$$

maintains perturbative renormalizability and acts together with suitable boundary conditions $\rightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(26) $$\frac{1}{4}\,(:)\,F^{A}_{\mu\nu}\,F^{\mu\nu A}\,(:) \quad , \quad \frac{1}{4}\,(:)\,F^{A}_{\mu\nu}\,\widetilde{F}^{\mu\nu A}$$

We will use the following definitions relative to the rescaling function β

$$-\beta/g = X\,B(X) \;\; ; \;\; B(X) = b_0\,A(X)$$

$$B(X) \sim \sum_{n=0}^{\infty} b_n\,X^n \quad , \quad A(X) \sim \sum_{n=0}^{\infty} a_n\,X^n$$

$\kappa = g^2/(16\,\pi^2)$ generic $X\,,\,Y$

(27) $$b_0 = \frac{1}{3}\,(33 - 2\,N_{fl}) \quad , \quad a_0 = 1 \quad , \quad a_n = b_n/b_0$$

$$b_1 = \frac{2}{3}\,(153 - 19\,N_{fl})$$

$$b_2 = \frac{1}{54}\left(77139 - 15099\,N_{fl} + 325\,N_{fl}^2\right)$$

$$b_3 \sim 29243 - 6946.3\,N_{fl} + 405.089\,N_{fl}^2 + 1.49931\,N_{fl}^3$$

$\longrightarrow$

2-1 The perturbatively accessible region and the long way back to hadronic scales

$$0.1 \text{ fm} \leq D \leq (\sim) 2 \text{ fm}$$

Renormalization group equation in QCD

We take the short distance expansion for the current product , subject to the renormalization group or rescaling equations, the latter representing *exact, anomalous* **Ward identities for the dilatation current [A21-1976]**

(28)
$$\begin{matrix} T \\ (\Pi) \end{matrix} \left\{ J_\mu \left(x + \tfrac{1}{2} z \right) J_\nu \left(x - \tfrac{1}{2} z \right) \right\} \underset{z \to 0}{\sim} \sum_{\mathcal{O}} C^{T\,(\Pi)}_{\mu\,\nu\,\mathcal{O}} (z) \, \mathcal{O} (x)$$

In the triple association

(29)
$$J_\mu (x_1) , \; J_\nu (x_2) \to \mathcal{O} (x_3)$$

we will assume that all three local fields are multiplicatively – *perturbatively* **– renormalizable for simplicity. Mixing effects of** *finite* **groups of operators**

(30)
$$\{ \cup \mathcal{O} \mid (\mathcal{O}_1 , \cdots , \mathcal{O}_n) \}$$

do arise and can easily be incorporated [A22-1974] .

We are mainly interested in the $tw = 4 \, ; \, dim \, 4$ **operators** *later*

(31)
$$\{ \mathcal{O} \}_4 = \left\{ \vartheta^{\,\mu}_{\,\mu} , \; \tfrac{1}{4} g^2 \; : F_{\mu\nu\, a} F^{\,\mu\nu}_{\,a} \; : , \cdots \right\}$$

$\vartheta^{\,\mu}_{\,\nu}$ **: suitable, conserved energy momentum tensor**

$\longrightarrow$

but discuss the general simply multiplicatively renormalizable case first .

In terms of unrenormalized quantities , generically denoted by the suffix $^{(0)}$ **, and renormalization constants including a Fermi-gauge fixing parameter** η **in order to controle the gauge invariant character of the so defined operators we set**

(32)
$$J_\alpha = (Z_J)^{-1} J_\alpha^{(0)} \quad , \quad \mathcal{O} = (Z_{\mathcal{O}})^{-1} \mathcal{O}^{(0)}$$
$$g = (Z_3)^{3/2} (Z_1)^{-1} g^{(0)} \quad , \quad \eta = (Z_3)^{-1} \eta^{(0)}$$

As renormalization conditions we use a ***finite*** **dummy scale** μ **, as it appears** ***also*** **naturally in dimensional renormalization , with respect to which unrenormalized quantities are insensitive**

(33)
$$d / d\mu \left\{ g^{(0)} , \eta^{(0)} ; J_\alpha^{(0)} , \mathcal{O}^{(0)} , \ldots^{(0)} \right\} = 0$$

For the choice of currents in eq. 28 and $d = 4$ **scalar operators** $\{ \mathcal{O} \}$ **in eq. 31 it follows , always within the (asymptotically) perturbative logic**

(34)
$$C^{T\,(\Pi)}_{\alpha\,\beta\,\mathcal{O}} (z) = \left(g_{\alpha\beta} \Box - \partial_\alpha \partial_\beta \right) C^{T\,(\Pi)}_{J\,\mathcal{O}} (z ; \mu , g , \eta)$$

with $dim\, C^{T\,(\Pi)}_{J\,\mathcal{O}} = 0 \;\; ; \;\; Z_J = 1$

$\longrightarrow$

The short distance distributions $C^{T\,(\Pi)}_{J\,\mathcal{O}}$ then are of the form

(35)
$$C^{T\,(\Pi)}_{J\,\mathcal{O}}\,(z\,;\,\mu\,,\,g\,,\,\eta\,)\;=\;(\,Z_J\,)^{2}\,(\,Z_{\mathcal{O}}\,)^{-1}\,\widehat{C}^{\,T\,(\Pi)}_{J\,\mathcal{O}}\,(z\,;\,\mu\,,\,g\,,\,\eta\,)$$
$$g\;=\;(\,Z_3\,)^{3/2}\,(\,Z_1\,)^{-1}\,g^{(0)}\quad,\quad\eta\;=\;(\,Z_3\,)^{-1}\,\eta^{(0)}$$

The μ rescaling equation now follows from eq. 33

$$\left(\begin{array}{l}\mu\,\partial_{\mu}\,+\,\beta\,(\,g\,)\,\partial_{g}\,-\,\gamma_{m_{\alpha}}\,m_{\alpha}\,\partial_{m_{\alpha}}\\ -\,2\,\gamma_3(\,\eta\,\partial_{\eta}\,)\,-\,\gamma_{J\,\mathcal{O}}\end{array}\right)\;C^{T\,(\Pi)}_{J\,\mathcal{O}}\,(\,z\,;\,\mu\,,\,g\,,\,m_{\beta}\,,\,\eta\,)\;=\;0$$

(36)
$$\left\{\begin{array}{l}\beta\,(\,g\,)\;=\;-\,g\,b\,(\,g^{2}\,)\\ \gamma_{m_{\beta}}\,(\,g^{2}\,)\\ \gamma_{J\,\mathcal{O}}\,(\,g^{2}\,,\,(\,\eta\,)\,)\\ \gamma_3\,(\,g^{2}\,,\,\eta\,)\end{array}\right\}\;=\;\mu\,d\,/\,d\,\mu\,\left\{\begin{array}{c}\log\,\left(\,(\,Z_3\,)^{3/2}\,(\,Z_1\,)^{-1}\,\right)\\ Z_{m_{\beta}}\\ \log\,\left(\,Z_{\mathcal{O}}\,/\,Z_J^{2}\,\right)\\ \log\,(\,Z_3\,)^{1/2}\end{array}\right\}$$

The brackets in red in eq. 36 shall indicate that upon establishing that $\{\,J\,,\,\mathcal{O}\,\}$ are indeed gauge invariant operators the derivative with respect to the gauge parameter η in the rescaling equation gives zero and also the ***combined*** **anomalous dimensions $\gamma_{J\,\mathcal{O}}$ do not depend on η . For the operators in the group of interest here (eq. 31) the determination of the correct gauge invariant ones has been derived by H. Kluberg-Stern and J. B. Zuber [A24-1975] .** $\longrightarrow$

From the *partial* renormalization systematics given given in eqs. 32 - 33 the *redundant* scale μ becomes an essential argument of renormalized quantities

$$\begin{array}{lcl} J_{\alpha} & : & \mu \, d / d \mu \, (J_{\alpha})_{\mu} = - \gamma_{J} \, (J_{\alpha})_{\mu} = 0 \\ \mathcal{O} & : & \mu \, d / d \mu \, (\mathcal{O})_{\mu} = - \gamma_{\mathcal{O}} \, \mathcal{O}_{\mu} \\ g & : & \mu \, d / d \mu \, (g)_{\mu} = \beta \, (g_{\mu}) \\ \eta & : & \mu \, d / d \mu \, (\eta)_{\mu} = - \gamma_{3} \, \eta_{\mu} \end{array} \qquad \text{with :} \left\{ \begin{array}{l} \gamma_{3} = \gamma_{3} \, (g^{2} , \eta) \\ \gamma_{J} = 0 \;\; , \;\; Z_{J} = 1 \\ \gamma_{\mathcal{O}} = \gamma_{\mathcal{O}} \, (g^{2} , (\eta)) \end{array} \right.$$

(37)

Three remarks are in order :

1) Perturbative accessibility of renormalization in asymptotically free theories

While the entire renormalization procedure thus (eq. 37) becomes within *perturbative accessibility* – as explained in textbooks [A25] , [A26-1982] – the associated renormalization group equation serves to restore renormalization group invariant properties, in particular such definitions of operators .

2) Infrared instability

is associated with all physical scales *not* accessible to perturbative approximations .

→

3) Quark mass dependence

We neglect in the considerations followed here the quark mass dependence of all Green functions in the deep Euclidean region on quark masses , the latter also *to be renormalized* **and thus** *not* **renormalization group independent [A27-1975] . This is in line with the main short distance contributions , which are sorting out by the twist characteristic leading contributions** *modulo less dominant ones modulo powers of inverse Euclidean distance* **. These dimensional hierarchies** *also* **break down whence the region of perturbative accessibility is transgressed . For small quark masses at a generic scale of** $\sim\ 1\,GeV$ **the quark mass associated mixing of operators with different dimensions sets in in subtle ways governed by approximate chiral symmetry also outside the deep Euclidean region .**

$\longrightarrow$

A2-slsc The sliding scale coupling constant

From eq. 37 the sliding scale coupling constant follows , considering a variation of μ and g_μ

(38)
$$t = \log(\overline{\mu}/\mu) \quad ; \quad \partial_t = \overline{\mu}\, d/d\overline{\mu}$$

$$\overline{g} = \overline{g}(t, g_\mu) \quad ; \quad \text{generic variablei is : } g' \to x \quad ; \quad t \to \tau$$

$$\partial_t \overline{g} = \beta(\overline{g}) \quad ; \quad \overline{g}(t = 0, g_\mu) = g_\mu$$

Using the generic variables x , τ the differential equation (eq. 38) becomes

(39)
$$(d/d\tau)\, x = \beta(x) \to d\tau = dx/\beta(x) \quad \to$$

$$\tau_2 - \tau_1 = \int_{x_1}^{x_2} dx\, (\beta(x))^{-1} = F(x_2) - F(x_1)$$

$$F(x) = \int_{x_0}^{x} dx'/\beta(x') ; \quad \text{generic : } x_0 \text{ independent of } x_1 , x_2$$

$$x = x(\tau) \to \begin{cases} x_1(\tau_1) \\ x_2(\tau_2) \end{cases}$$

The function $F(x)$ satisfies the (generic) equation

(40)
$$\beta(x)\, \partial_x F(x) = 1$$

$\rightarrow$

From the generic identities a consequence follows, relevant for the short distance limit of the rescaling equation (eq. 36) , for the solution to the associated homogeneous partial differential equation [a]

(41)
$$(\partial_{\tau} - \beta (g) \partial_{g}) \, h (\tau , g) = 0 \quad \longrightarrow$$
$$h = h \, (\overline{g} (\tau = t , g = g_{\mu})) \quad \text{with} \quad \overline{g} \quad \text{as defined in eq. 38}$$

Leaving aside quark mass dependence for simplicity here , bearing in mind remark 3) above, we turn to the properties of the sliding scale coupling constant , i.e. the function $\overline{g} = \overline{g} (t , g_{\mu})$ and the associated differential equation defined in eq. 38 readapted below. The universal independence of the sliding scale coupling constant can be maintained independent of quark masses .

(42)
$$t = \log (\overline{\mu} / \mu) \quad ; \quad \partial_{t} = \overline{\mu} \, d / d \, \overline{\mu}$$
$$\partial_{t} \overline{g} = \beta (\overline{g}) \quad ; \quad \overline{g} (t = 0 , g_{\mu}) = g_{\mu}$$

Thus we are to determine the function $F (x)$ as defined in eq. 38 such that

(43)
$$t = F (\overline{g}) - F (g) \quad ; \quad g = g_{\mu} \quad ; \quad F (x) = \int_{x_0}^{x} d y / \beta (y)$$

$\longrightarrow$

[a] Note the - sign in the first line of eq. 41 .

We have followed the concise treatment and notation of C. G. Callan [A28-1970] and K. Symanzik [A29-1970] on whose formulation and structural discussion the modern form of the renormalization group equation(s) relies . The two loop renormalizability of nonabelian gauge theories is due to G. t'Hooft [A210-1971(2)] and contain all elements which determine the sliding scale function discussed here. These quotations duely made including [A211-1969] , [A212-1972] , I continue using selected changes of variables, transforming eq. 43

$$s = 2t = \log\left[(\overline{\mu} / \mu)^2 \right]$$

$\kappa = g^2 / \left(16 \pi^2 \right)$ **and** $\kappa \rightarrow \overline{\kappa}$ **generic** $\kappa \rightarrow X , Y$

(44)
$$s = F(\overline{\kappa}) - F(\kappa_\mu) \; ; \; \kappa_\mu = g_\mu^2 / \left(16 \pi^2 \right)$$
$$F(X) = \int_X^{X_0} \left(dY / Y^2 \right) (B(Y))^{-1}$$
$$\beta(y) = -y\, b(y^2) \; ; \; B(Y) = b(y^2)/Y \leftrightarrow Y = y^2 / \left(16 \pi^2 \right)$$

With the substitutions in eq. 44 we have

(45)
$$s = F(\overline{\kappa}) - F(\kappa_\mu) = \int_{\overline{\kappa}}^{\kappa_\mu} \left(dY / Y^2 \right) (B(Y))^{-1}$$
$$B(Y) = b_0 + b_1 Y + b_2 Y^2 + \cdots$$

→

There exist many ways to separate the limiting part $\overline{\kappa} \rightarrow 0 \leftrightarrow s \rightarrow +\infty$ of the integral in eqs. 44, 45 . We use (two) partial integrations

(46)
$$s = (Y B(Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} + \int_{\overline{\kappa}}^{\kappa_\mu} (dY/Y)(d/dY)(B(Y))^{-1}$$
$$\Delta_1 s = s - (Y B(Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} \quad ; \quad \Delta_0 B^{-1} = (B(Y))^{-1}$$

Upon the substitution $s \rightarrow \Delta_1 s$ we obtain

(47)
$$\Delta_1 s = (\log(Y))(-d/dY)(B(Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} + \Delta_2 F$$
$$\Delta_2 F = \int_{\overline{\kappa}}^{\kappa_\mu} dY (-\log Y)(-d/dY)^2 (B(Y))^{-1}$$

Going step by step we evaluate first the derivatives as acting on $B^{-1}(Y)$

$$(-d/dY)(B(Y))^{-1} = B'(Y)/B^2(Y) = \Delta_1 B^{-1} = -\left(B^{-1}\right)'$$
$$(-d/dY)^2 (B(Y))^{-1} = \left(2\left(B'(Y)\right)^2 - B''(Y) B(Y)\right)(B(Y))^{-3}$$
$$= \Delta_2 B^{-1} = \left(B^{-1}\right)''$$
$${}' = d/dY \, , \; {}'' = (d/dY)^2 \, , \; \cdots$$
(48)

$\longrightarrow$

Collecting the definitions in eqs. 46 - 48 we restate

(49)
$$\begin{aligned} n &= 0 \quad \Delta_0 B^{-1} &&= (B(Y))^{-1} \\ n &\geq 1 \quad \Delta_n B^{-1} &&= (-d/dY)^n (B(Y))^{-1} \end{aligned}$$

and

(50)
$$\begin{aligned} s &= \left[\left(Y^{-1} \right) \Delta_0 B^{-1}(Y) + (\log Y) \Delta_1 B^{-1}(Y) \right]_{\kappa_\mu}^{\overline{\kappa}} + \Delta_2 F \\ \Delta_2 F &= \int_{\overline{\kappa}}^{\kappa_\mu} dY \, (-\log Y) \, \Delta_2 B^{-1}(Y) \end{aligned}$$

We add two representations valid in the perturbatively accessible region $0 \leq Z \leq \overline{X} = \overline{\kappa}$

---

(51)
$$\left(Y^{-1} \right) \Delta_0 B^{-1}(Y) \big|_{Y = \overline{X}} = \overline{X}^{-1} \left[\begin{array}{c} B^{-1}(0) + \\ + \int_0^{\overline{X}} dZ \, \left(-\Delta_1 B^{-1} \right) (Z) \end{array} \right]$$

$$\overline{X} \rightarrow \overline{\kappa_\mu}$$

$\longrightarrow$

Also we anchor $\Delta_2 F$ in eq. 50 at $X \to 0$

(52)
$$\begin{aligned} \Delta_2 F &= \int_{\overline{\kappa}}^{\kappa_\mu} dY \, (-\log Y) \, \Delta_2 B^{-1}(Y) \\ &= F_2(X_\mu) - F_2(\overline{X}) \end{aligned}$$

$$F_2(X) = \int_0^X dY \, (-\log Y) \, \Delta_2 B^{-1}(Y) \quad ; \quad \begin{cases} X_\mu = \kappa_\mu \\ \overline{X} = \overline{\kappa} \end{cases}$$

We decompose s in eq. 50

(53)
$$\begin{aligned} &s = \Sigma(\overline{X}) - \Sigma(X_\mu) \\ &\Sigma(X) = \left(X^{-1}\right) \Delta_0 B^{-1}(X) + (\log X) \, \Delta_1 B^{-1}(X) - F_2(X) \end{aligned}$$

For $X \to X_\mu$ we do not know the form of the functions determining $\Sigma(X)$, ***in particular*** **if we choose the scale μ** ***outside*** **the region of perturbative accessibility . But for $X \to \overline{X}$ we can perform an asymptotic expansion for $X \to 0$, assuming a pure power expansion for the functions $\left(B, B^{-1}, \cdots (X)\right)$, as they appear in the asymptotic expressions for $\Sigma(\overline{X})$ as defined in eqs. 52 - 53 .**

To this end it is enough to determine $\longrightarrow$

the power expansion of $B^{-1}(X)$ up to second order in X , to which we turn below.

(54)
$$B(X) = b_0 + b_1 X + b_2 (X)^2 + R_2(X)$$
$$R_2(X) = o(X^2) \text{ for } X \to +0$$

To this end it is convenient to rescale $B(X)$

(55)
$$B(X) = b_0 A(X) \;;\; b_0 = \frac{1}{3}(33 - 2N_{fl}) > 0$$
$$A(X) = 1 + a_1 X + a_2 X^2 + \widehat{R}_2(X) \;;\; a_n = b_n / b_0$$
$$\widehat{R}_2 = (b_0)^{-1} R_2 \;;\; \widehat{R}_2(X) = o(X^2)$$

The rescaling by $(b_0)^{-1}$ is universal to all three terms on the right hand side of the expression for s in eq. 53 yielding

(56)
$$s = (b_0)^{-1} \left(\widehat{\Sigma}(\overline{X}) - \widehat{\Sigma}(X_\mu) \right)$$
$$\widehat{\Sigma}(X) = \left(X^{-1}\right) \Delta_0 A^{-1}(X) + (\log X) \Delta_1 A^{-1}(X) - \widehat{F}_2(X)$$
$$\widehat{F}_2(X) = \int_0^X dY\, (-\log Y) \Delta_2 A^{-1}(Y)$$

$\longrightarrow$

For A^{-1} we thus have

(57)
$$\begin{aligned}
\Delta_0 A^{-1}(X) &= 1 - a_1 X + \Big((a_1)^2 - a_2 \Big) X^2 && + R_2^{(\Delta_0 A^{-1})}(X) \\
\Delta_1 A^{-1}(X) &= a_1 - 2 \Big((a_1)^2 - a_2 \Big) X && + R_1^{(\Delta_1 A^{-1})}(X) \\
\Delta_2 A^{-1}(X) &= 2 \Big((a_1)^2 - a_2 \Big) && + R_0^{(\Delta_2 A^{-1})}(X)
\end{aligned}$$

$$\Delta_0 A^{-1} = A^{-1} \; ; \quad R_n^{(.)}(X) = o(X^n)$$

Here we list the three coefficients of the function $B(\kappa) = -\beta(g)/(g\kappa)$ [A213-1988] (eqs. 44 - 45 , 55) , to which the present asymptotic expansion at short distances is restricted, in the $\overline{MS}$ renormalization scheme

(58)
$$\begin{aligned}
b_0 &= \tfrac{1}{3} \left(33 - 2 N_{fl} \right) \\
b_1 &= \tfrac{2}{3} \left(9 \times 17 - 19 N_{fl} \right) \\
b_2 &= \tfrac{1}{54} \Big(27 \times 2857 - 21 \times 719 N_{fl} + 25 \times 13 N_{fl}^2 \Big)
\end{aligned}$$

$$5033 = 7 \times 719 \; , \quad 325 = 25 \times 13$$

b_3 **has been calculated in ref. [A214-1997] .** $\rightarrow$

Back to the asymptotic expansion of the sliding scale coupling constant

We invert the functional relation

$$\Sigma_{asy}(X) = Z \quad \leftrightarrow \quad X = \overline{X}(Z)$$

(59)

$$Z = b_0 \log\left(\overline{\mu}^2/\Lambda^2\right) \equiv b_0\, s$$

$$\Sigma_{asy}(X) = \sigma_{asy}(X) + \mathcal{R}_\sigma(X)$$

$$\sigma_{asy} = X^{-1} - a_1 \log(X^{-1}) \;\; ; \;\; \mathcal{R}_\sigma(X) \sim -\sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

in the form suitable for successive approximations

(60)

$$X^{-1} = Z + a_1 \log(X^{-1}) - \mathcal{R}_\sigma(X) = Z + f(X) \longrightarrow$$

$$f(X) \sim a_1 \log(X^{-1}) + \sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

$$1/X_{\nu+1}(Z) = Z + f\,(X_\nu(Z))$$

starting with the substitution for $\nu = 0 \;\; : \;\; f\,(X_{\nu=0}(Z)) = 0$

In the successive approximation procedure furthermore the function $f(X \ldots)$ **can be evaluated in various suitable approximations .** $\longrightarrow$

We first give the two diverging terms for $Z \rightarrow \infty$ keeping only the first term in

$f \approx f_{(1)} = a_1 \log (X^{-1})$

$$\begin{array}{rcl} 1 / X^{(0)}_{\nu=1} (Z) & = & Z \\ 1 / X^{(1)}_{\nu=2} & = & Z + a_1 \log [Z] = b_0 s + a_1 \log [b_0 s] \\ \ldots & & a_1 = b_1 / b_0 \end{array} \tag{61}$$

$$f \rightarrow f_{(\varrho)} (X) \quad \textbf{with} \quad f_{(1)} (X) = a_1 \log (X^{-1})$$

The above diverging terms for $Z \rightarrow \infty$ entail the universal character of the first *two* coefficients – b_0 , b_1 – of the $\beta -$ function in any renormalization scheme .

While the above path of successive approximations may not be optimally converging whence extended to terms vanishing for $Z \rightarrow \infty$, these emerge as a double sequence

$$1 / X^{(\varrho)}_{\nu+1} (Z) = Z + f_{(\varrho)} \left(X^{(\leq \varrho)}_{\nu} (Z) \right) \tag{62}$$

We are here not interested in a high level of precision of the approximations, only illustrating within the perturbatively accessible region of QCD the structure of asymptotic expansions. For the evaluations involving the first four orders (in X) of the beta-function I refer to ref. [A215-2009] . $\rightarrow$

For the purpose of illustration we give the next approximation corresponding to $\nu = 3\,,\ (\varrho) = 1$

$$\begin{aligned} 1\,/\,X^{(\varrho)=1}_{\nu=3}(Z) &= Z + a_1 \log\left[Z + a_1 \log\left[Z\right]\right] \\ &= Z + a_1 \log\left[Z\right] + a_1 \log\left[1 + a_1 Z^{-1} \log\left[Z\right]\right] \\ &\sim Z + a_1 \log\left[Z\right] + a_1^2 Z^{-1} \log\left[Z\right] + o\left(Z^{-1} \log Z\right) \end{aligned}$$

(63)

The third term in the third line of eq. 63 is the first of its kind vanishing for $Z \to +\infty$.

A2-mass **Quark mass renormalization transposed to quark bilinear operator insertion** $:\ \overline{q}^{\,c} q^{\,c}\ :\big|_0$
and renormalization **using QCD with exactly vanishing quark mass(es)**

A well known problem of electron mass - and analogously quark mass induced effects goes back to the general operator product expansion discussed by K. Wilson [A216-1969] in the light of QED and the renormalization group equation as specifically formulated by M. Gell-Mann and F. Low [A217-1954] *and* extended to QCD . $\longrightarrow$

We rescale the renormalization group equation relative to its conventional form in eq. 36

$$\begin{pmatrix} \mu^2 \partial_\mu^2 + (-\kappa^2 B)\,\partial_\kappa \\ -\kappa\,\Gamma_{m_\alpha} m_\alpha \partial_{m_\alpha} \\ -\gamma_3 (\eta\,\partial_\eta) - \Gamma_{J\mathcal{O}} \end{pmatrix} C_{J\,\mathcal{O}}^{T\,(\Pi)}\left(z\,;\,\mu^2, \kappa, m_\beta, \eta\right) = 0$$

$$\left\{ \begin{array}{l} \kappa B(\kappa) = -\beta(g)/g \\ \Gamma_{m_\beta}(\kappa) = \frac{1}{2}\gamma_{m_\beta} \\ \Gamma_{J\mathcal{O}}(\kappa, (\eta)) = \frac{1}{2}\gamma_{J\mathcal{O}} \\ \gamma_3(\kappa, \eta) \equiv \gamma_3(g^2, \eta) \end{array} \right\} = \mu^2 \partial_{\mu^2} \left\{ \begin{array}{c} 2\log\left((Z_3)^{3/2} (Z_1)^{-1} \right) \\ Z_{m_\beta} \\ \log\left(Z_{\mathcal{O}} / Z_J^2 \right) \\ 2\log(Z_3)^{1/2} \end{array} \right\}$$

$$\kappa = \alpha_s/(4\pi) = g^2/(16\pi^2)$$

(64)

In the $\overline{\text{MS}}$ scheme – and ignoring the precise form of normal orderings – or rather following the most thoughtful suggestion of S. Weinberg [A218-1973] $\longrightarrow$

the mass rescaling functions $\Gamma_{\beta} \rightarrow \Gamma_{m}$ become independent of quark flavor β and of the quark masses [A219-1982] , [A220-1994]

$$
\begin{aligned}
&\Gamma_{m\,\beta} \rightarrow \Gamma_{m} \equiv \kappa\, G_{m} \\
&G_{m} = g_{0} + g_{1}\,\kappa + \cdots \sim \textstyle\sum_{n=0}^{\infty} g_{n}\,\kappa^{n} \\
&g_{0} = 4 \quad , \quad g_{1} = \tfrac{2}{9}\,\left(101 - 10\,N_{fl} \right) \\
&g_{2} = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta\,(3) \right] N_{fl} - \frac{140}{81} N_{fl}^{2} \\
&\ldots
\end{aligned}
\tag{65}
$$

The pertinent rearrangement of normal orderings and of renormalizatiun group invariant quantities to five loop order has been carried out in ref. [A221-2006] .

The obstacles thus outlined and surpassed the *sliding scale* quark mass function(s) inherit universality and perturbative accessibility equal to the *sliding scale* coupling constant .

We are led to consider the pair of rescaling equations $\longrightarrow$

using the same notations as defined in eq. 44

(66)

$$
\begin{aligned}
& s \;=\; 2\,t \;=\; \log \left[\, (\, \overline{\mu} \,/\, \mu \,)^{\,2} \, \right] \\
& \kappa \;=\; g^{\,2} \,/\, \left(16\,\pi^{\,2} \right) \quad \textbf{generic} \quad \left\{ \begin{array}{c} \kappa \\ \overline{\kappa} \end{array} \right\} \;\rightarrow\; \left\{ \begin{array}{c} X \,,\, Y \\ \overline{X} \,,\, \overline{Y} \end{array} \right\} \\
& G_{\,m} \;\rightarrow\; G \\
& m_{\,q} \;\rightarrow\; m \quad \textbf{generic} \quad \left\{ \begin{array}{c} m \\ \overline{m} \end{array} \right\} \;\rightarrow\; \left\{ \begin{array}{l} m\,(\,\mu\,) \;\rightarrow\; m_{\,0} \\ \overline{m}\,(\,s\,;\, m_{\,0}\,) \end{array} \right\}
\end{aligned}
$$

$$
\longrightarrow \; \frac{d}{d\,s} \left\{ \begin{array}{c} \overline{X} \\ \overline{m} \end{array} \right\} \;=\; - \left\{ \begin{array}{l} \overline{X}^{\,2}\, B\,(\,\overline{X}\,) \\ \overline{X}\, G\,(\,\overline{X}\,)\; \overline{m} \end{array} \right\}
$$

It is *obvious* that the universal sliding scale mass function cannot be calculated also in the perturbatively accessible region using any version of a quark mass *dependent* propagator . This makes comparison with data , where quark mass dependent thresholds of *hadrons* appear, which depend even nonperturbatively on quark masses, a step more remote – yet not impossible – . This said we proceed $\longrightarrow$

to solve the differential equations as defined in eq. 66 but using the results already established for the sliding scale coupling constant in the last subsection. Thus we introduce the dimensionless quark mass function

(67) $$f\,(Y) = \log\left[\frac{\overline{m}}{m_0}\right] \;;\; Y \to \overline{X}$$

f **still depends through the initial conditions on the** *a priori* **arbitrary scale** μ **, which is however replaced by – an appropriate multiple of – the renormalization group invariant scale** Λ **in a way** *related* **to the asymptotic expansion of the running coupling constant .**
The function f **defined in eq. 67 satisfies the differential equation**

(68) $$\frac{d}{dY} f\,(Y) = \mathcal{Q}\,(Y) \;;\; \mathcal{Q} = Y^{-1}\,\frac{G\,(Y)}{B\,(Y)}$$

$$G = g_0\,H \;,\; B = b_0\,A$$

which can be integrated →

yielding the initial value dependent relation

(69)
$$f\left(\overline{X}\right) - f\left(X_0\right) = \frac{g_0}{b_0}\int_{X_0}^{\overline{X}} \frac{dY}{Y}\,\mathcal{Q}_{red}(Y)$$

$$\mathcal{Q}_{red} = H(Y)/A(Y) \sim \sum_{n=0}^{\infty} q_n Y^n \;\; ; \;\; q_0 = 1$$

We proceed the same way as in the asymptotic expansion of the coupling constant , integrating the first term in the expansion of the reduced function $\mathcal{Q}_{red}$ in eq. 69

$$\frac{1}{Y}\,\mathcal{Q}_{red} = \frac{1}{Y} + \mathcal{R}_1\left(Y^{-1}\mathcal{Q}_{red}\right)$$

(70)
$$\mathcal{R}_1 \to \mathcal{R}_Q \;\; ; \;\; \mathcal{R}_Q \sim \sum_{n=1}^{\infty} q_n Y^{n-1}$$

$$q_1 = \frac{g_1}{g_0} - \frac{b_1}{b_0} \;\; , \;\; q_2 \cdots$$

Thus eq. 69 $\longrightarrow$

becomes, anchoring the integrals of $\mathcal{R}_Q$ **at** $X = 0$

$$\int_0^X \mathcal{R}_Q(Y)\, dY \equiv M(X)$$

$$f(\overline{X}) - f(X_0) = \frac{g_0}{b_0} \left[\log(\overline{X}) + M(\overline{X}) - \log(X_0) - M(X_0) \right]$$

(71)

$$M(\overline{X}) \sim \sum_{n=1}^{\infty} \frac{1}{n} q_n (\overline{X})^n$$

remembering $f(Y) = \log\left[\frac{\overline{m}}{m_0}\right] \; ; \; Y \rightarrow \overline{X}, X_0$

We note the values of the critical mass rescaling exponent – denoted c_{mar} **– for** $N_{fl} = 3$ **to** 6

(72) $$c_{mar}(N_{fl}) = \frac{g_0}{b_0} = \frac{4 \times 3}{33 - 2 N_{fl}}$$

$\longrightarrow$

N_{fL}	3	4	5	6
c_{mar}	$4/9 \sim 0.44$	$12/25 = 0.48$	$12/23 \sim 0.52$	$4/7 \sim 0.57$

(73)

$$\longrightarrow \quad 0.4 < c_{mar} < 0.6 \ \textbf{for} \ 3 \leq N_{fl} \leq 6$$

We rewrite eq. 71 *separating variables*

(74)
$$\log\left[\frac{\overline{m}}{m^*}\right] - c_{mar}\left[\log\left(\overline{X}\right) + M\left(\overline{X}\right)\right] =$$
$$= \log\left[\frac{m_0}{m^*}\right] - c_{mar}\left[\log\left(X_0\right) + M\left(X_0\right)\right]$$

The reference mass denoted m^* in eq. 74 is completely arbitrary , yet we *restrict* **it to be renormalization group invariant .**

Next we exponentiate both sides of eq. 74 $\longrightarrow$

$$\frac{\overline{m}}{m^*}\left[(\overline{X})^{-c_{mar}}\right] EM(\overline{X}) =$$

$$= \frac{m_0}{m^*}\left[(X_0)^{-c_{mar}}\right] EM(X_0)$$

$$EM(X) = \exp\left[-c_{mar} M(X)\right]$$

(75) $$M(X) : \begin{cases} \textbf{universal , quark mass independent function} \\ \sim \sum_{n=1}^{\infty} \frac{1}{n} \; q_n (X)^n \quad \textbf{in perturbatively accessible region} \end{cases}$$

$$\searrow$$

$$EM(X) = 1 + \mathcal{R}_{EM}(X) : \begin{cases} \textbf{universal , quark mass independent function} \\ \mathcal{R}_{EM} \sim \sum_{n=1}^{\infty} \; g_{EM\,n} (X)^n \\ \textbf{in perturbatively accessible region} \end{cases}$$

$$g_{EM\,1} = -c_{mar}\, q_1 , \cdots$$

$\longrightarrow$

We collect the expressions composing $g_{EM\,1}$ in the last relation of eq. 75 below, using eqs. 58 (b_0 , b_1) , 65 (g_0 , g_1) , 70 (q_1) , 72 and 73 (c_{mar})

$$g_{EM\,1} = -c_{mar}\,q_1 \quad ; \quad c_{mar}(N_{fl}) == \frac{g_0}{b_0} = \frac{4\times 3}{33 - 2N_{fl}}$$

(76) $$q_1 = \frac{g_1}{g_0} - \frac{b_1}{b_0} \;;$$

$$\begin{aligned} b_0 &= \tfrac{1}{3}\left(33 - 2N_{fl}\right) \\ b_1 &= \tfrac{2}{3}\left(9\times 17 - 19N_{fl}\right) \end{aligned} \quad , \quad \begin{aligned} g_0 &= 4 \\ g_1 &= \tfrac{2}{9}\left(101 - 10N_{fl}\right) \end{aligned}$$

We evaluate the ratios forming the expresion for $g_{EM\,1}$ in eq. 76 only for $N_{fl} = 3$ and 5 and also neglect all $g_{EM\,n>1}$ for simplicity and to show the structural effects in a coherent way, leaving *subsequent* **systematic approximations aside . The latter include heavy flavor matching if we go deep enough inside the region of perturbative accessibility .** $\longrightarrow$

$$N_{fl} = 3,5 \; : \; c_{mar} = \tfrac{4}{9} \, , \, \tfrac{12}{23}$$

(77)
$$\left\{ \begin{array}{lllll|lllll} b_0 & = & 9 & , & \frac{23}{3} & g_0 & = & 4 & , & 4 \\ b_1 & = & 64 & , & \frac{116}{3} & g_1 & = & \frac{142}{9} & , & \frac{34}{3} \end{array} \right.$$

It follows

(78)
$$c_{mar} = \tfrac{4}{9} \, , \, \tfrac{12}{23} \quad , \quad \frac{g_1}{g_0} = \tfrac{71}{18} \, , \, \tfrac{17}{6} \quad , \quad \frac{b_1}{b_0} = \tfrac{64}{9} \, , \, \tfrac{116}{23}$$

$$q_1 = - \tfrac{19}{6} \, , \, - \tfrac{305}{138}$$

$$g_{EM\,1} = - c_{mar} \, q_1 = \tfrac{38}{27} \sim 1.41 \, , \, \tfrac{610}{23*23} \sim 1.15$$

Universal quark mass rescaling – strengths and limits

With the criteria layed out in the last subsection we cast eq. 75 into the form

(79)
$$\overline{m} = m_0 \; \frac{EM\,(X_0)}{(X_0)^{c_{mar}}} \; \frac{\left(\overline{X}\right)^{c_{mar}}}{EM\,\left(\overline{X}\right)}$$

$\longrightarrow$

It is here important to maintain clarity of notions and I repeat the generic form of the function $EM\,(\,X\,)$ in eq. 75 below , using the values og the first two sets of renormalization coefficients for $N_{fl}\ =\ 3$ in eq. 78

$$EM\,(\,X\,)\ =\ \exp\,[\,-\,c_{mar}\,M\,(\,X\,)\,]$$

$$M\,(\,X\,)\ :\ \left\{\begin{array}{l}\text{universal , quark mass independent function}\\ \sim\ \sum_{n=1}^{\infty}\dfrac{1}{n}\ q_{n}\,(\,X\,)^{\,n}\quad\text{in perturbatively accessible region}\end{array}\right.$$

$\searrow$

(80)

$$EM\,(\,X\,)\ =\ 1\ +\ \mathcal{R}_{EM}\,(\,X\,)\ :\ \left\{\begin{array}{l}\text{universal , quark mass independent function}\\ \mathcal{R}_{EM}\ \sim\ \sum_{n=1}^{\infty}\ g_{EM\,n}\,(\,X\,)^{\,n}\\ \text{in perturbatively accessible region}\end{array}\right.$$

$$g_{EM\,1}\ =\ -\,c_{mar}\,q_{1}\ ,\ \cdots$$

$$N_{fl}\ =3\ :\ c_{mar}\ =\ \frac{4}{9}\ \ ;\ \ g_{EM\,1}\ =\ -\,c_{mar}\,q_{1}\ =\ \frac{38}{27}\ \ ,\ \ q_{1}\ =\ -\ \frac{19}{6}$$

$$N_{fl}\ =\ 5\ :\ c_{mar}\ =\ \frac{12}{23}\ ;\ \ g_{EM\,1}\ =\ -\,c_{mar}\,q_{1}\ =\ \frac{610}{529}\ ,\ q_{1}\ =\ -\ \frac{305}{138}$$

$\longrightarrow$

After an apparent detour we separate the universal repsonse function in the 'deep euclidean $\leftrightarrow$ perturbatively accessible' rescaling function of mass versus coupling constant and thus versus scale in the following way

(81)
$$\frac{\overline{m}}{m^{*}} = \exp\left(- c_{mar} \left[\log \frac{1}{\overline{X}} - M(\overline{X}) \right] \right)$$

$$M(X) \sim \sum_{n=1}^{\infty} \frac{1}{n} \, q_{n} (X)^{n}$$

The sliding scale is related to the coupling constant asw

(82)
$$Z = b_{0} \log\left(\overline{\mu}^{2} / \Lambda^{2}\right) \equiv b_{0}\, s \;\; ; \;\; X = \overline{\kappa}_{\overline{\mu}}$$

$$\Sigma_{asy}(X) = Z \quad \leftrightarrow \quad X = \overline{X}(Z)$$

$$\Sigma_{asy}(X) = \sigma_{asy}(X) + \mathcal{R}_{\sigma}(X)$$

$$\sigma_{asy} = X^{-1} - a_{1} \log(X^{-1}) \;\; ; \;\; \mathcal{R}_{\sigma}(X) \sim - \sum_{m=1}^{\infty} m^{-1} I_{m+1} X^{m}$$

$\longrightarrow$

The adopted two loop or second order approximation eqs. 81 and 82 amounts (for $N_{fl} = 3$) to the substitutions

(83)

$$M(X) \sim q_1 X = -\frac{19}{6} X$$

$$Z \sim X^{-1} - a_1 \log(X^{-1}) \longrightarrow$$

$$X^{-1} \sim Z + a_1 \log Z$$

$$q_1 = -\frac{19}{6}, -\frac{305}{138}, \quad a_1 = \frac{64}{9}, \frac{116}{23}$$

Eqs. 81 , 82 become

(84)

$$\frac{\overline{m}}{m^*} \sim \exp\left(-c_{mar}\left[\log\frac{1}{\overline{X}} - q_1 \overline{X}\right]\right)$$

$$Z = b_0 \log(\overline{\mu}^2 / \Lambda^2) \sim \overline{X}^{-1} - a_1 \log\left(\overline{X}^{-1}\right)$$

$$N_{fl} = 3 : b_0 = 9, \quad c_{mar} = \frac{4}{9}, \quad q_1 = -\frac{19}{6}, \quad a_1 = \frac{64}{9}$$

$$N_{fl} = 5 : b_0 = \frac{23}{3}, \quad c_{mar} = \frac{12}{23}, \quad q_1 = -\frac{305}{138}, \quad a_1 = \frac{116}{23}$$

We proceed to transform the second relation in eq. 84 $\rightarrow$

anchoring the mass scale at $\mu^* = 1$ GeV and normalizing the running strong coupling constant (square) relative to $\overline{\alpha}_s = 4\pi\overline{X}$

$$Z = 2 b_0 \left[\log \frac{\overline{\mu}}{\mu^*} + A \right] \quad ; \quad EA \equiv e^A = \frac{\mu^*}{\Lambda} \quad ; \quad \mu^* = 1 \text{ GeV}$$

(85) $$X = \frac{\alpha_s}{4\pi} \text{ and generic } X \rightarrow \overline{X} \ , \ \alpha_s \rightarrow \overline{\alpha}_s$$

$$\log \left[\frac{\overline{\mu}}{\mu^*} + A \right] \sim \left(\frac{2\pi}{b_0} \right) (\overline{\alpha}_s)^{-1} - \frac{a_1}{2 b_0} \log \left(\frac{4\pi}{\overline{\alpha}_s} \right)$$

We first show three figures : (1) repeating Fig A21 , (2) Fig A22 : comparing with the two loop approximate rescaling with the four loop based $\alpha_s(Q)$ on Fig A21 from ref. [A215-2009] , (3) Fig A23 : universally rescaled running quark masses with unspecified reference scale m^* and fixed ratios $m_d : \frac{1}{2}(m_d + m_u) : m_u = 5 : 4 : 3$.

More detailed description of these three figures is given subsequently . $\longrightarrow$

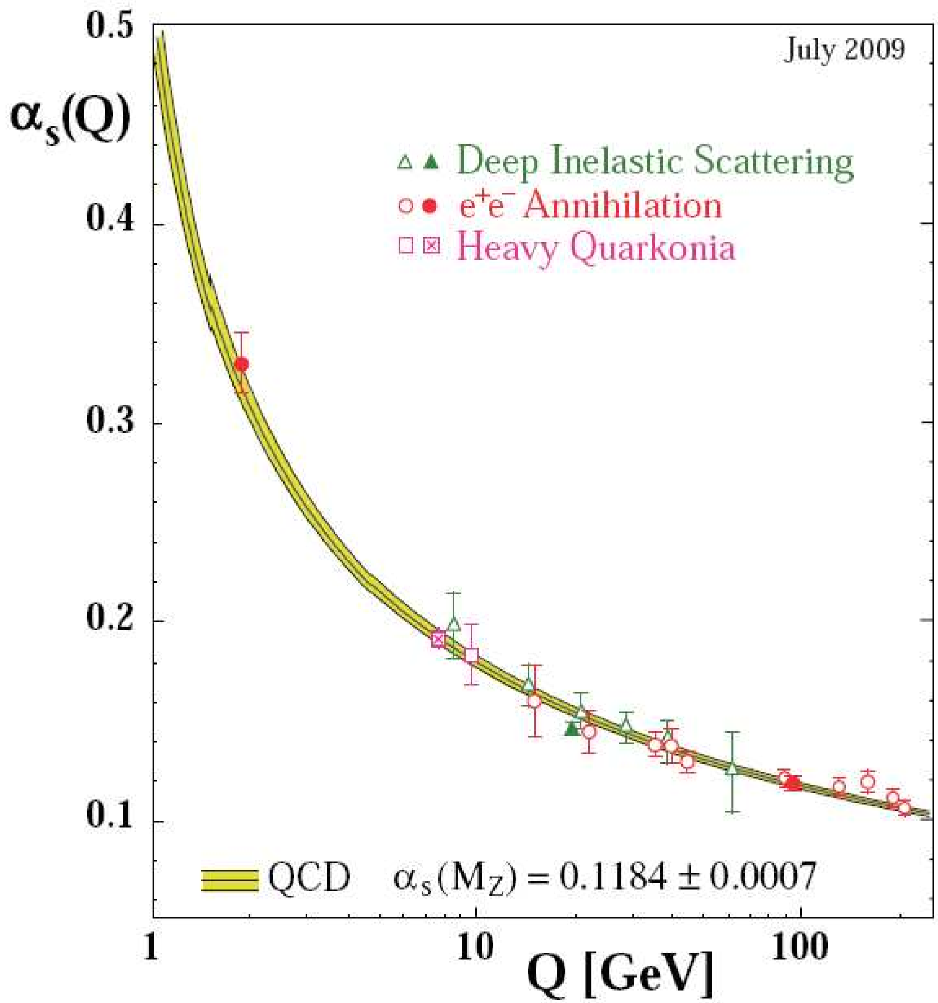

Fig A21 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from ref. [A215-2009] .**

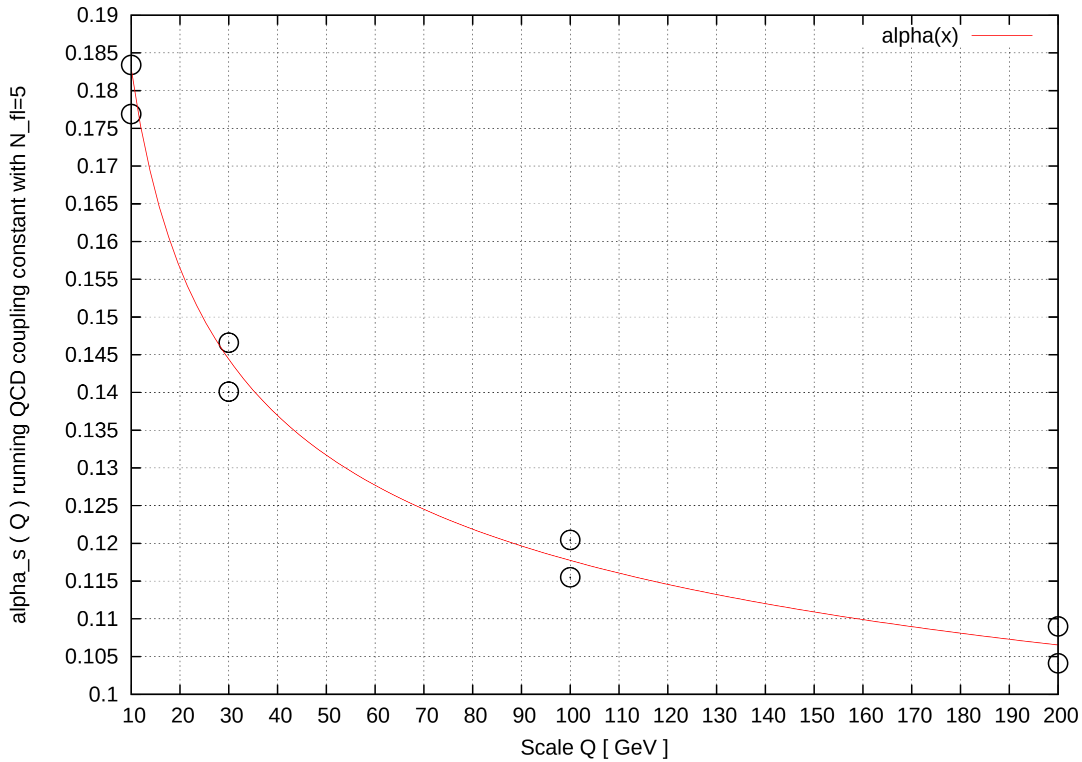

Fig A22 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from eq. 83 compared with Fig. A21 .**

$\longrightarrow$

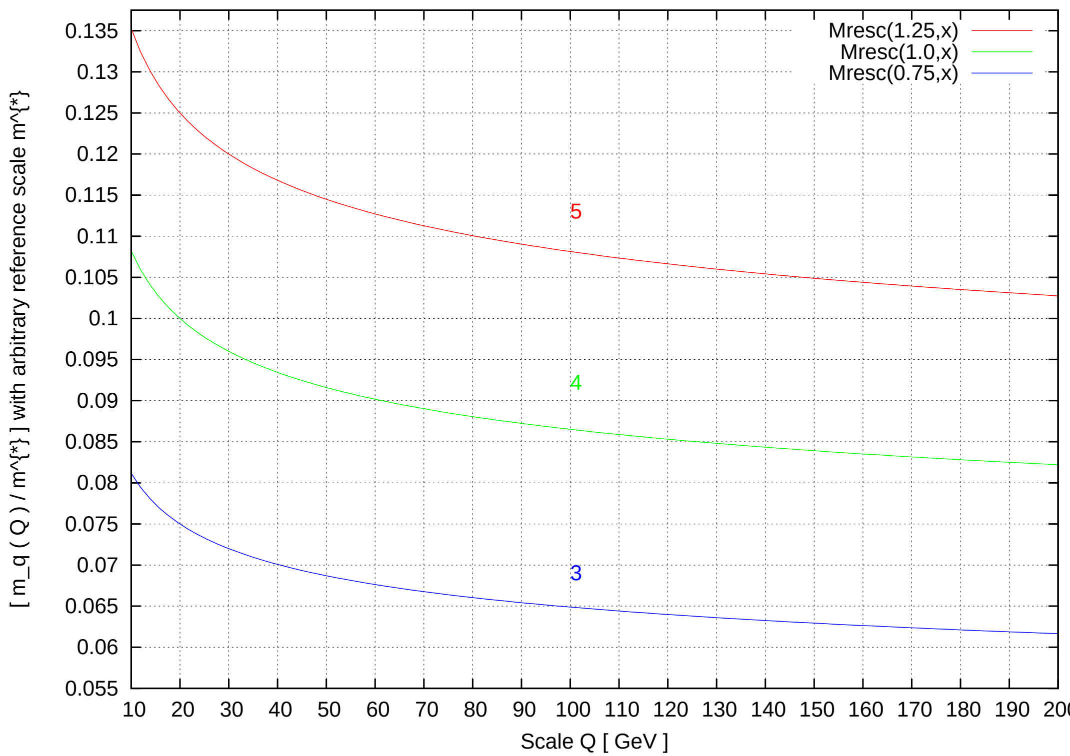

Fig A23 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$ **from eq. 81 .**

$\longrightarrow$

To Fig A21 : In the four loop evaluation of the running coupling constant in ref. [A215-2009] the renormalization group invariant quantity is obtained in the $\overline{MS}$ renormalization scheme

$$\Lambda_5^{(4)} = 213 \pm 9 \text{ MeV} \longleftrightarrow \alpha_s(m_Z) = 0.1184 \pm 0.0007$$

(86)
$$\lambda_5^{(4)} = 213 \text{ MeV} \rightarrow \begin{cases} \Lambda_4^{(4)} = 296 \text{ MeV} \\ \Lambda_3^{(4)} = 338 \text{ MeV} \end{cases}$$

The matching between $N_{fl} = 5 \rightarrow 4 \rightarrow 3$ in ref. [A215-2009] involves the modeling of the b- and c-flavor associated thresholds through the perturbatively assigned b- and c-quark *pole-* **masses $m_b = 4.7$ GeV , $m_c = 1.5$ GeV . This is a nonuniversal way to rescale quark masses , and thus does not follow the strict quark mass rescaling at zero quark mass , used here . As a comparison in determining up and down quark masses at an $\overline{MS}$ scale of 2 GeV , Dominguez, Nasrallah, Röntsch and Schilcher [A223-2008] use**

(87)
$$\Lambda_3^{(4)} = 381 \pm 16 \text{ MeV} \leftrightarrow \alpha_s(m_\tau) = 0.344 \pm 0.009$$

and adopting the scheme of quark mass rescaling at zero mass obtain for the u,d,s quark mass ratios

(88)
$$\begin{array}{ccccccc} m_u & : & \frac{1}{2}(m_d + m_u) & : & m_d & : & m_s \\ 2.9 \pm 0.2 & : & 4.1 \pm 0.2 & : & 5.3 \pm 0.5 & : & 102 \pm 8 \end{array}$$

$\rightarrow$

To Fig A22 : The following value was used : $\alpha_s(M_Z) = 0.184$ **corresponding – for the two loop running as defined in eq. 83 – to** $\Lambda_5^{(2)} \sim 408$ **MeV . The so determined running coupling constant is compared with the 1 -** σ **limits of the same quantity as determined in 4 loop order in ref. [A215-2009] confirming the validity of the two loop approximation in the range** 10 **GeV** $\leq Q \leq 200$ **GeV within the accuracy claimed in ref. [A215-2009] .**

To Fig A23 : Here the strength and weakness of the mass rescaling at zro mass within the perturbatively accessible region is illustrated using as a guide *only* the ratio of u,d quark masses

$$\begin{array}{ccccc} m_u & : & \frac{1}{2}(m_d + m_u) & : & m_d \\ 3 & : & 4 & : & 5 \end{array} \tag{89}$$

It seems appropriate to me to refer to the *in principle* approach of rescaling in a universal way the coupling constant and quark masses *initially* restricting all analysis to the perturbatively accessible region , citing (adapting) the pertinent comment by Murray Gell-Mann :
'Rising when last (first) seen .'
On the other hand the progress achieved in transgressing the perturbatively accessible region , using universal mass rescaling , in refs. [A223-2008] , [A221-2006] and references cited therein, is significant, based on improved treatment of finite energy sum rules pioneered by Shifman , Vainshtain and Zakharov [A224-1979] . $\longrightarrow$

To Fig A23 *continued* **: It is worth noting that the value of the gauge boson condensate, found in ref. [A223-2008] , approximated as**

(90) $$\langle \Omega \mid \frac{\alpha_s}{\pi} : F^A_{\mu\nu} F^{\mu\nu\, A} : \mid \Omega \rangle \to 0.06 \text{ GeV}^4$$

is 5 times larger , than its original estimate in ref. [A224-1979] .

The basics of chiral expansions in assessing ratios of the u,d,s quark masses continue to provide additional benchmarks at low hadron energies [A225-2001] and references cited therein, while fine details of these ratios can be subject to improvement . Finally the validity of chiral expansions as guidelines for lattice calculations present another *strategy in principle* **[A226-2008] .**

We add here a few representative determination of $\alpha_s(m_Z)$

	$\alpha_s(m_Z)$	processes	source	authors
(91)	0.1176 ± 0.0020	average	[A227-2008]	PDG
	0.1172 ± 0.0022	thrust distributions at LEP	[A228-2008]	Becher , Schwartz
	0.1184 ± 0.0007	average	[A215-2009]	Bethke

Outlook

1) The consolidation of the perturbatibely accessible region and analyzable hard processes up to 4 and 5 loop order has seen a remarkable consolidation .

2) The asymptotic expansion brings about universal rescaling of running coupling constant and mass renormalized 'at zero quark masses' .

3) Nevertheless the need for renormalization group invariant definitions of operators determining the two central anomalies – scale and axial U1 current anomalies – necessitates to transgress the perturbative domain , which by itself yields at first sight very distorted such renormalizatin group invariant scales .

4) This leaves demanding questions, hitherto unanswered, and thus opens aspiration and drive towards further insights .

— Thank you —

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