THE HAMILTONIAN APPROACH TO YANG-MILLS (2+1)

UPDATE & CORRECTIONS TO STRING TENSION

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Why is YM(2+1) interesting?

- Interesting in its own right
  - YM(1+1) is exactly solvable, but has no propagating degrees of freedom
  - YM(3+1) is highly nontrivial and difficult
  - YM(2+1) has propagating degrees of freedom, it is nontrivial. Can be amenable to a Hamiltonian analysis.
  - It has a dimensional coupling constant and is super-renormalizable. This helps to simplify it.
- A real physical context for YM(2+1)
  - Mass gap of YM(2+1) $\approx$ Magnetic screening mass of YM(3+1) at high temperatures

• We will use a Hamiltonian approach because some exact calculations are possible

Collaborators: Dimitra Karabali, Chanju Kim, Abhishek Agarwal, Alexander Yelnikov

The key points of our analysis are:

1. Parametrization of potentials: $A_0 = 0$ and we use complex coordinates $z = x_1 - ix_2$ with

$$\tfrac{1}{2}(A_1 + iA_2) = -\partial M\, M^{-1}, \quad \tfrac{1}{2}(A_1 - iA_2) = M^{\dagger -1}\bar{\partial} M^{\dagger}$$

$M \in SL(N, \mathbb{C})$, for gauge group $SU(N)$. (More generally, $G \Rightarrow G^{\mathbb{C}}$.)

2. Wave functions are gauge-invariant (Gauss law) and depend on $H = M^{\dagger}M$ with the inner product

$$\langle 1|2\rangle = \int d\mu(H) \exp[2\, c_A\, S_{wzw}(H)]\; \Psi_1^* \Psi_2$$

$$S_{wzw}(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H\, \bar{\partial} H^{-1}) + \frac{i}{12\pi}\int \epsilon^{\mu\nu\alpha}\mathrm{Tr}(H^{-1}\partial_\mu H\, H^{-1}\partial_\nu H\, H^{-1}\partial_\alpha H)$$

$d\mu(H)$ = Haar measure for H; $c_A = N$ = adjoint Casimir value.

3. $H \in SL(N, \mathbb{C})/SU(N)$ is the basic gauge-invariant variable for the theory.

4. The Hamiltonian and the wave functions can be expressed as functions of the (scaled version of the) current $J = (2/e)\partial H\, H^{-1}$.

5. The Wilson loop operator is given by

$$W(C) = \mathrm{Tr}\, \mathcal{P} e^{-\oint A} = \mathrm{Tr}\, \mathcal{P} \exp\left(\frac{e}{2} \oint J \right)$$

All gauge-invariant quantities can be made from J.

6. The Hamiltonian is $\mathcal{H} = \mathcal{H}_0 \;+\; \mathcal{H}_1$ with

$$\mathcal{H}_0 = m \int_z J_a(\vec{z}) \frac{\delta}{\delta J_a(\vec{z})} + \frac{2}{\pi} \int_{z,w} \frac{1}{(z-w)^2} \frac{\delta}{\delta J_a(\vec{w})} \frac{\delta}{\delta J_a(\vec{z})}$$
$$+ \frac{1}{2} \int_x :\bar{\partial} J^a(x)\, \bar{\partial} J^a(x):$$
$$\mathcal{H}_1 = i\, e\, f_{abc} \int_{z,w} \frac{J^c(\vec{w})}{\pi(z-w)} \frac{\delta}{\delta J_a(\vec{w})} \frac{\delta}{\delta J_b(\vec{z})}$$

where $m = e^2 c_A / 2\pi$.

All calculations have to be done with proper regularization.

- We start with a regularization of the δ-function

$$\delta^{(2)}(u,w) \Longrightarrow \sigma(\vec{u},\vec{w},\epsilon) = \frac{1}{\pi\epsilon}\exp\left(-\frac{|u-w|^2}{\epsilon}\right)$$

- This is equivalent to

$$\begin{aligned} \bar{G}(\vec{x},\vec{y}) &= \frac{1}{\pi(x-y)} \\ \Longrightarrow \quad \bar{\mathcal{G}}(\vec{x},\vec{y}) &= \int_u \bar{G}(\vec{x},\vec{u})\,\sigma(\vec{u},\vec{y};\epsilon)H(u,\bar{y})H^{-1}(y,\bar{y}) \end{aligned}$$

- This simplifies as

$$\overline{\mathcal{G}}_{ma}(x,y) = \frac{1}{\pi(x-y)}\left[\delta_{ma} \;-\; e^{-\frac{(x-y)^2}{\epsilon}}\;[H(x,\bar{y})H^{-1}(y,\bar{y})]_{ma}\right]$$

All results checked using regularized expressions, with a single regulator from beginning to end.

7. By comparison to a resummed perturbation theory, m can be identified as the magnetic mass, relevant for the QCD plasma.
8. For $SU(2)$, this value is $m \approx 0.32\, e^2$
9. Compare with other estimates:

$m/e^2 =$	0.35	Common factor for glueball masses (lattice, PHILIPSEN)
	0.51	Max. Abelian gauge (lattice, KARSCH ET AL)
	0.52	Landau gauge (″)
	0.44	$\lambda_3 = 2$ gauge (″)
	0.38	Resummation of P.T. (ALEXANIAN & VPN)
	0.28	Resummation of P.T. (BUCHMULLER & PHILIPSEN, JACKIW & PI)
	0.37	Gauge-invariant lattice definition (PHILIPSEN)

10. The vacuum wave function was found as $\Psi_0 = \exp\left[\frac{1}{2}F(H)\right]$,

$$F(H) = -\int \bar{\partial} J_a \left[\frac{1}{(m + \sqrt{m^2 - \nabla^2})}\right] \bar{\partial} J_a + 2\, f_{abc} \int f^{(3)}(\vec{x}, \vec{y}, \vec{z})\, J_a(\vec{x})\, J_b(\vec{y})\, J_c(\vec{z}) + \mathcal{O}(J^4)$$

11. This gave values of string tension

$$\sqrt{\sigma}_R = e^2 \sqrt{\frac{c_A c_R}{4\pi}},$$

in agreement with lattice values to within $1 - 3\%$.

Group	Representations					
	k=1 Fund.	k=2 antisym	k=3 antisym	k=2 sym	k=3 sym	k=3 mixed
$SU(2)$	0.345 0.335					
$SU(3)$	0.564 0.553					
$SU(4)$	0.772 0.759	0.891 0.883		1.196 1.110		
$SU(5)$	0.977 0.966					
$SU(6)$	1.180 1.167	1.493 1.484	1.583 1.569	1.784 1.727	2.318 2.251	1.985 1.921
$SU(N)$ $N \to \infty$	0.1995 N 0.1976 N					

Comparison of $\sqrt{\sigma}/e^2$ with lattice estimates (lower entry, in red) from LUCINI &TEPER, BRINGOLTZ & TEPER. k is the rank of the representation.

- Absorb $\exp(2c_A S_{wzw})$ from the inner product into the wave function by $\Psi = e^{-c_A S_{wzw}(H)} \Phi$.

  The Hamiltonian acting on Φ is

$$\mathcal{H} \to e^{-c_A S_{wzw}(H)}\, \mathcal{H}\, e^{-c_A S_{wzw}(H)}$$

- Consider $H = e^{t^a \varphi^a} \approx 1 + t^a \varphi^a + \cdots$, a small φ limit appropriate for a (resummed) perturbation theory. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \int \left[-\frac{\delta^2}{\delta \phi^2} + \phi(-\nabla^2 + m^2)\phi + \cdots \right]$$

  where $\phi_a(\vec{k}) = \sqrt{c_A k \bar{k} / (2\pi m)}\, \varphi_a(\vec{k})$.

- The vacuum wave function is

$$\Phi_0 \approx \exp \left[-\frac{1}{2} \int \phi^a \sqrt{m^2 - \nabla^2}\, \phi^a \right]$$

- Transforming back to Ψ,

$$\Psi_0 \approx \exp\left[-\frac{c_A}{\pi m} \int (\bar{\partial}\partial\varphi^a) \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right] (\bar{\partial}\partial\varphi^a) + \cdots \right]$$

- The full wave function must be a functional of J. The only form consistent with the above is

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2 c_A^2} \int \bar{\partial} J^a(x) \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right]_{x,y} \bar{\partial} J^a(y) \; + \cdots \right]$$

since $J \approx (c_A/\pi)\partial\varphi + \mathcal{O}(\varphi^2)$.

- This indicates the robustness of the Gaussian term in Ψ_0, since this argument only presumes
  1. Existence of a regulator, so that the transformation $\Psi \Longleftrightarrow \Phi$ can be carried out
  2. The two-dimensional anomaly calculation

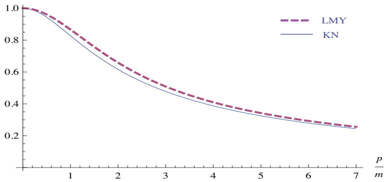
12. Other developments include

- Analysis of glueball states (LEIGH, MINIC, YELNIKOV)
  - The vacuum wave function is taken to be

$$\Psi_0 = \exp\left[-\frac{1}{4m} \int \bar{\partial}J \, K[L] \, \bar{\partial}J\right]$$

$$K[L] = \frac{J_2(4\sqrt{L})}{\sqrt{L}\, J_1(4\sqrt{L})}$$

    Here $L = \mathcal{D}\bar{\partial}/m^2$, and J_1, J_2 are Bessel functions of orders 1 and 2 respectively.
  - The kernel, despite appearances, is very close to the KKN kernel.



A number of glueball masses are obtained in this way

State	LMY Calculation	Lattice
0^{++}	4.098	4.065 ± 0.055
0^{++*}	5.407	6.18 ± 0.13
0^{++**}	6.716	7.99 ± 0.22
0^{++***}	7.994	9.44 ± 0.38
0^{--}	6.15	5.91 ± 0.25
0^{--*}	7.46	7.63 ± 0.37
0^{--**}	8.77	8.96 ± 0.65
2^{++}	6.72	6.88 ± 0.16
2^{++*}	7.99	8.62 ± 0.38
2^{++**}	9.26	9.22 ± 0.32
2^{+-}	8.76	8.04 ± 0.50
2^{--}	8.76	7.89 ± 0.35
2^{+-*}	10.04	9.97 ± 0.91
2^{--*}	10.04	9.46 ± 0.46

Lattice values are from MEYER & TEPER and TEPER

13. Extension to YM-Chern-Simons theory (KARABALI, KIM, NAIR)
14. Screening of adjoint representation and string breaking (AGARWAL, KARABALI, NAIR)
    - A glue-lump state is given by

$$\Psi_G = \int d^2x d^2y\, f(\vec{x}, \vec{y})\, \bar{\partial} J^a(\vec{x}) \left[\mathcal{P} e^{\int_y^x \partial H H^{-1}} \right]^{ab} \chi^b(\vec{y}) \;\; \Psi_0$$

    $\chi = M^\dagger \phi M^{\dagger -1} \;=\;$ A heavy scalar field of mass μ

    - We obtain an eigenstate of the Hamiltonian if $f(x, y)$ obeys the equation

$$\left[\mu + m - \frac{\nabla_x^2}{2m} + \sigma_A |\vec{x} - \vec{y}| \right] f(\vec{x}, \vec{y}) \approx E\, f(\vec{x}, \vec{y})$$

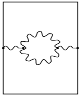
    (This equation is valid when e^2 is large, $\mu \to \infty$ and $N \to \infty$.)

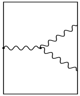
    - A variational estimate gives $E_* \approx 7.92\, m$.
    - Differs by $\sim 8.8\%$ from the lattice estimate of $\approx 8.68\, m$ for $SU(2)$ (DE FORCRAND & KRATOCHVILA).
15. Extension to $\mathbb{R} \times S^2$ (AGARWAL, NAIR)
16. Attempts to understand deconfinement (ABE)

- The large N limit of string tension differs from lattice value by only $\sim 1\%$ (BRINGOLTZ & TEPER). A different lattice calculation, by a different method, gives $0.1964\,N$ at large N, deviation of $\sim 1.55\%$ (KISKIS & NARAYANAN).
- The difference is small, but considered statistically significant.
- Two types of corrections possible
  - Corrections to coupling, numerical and R-independent $\Rightarrow$ ratios σ_R/σ_F unaffected.

    For this, we will calculate the correction to the coefficient of $\bar{\partial}J\bar{\partial}J$ in Ψ_0 in a systematic expansion scheme (KARABALI, NAIR, YELNIKOV, arXiv: 0906.0783)

  - Corrections via new diagrams to Wilson line expectation value (under investigation, related to screening and string breaking)





- Solve Schrödinger equation as a power series in e, treating m and e as independent parameters
- This gives $\Psi_0^* \Psi_0 = e^F$ with

$$F = \int f^{(2)}_{a_1 a_2}(x_1, x_2)\; J^{a_1}(x_1) J^{a_2}(x_2) \;+\; \frac{e}{2} f^{(3)}_{a_1 a_2 a_3}(x_1, x_2, x_3)\; J^{a_1}(x_1) J^{a_2}(x_2) J^{a_3}(x_3)$$
$$+\; \frac{e^2}{4} f^{(4)}_{a_1 a_2 a_3 a_4}(x_1, x_2, x_3, x_4)\; J^{a_1}(x_1) J^{a_2}(x_2) J^{a_3}(x_3) J^{a_4}(x_4) \;+\; \cdots$$

  $f^{(2)}, f^{(3)}$, etc., are determined recursively.
- Integration over J's can be expressed as the functional integral of a two-dimensional field theory for a complex unconstrained bosonic field.
- Evaluate loop corrections to the action in this two-dimensional field theory
- Group loop corrections by powers of $m/E_k \equiv m/\sqrt{k^2 + m^2}$, calculate all contributions to a given order in m/E_k.
- Set $m = e^2 c_A / 2\pi$ at the end.

- Each function $f^{(n)}$ has a series expansion in powers of e^2,

$$f^{(2)}_{\;a_1a_2}(x_1,x_2) = f^{(2)}_{0\;a_1a_2}(x_1,x_2) \;+\; e^2 f^{(2)}_{2\;a_1a_2}(x_1,x_2) \;+\cdots$$
$$f^{(3)}_{0\;a_1a_2a_3}(x_1,x_2,x_3) = f^{(3)}_{0\;a_1a_2a_3}(x_1,x_2,x_3) \;+\; e^2 f^{(3)}_{2\;a_1a_2a_3}(x_1,x_2,x_3) \;+\cdots$$
$$f^{(4)}_{0\;a_1\ldots a_4}(x_1,\ldots,x_4) = f^{(4)}_{0\;a_1\ldots a_4}(x_1,\ldots,x_4) \;+\; e^2 f^{(4)}_{2\;a_1\ldots a_4}(x_1,\ldots,x_4) \;+\cdots$$

- To the lowest order,

$$f^{(2)}_{0\;a_1a_2}(x_1,x_2) = \delta_{a_1a_2}\left[-\frac{\bar{q}^2}{m+E_q}\right]_{x_1,x_2}$$

- We calculate corrections to order e^2. For this, we need $f^{(3)}$ and $f^{(4)}$ to the lowest nontrivial order in e. For example,

$$f^{(3)}_{0\;a_1a_2a_3}(k_1,k_2,k_3) = -\frac{f^{a_1a_2a_3}}{24}\,(2\pi)^2\delta(k_1+k_2+k_3)\,g^{(3)}(k_1,k_2,k_3)$$
$$g^{(3)}(k_1,k_2,k_3) = \frac{16}{E_{k_1}+E_{k_2}+E_{k_3}}\left\{\frac{\bar{k}_1\bar{k}_2(\bar{k}_1-\bar{k}_2)}{(m+E_{k_1})(m+E_{k_2})} + cycl.\,perm.\right\}$$

There is a similar, but more complicated, formula for $f^{(4)}$.

- The correction to $f^{(2)}$ from the recursive solution of the Schrödinger equation is

$$e^2 f_2^{(2)}(q) = \frac{m}{E_q}\left(\int \frac{d^2k}{32\pi}\ \frac{1}{\bar{\bar{k}}}\, g^{(3)}(q,k,-k-q)\ +\ \int \frac{d^2k}{64\pi}\ \frac{k}{\bar{\bar{k}}}\, g^{(4)}(q,k;-q,-k)\right)$$
$$\approx \frac{\bar{q}^2}{2m}\,(1.1308) + \ldots$$

(We have set $e^2 = 2\pi m/c_A$.)

- Seemingly, this is about 113% correction, but other "loop-contributions" are important.
- The expectation value of an observable $\mathcal{O}$ is

$$\langle \mathcal{O} \rangle = \int d\mu(H) e^{2c_A S_{wzw}(H)}\ \Psi_0(J)\Psi_0(J)\ \mathcal{O}(J)$$
$$= \int \underbrace{d\mu(H)\ e^{2c_A S_{wzw}(H)}}\ e^{F(J)}\ \mathcal{O}(J)$$
$$= \int [d\varphi d\bar{\varphi}]\ e^{-S(\varphi)}\ \mathcal{O}(\sqrt{2\pi/mc_A}\ \bar{\varphi} t^a \varphi)$$

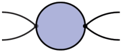
- The action $S(\varphi)$ is given by

$$S(\varphi) = \int \left(Z_2 \bar{\varphi} \bar{\partial} \varphi + Z_1 \bar{\varphi} \bar{C} \varphi\right) - F(Z_1 \sqrt{2\pi / m c_A}\, \bar{\varphi} t^a \varphi)$$

($\bar{C}$ can be set to 0 at this point.)

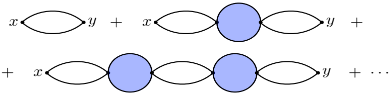
- Notice that $J^a \to Z_1 \sqrt{2\pi / m c_A}\, \bar{\varphi} t^a \varphi$.
- $F(Z_1 \sqrt{2\pi / m c_A}\, \bar{\varphi} t^a \varphi)$ contains vertices, $F^{(2)}$ with two currents (quartic in φ, $\bar{\varphi}$), $F^{(3)}$ with three currents, etc. For example,

$$F^{(2)} = \frac{2\pi}{m c_A} \int (\bar{\varphi} t^a \varphi)_x f^{(2)}(x, y) (\bar{\varphi} t^a \varphi)_y =$$



- We should calculate the corrected quartic vertex, i.e., correction to $F^{(2)}$ (like a Wilsonian RG reduction).
- Vertices $F^{(3)}$, $F^{(4)}$, etc., can be treated in a perturbation series (loop expansion)
- But we need to sum all contributions due to $F_0^{(2)}$, because it does not have e^2.

$\langle\bar{\varphi}t^a\varphi(x)\ \bar{\varphi}t^b\varphi(y)\rangle =$



- The current correlator becomes

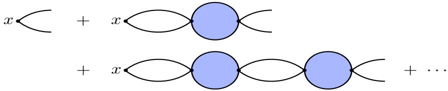
$$\langle\bar{\varphi}t^a\varphi(x)\ \bar{\varphi}t^b\varphi(y)\rangle = \delta^{ab}\frac{c_A}{\pi}\int\frac{d^2k}{(2\pi)^2}e^{ik(x-y)}\ \frac{k}{\bar{k}}\left(\frac{m}{E_k}\right)$$

- The effective current vertex is

$$\bar{\varphi}t^a\varphi(x)\Big]_{eff} = \int\frac{d^2k}{(2\pi)^2}e^{ik(x-z)}\ \frac{m}{E_k}\ (\bar{\varphi}t^a\varphi)(z)$$

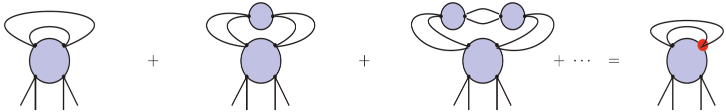
as given by the diagram

$\bar{\varphi}t^a\varphi(x)\Big]_{eff} =$



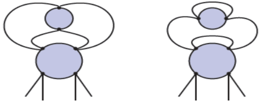
STRATEGY:

- Construct loop diagrams generated by $F^{(3)}$ (3 factors of $\bar{\varphi}t^a\varphi$) and $F^{(4)}$ (4 factors of $\bar{\varphi}t^a\varphi$).
- They can have arbitrary insertions of $F^{(2)}$'s; e.g.,



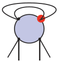
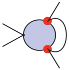
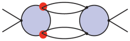
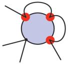
This leads to a factor of m/E_k.

- There are also renormalizations (due to $F^{(2)}$) we have to take into account; e.g.,



- Sum up $F^{(2)}$ insertions in all diagrams (of order e^2) generated by $F^{(3)}$ and $F^{(4)}$.
- Classify and group these by the number of factors of m/E_k.
- We will compute corrections of order e^2 and up to 4 powers of m/E_k.

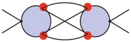
CORRECTIONS TO $F^{(2)}$

Diagram	*Powers of m/E_k*	*Low momentum value*
	m/E_k	$\frac{\bar{q}^2}{2m}(-0.58118)+\cdots$
	$(m/E_k)^2$	$\frac{\bar{q}^2}{2m}(-0.47835)+\cdots$
	$(m/E_k)^2$	$\frac{\bar{q}^2}{2m}(0.20169)+\cdots$
	$(m/E_k)^3$	$\frac{\bar{q}^2}{2m}(-0.23569)+\cdots$

Red dots indicate factors of m/E_k due to resummation of $F_0^{(2)}$ insertions.

CORRECTIONS TO $F^{(2)}$ (cont'd.)

There are 6 diagrams at the $(m/E_k)^4$-order:



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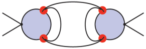
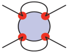
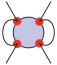


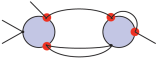
$\frac{\bar{q}^2}{2m}(0.02083) + \cdots$



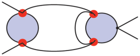
$\frac{\bar{q}^2}{2m}(-0.06893) + \cdots$



$\frac{\bar{q}^2}{2m}(-0.01216) + \cdots$



$\frac{\bar{q}^2}{2m}(0.06824) + \cdots$



$\frac{\bar{q}^2}{2m}(-0.1037)$ to (-0.166)

- We can now sum up the contributions, including the correction from the recursion rules,

$$\Delta f^{(2)}_{Rec} \approx \frac{\bar{q}^2}{2m}(1.1308) + \cdots$$

- Let C_n be the sum of corrections up to terms with $(m/E_k)^n$
- These partial sums are

$$C_0 = 1.13082$$

$$C_1 = 0.54964$$

$$C_2 = 0.27298$$

$$C_3 = 0.03729$$

$$C_4 = \begin{cases} -0.05843 \\ -0.00583 \end{cases}$$

- Remarks
  - The partial sums are systematically decreasing in value
  - The corrections to this order are small, much smaller than might be expected on parametric grounds or by computing one or two diagrams.

- The correction to string tension is then

$$\sqrt{\sigma_R} = e^2 \sqrt{\frac{c_A c_R}{4\pi}} \begin{cases} (1 - 0.02799 + \cdots) \\ (1 - 0.0029 + \cdots) \end{cases}$$

- This correction, of the order of -2.8% to -0.03% is entirely consistent with lattice calculations.
- Terms of order $(m/E_k)^5$ are expected to contribute only at the level of a fraction of 1%.
- Diagrams with two current-loops will also have powers of (m/E_k), so their contributions might also be small.

A systematic expansion scheme is possible for $YM(2+1)$, despite lack of obvious parameters, with calculable small corrections to string tension.

- A gauge-invariant Hamiltonian formulation
- Vacuum wave function is of the form

$$\Psi_0 = \exp\left[-\frac{2\pi^2}{e^2 c_A^2} \int \bar{\partial} J^a(x) \left[\frac{1}{m + \sqrt{m^2 - \nabla^2}} \right]_{x,y} \bar{\partial} J^a(y) \; + \cdots \right]$$

- A systematic expansion scheme for which this is the lowest order result
- String tensions: lowest order and corrections of order a few percent
- Possibility of screening of $W_R(C)$ via string-breaking in some representations
- Magnetic screening mass
- Some results on glueballs
- Yang-Mills-Chern-Simons Theory
- Formulation on $\mathbb{R} \times S^2$

- More accurate calculation of higher order corrections to string tension
- Better handle on glueballs
- Calculations on the torus can help understand the theory at finite temperature
- Connecting the formulation on $\mathbb{R} \times S^2$ to the duality-matrix model approach by changing the radius of S^2?
- Fermions, supersymmetric cases
- Geometrical properties of the configuration space $\mathcal{A}/\mathcal{G}_*$