Vacuum structure and Casimir scaling in Yang-Mills theories

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(results obtained in collaboration with
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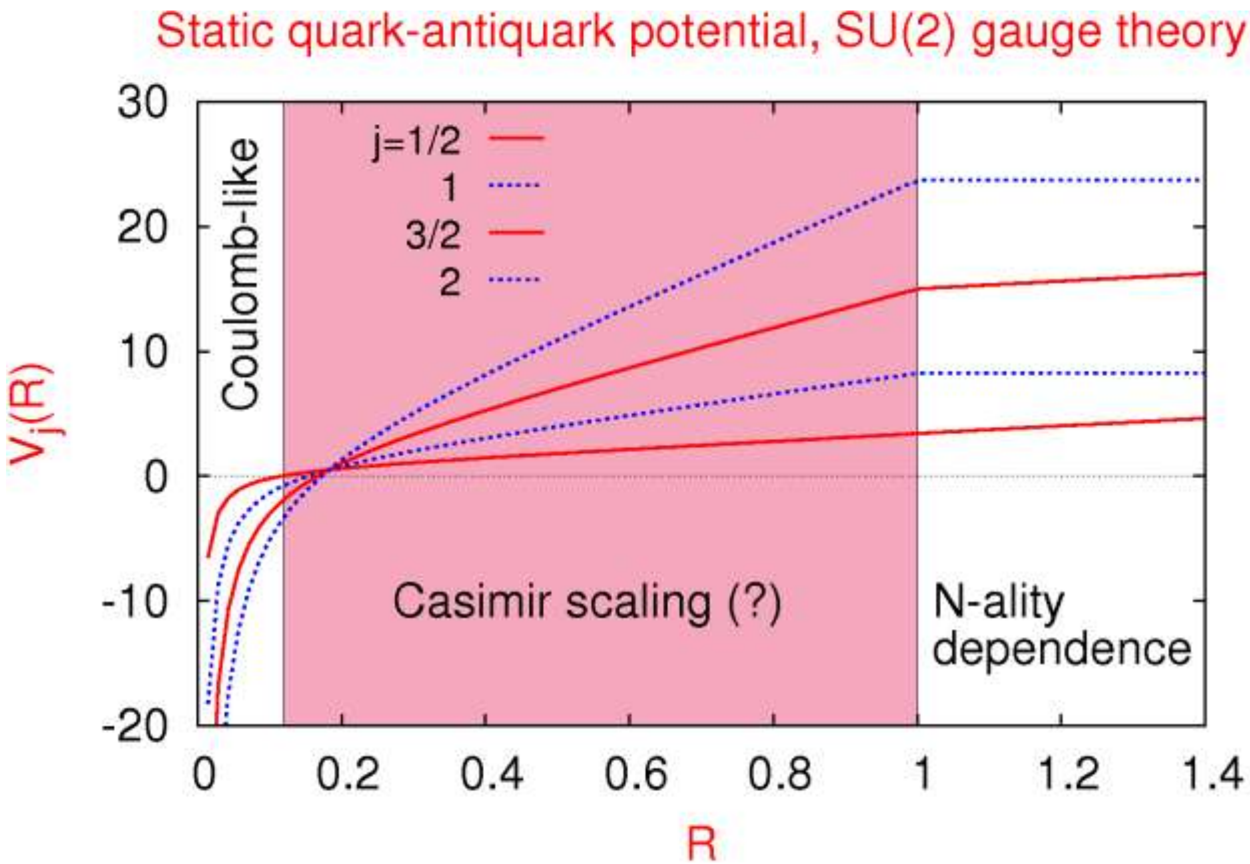

Static quark-antiquark potential, SU(2) gauge theory
j=1/2
1
3/2
2
Coulomb-like
Casimir scaling (?)
N-ality
dependence
$V_j(R)$
R
30
20
10
0
-10
-20
0
0.2
0.4
0.6
0.8
1
1.2
1.4

Casimir scaling hypothesis

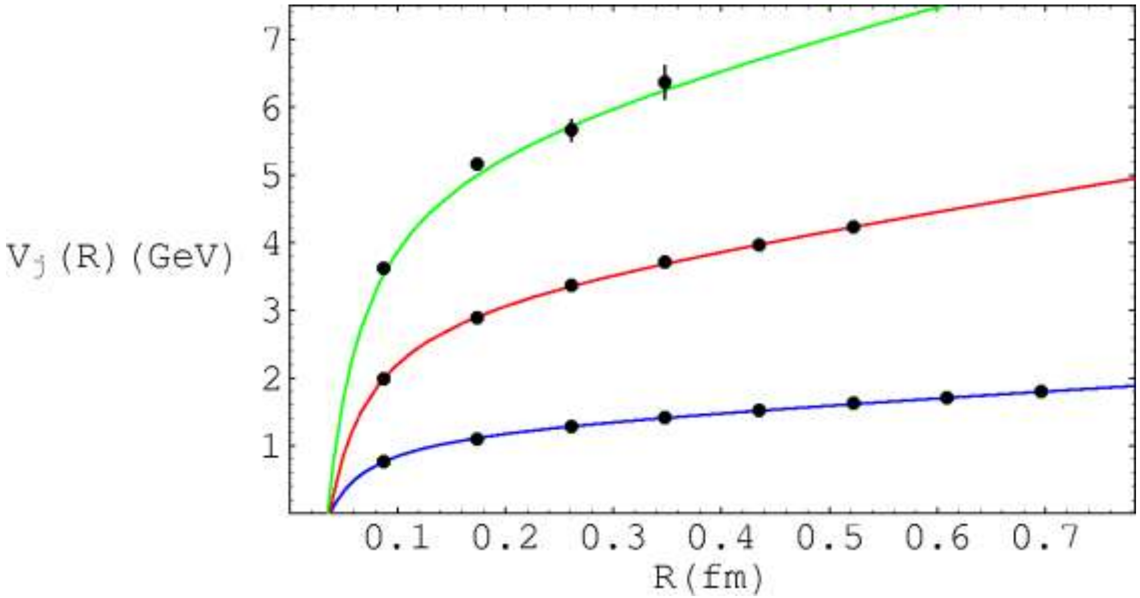
- At intermediate distances the string tension between charges in representation r is proportional to C_r.
- **Argument:** Take a planar Wilson loop, integrate out fields out of plane, expand the resulting effective action:

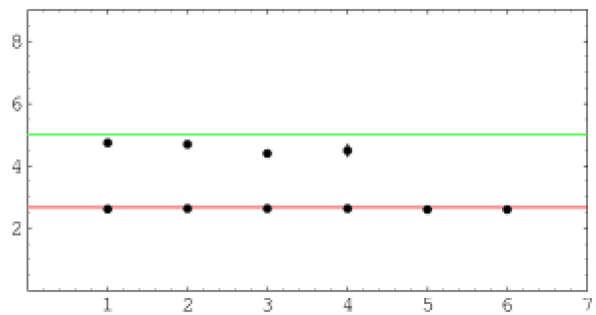
$$W_r(C) = \frac{1}{Z}\int DA_x(x,y)DA_y(x,y)\chi_r[U(C)]\,\mathrm{exp}(-S_{\mathrm{eff}})$$

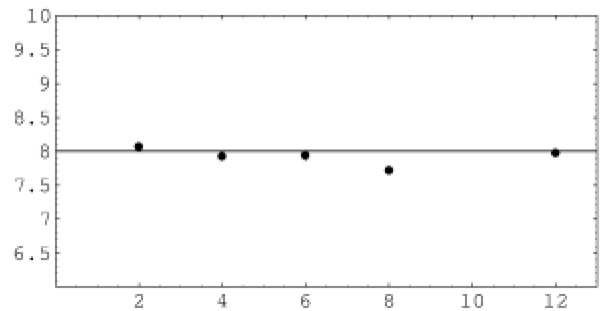
$$S_{\mathrm{eff}} = \int d^2x\left[c_0\mathrm{Tr}(F^2) + c_1\mathrm{Tr}(D_\mu F D_\mu F) + \ldots\right]$$

- Truncation to the first term gives Casimir scaling automatically.
- A challenge is to explain both Casimir and N-ality dependence in terms of vacuum fluctuations which dominate the functional integral.
  - Shevchenko, Simonov, arXiv:hep-ph/0104135

Casimir scaling – lattice evidence, SU(2)

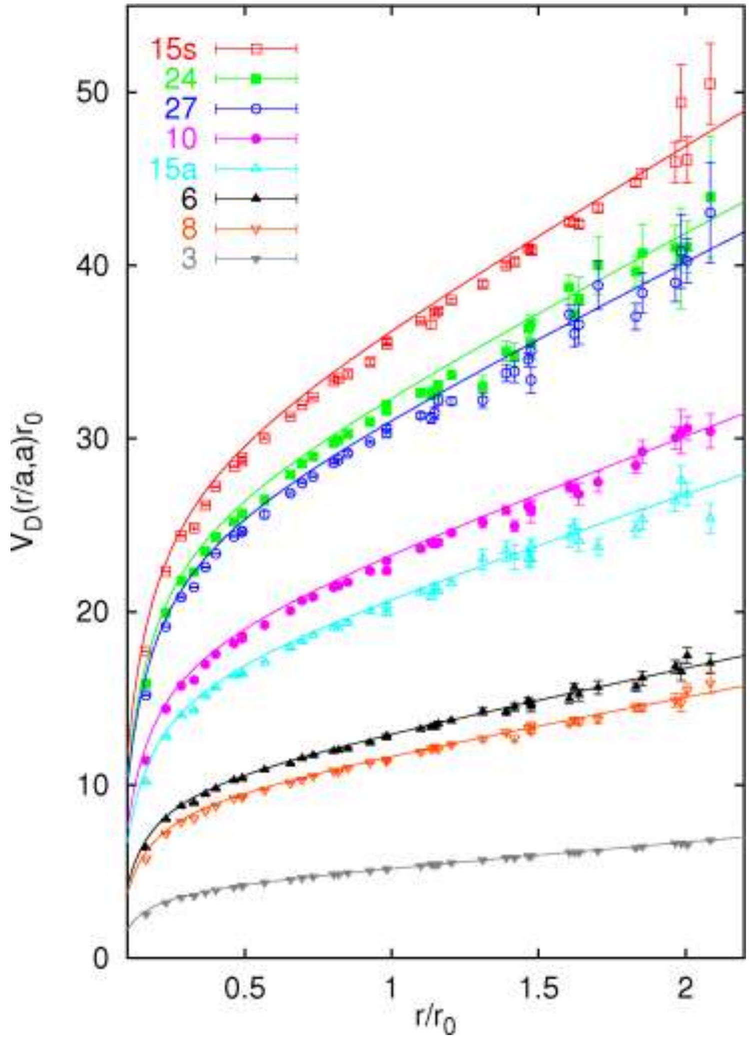


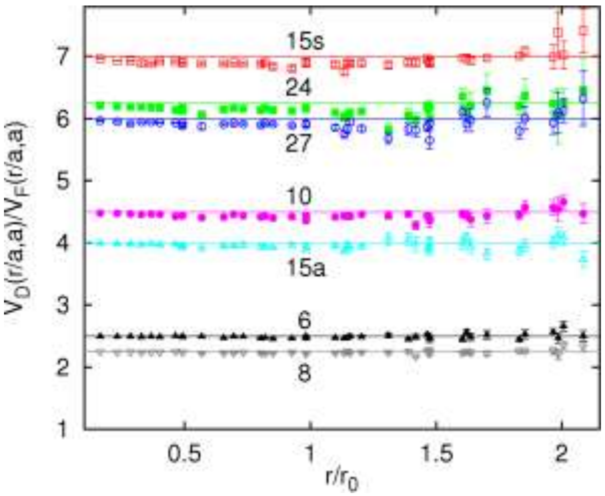




- Piccioni, arXiv:hep-lat/0503021

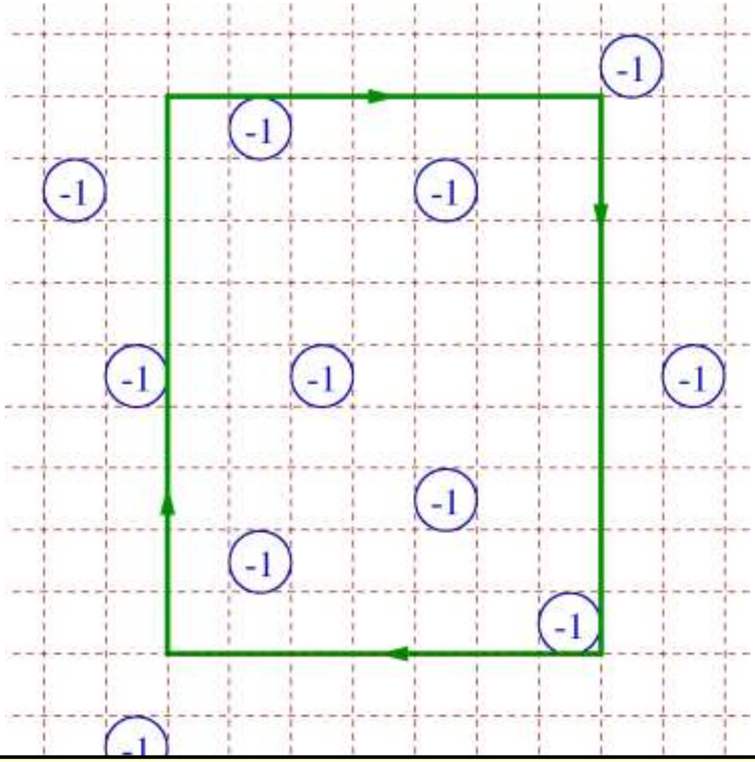
Casimir scaling – lattice evidence, SU(3)





- Bali, arXiv:hep-lat/0006022

N-ality ("biality") dependence from thin center vortices



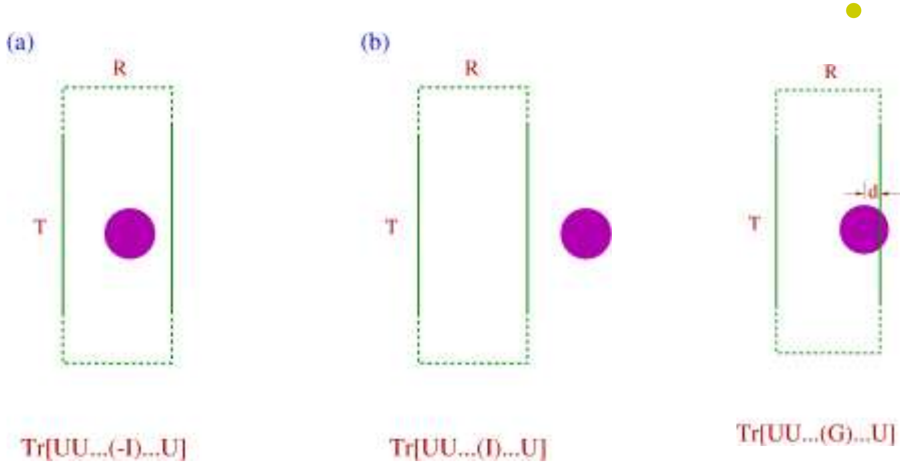
$$W_j(\mathcal{C}) = \left[p \cdot (-1)^{2j} + (1-p) \cdot (+1)\right]^{\mathcal{A}(\mathcal{C})}$$

$$\sigma_j = -\ln\left[1 - p + (-1)^{2j} p\right] \approx \begin{cases} 2p & \dots \text{ half-integer } j \\ 0 & \dots \text{ integer } j \end{cases}$$

A simple model of Casimir scaling and color screening from thick center vortices percolating in the QCD vacuum

- **Casimir scaling** results from uncorrelated (or short-range correlated) fluctuations on a surface slice (piercings of vortices with the Wilson loop).
- **Color screening** comes from center domain formation.
- **Idea:** On a surface slice, YM vacuum is dominated by overlapping center domains. Fluctuations within each domain are subject to the weak constraint that the total magnetic flux adds up to an element of the gauge-group center.

- Faber, Greensite, ŠO, arXiv:hep-lat/9710039



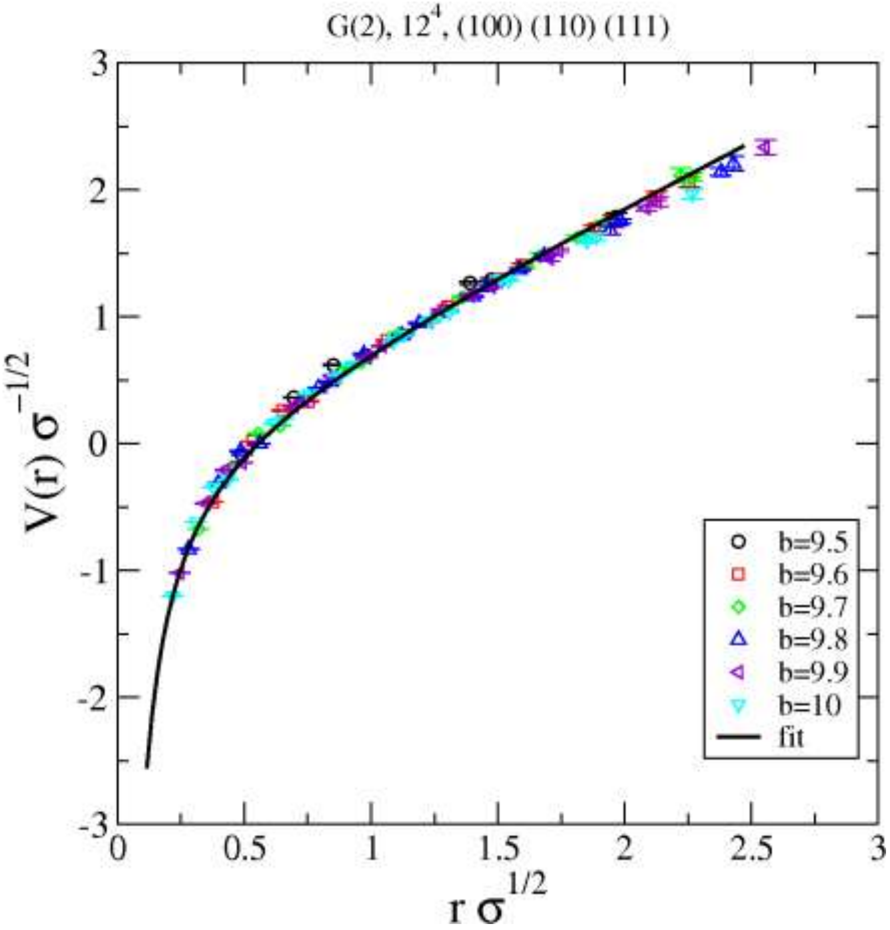
$$G_r(\alpha^n) = \frac{1}{d_r}\chi_r\left[\exp(i\vec{\alpha}^n\cdot\vec{H})\right]$$

A serious question: What about G_2 gauge theory?

Alice, in Lewis Carroll's *Through the Looking-Glass,and What Alice Found There*, enters a garden, where flowers speak to her and mistake her for a flower. **Does the gauge theory with the exceptional group G_2 belong to the same species of flowers with "ordinary" confining SU(N) gauge theories, or is it different and only mistaken for a flower?**

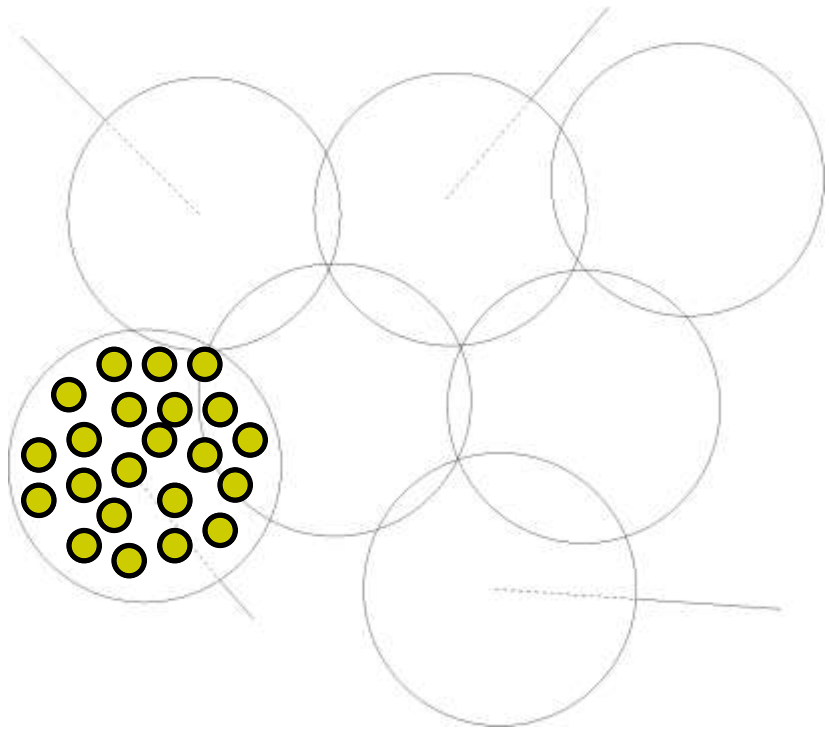
- Center-vortex confinement mechanism claims that the asymptotic string tension of a pure non-Abelian gauge theory results from random fluctuations in the number of center vortices.
- **No vortices** implies **no asymptotic string tension**!

- Is G_2 gauge theory a counterexample?
  - Holland, Minkowski, Pepe, Wiese, arXiv:hep-lat/0302023
  - Pepe, Wiese, arXiv:hep-lat/0610076
  - **We believe not.**
  - The asymptotic string tension of G_2 gauge theory is zero, in accord with the vortex proposal.
  - G_2 gauge theory however exhibits temporary confinement, i.e. the potential between fundamental charges rises linearly at intermediate distances. This can be qualitatively explained to be due to the group center, albeit trivial.
    - Greensite, Langfeld, ŠO, Reinhardt, Tok, arXiv:hep-lat/0609050

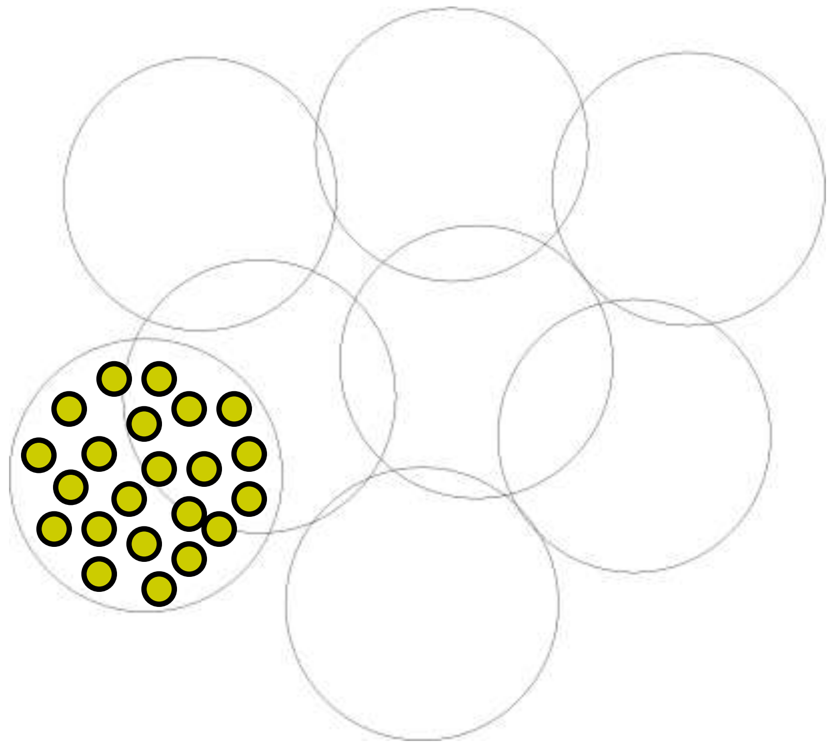


- Greensite, Langfeld, ŠO, Reinhardt, Tok, arXiv:hep-lat/0609050

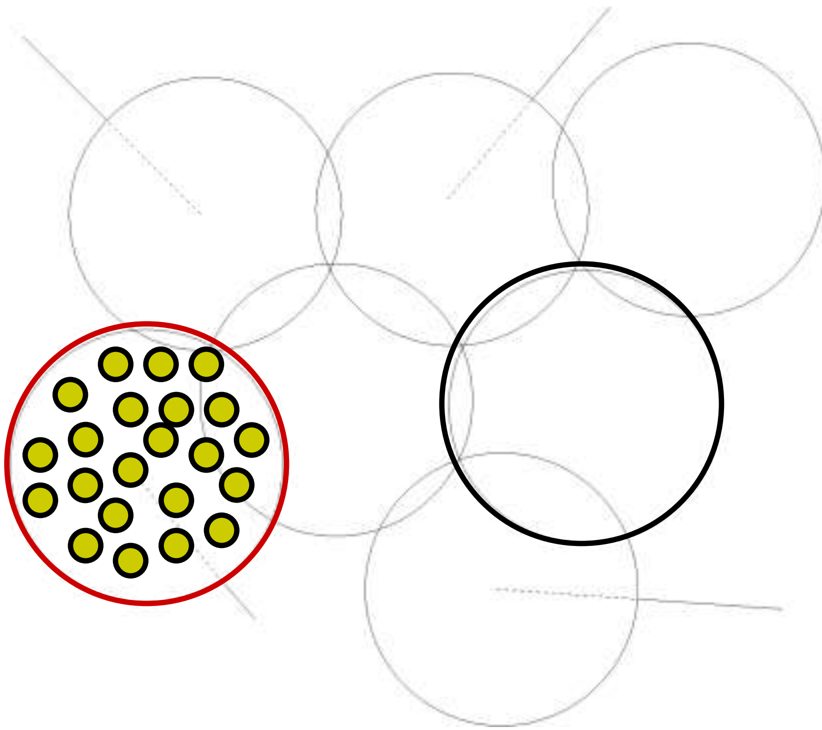
- SU(2)



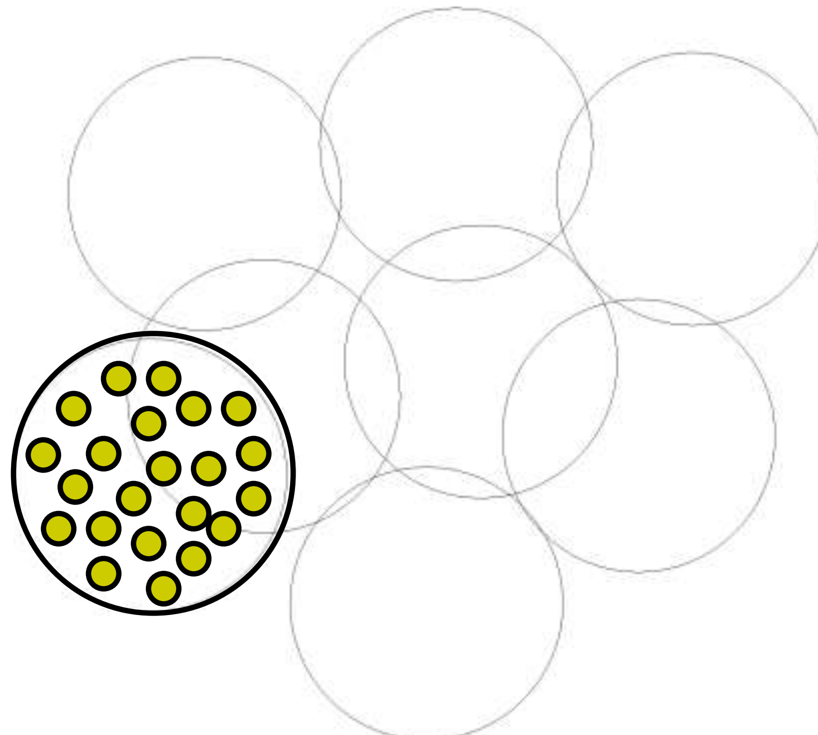
- G_2



- SU(2)



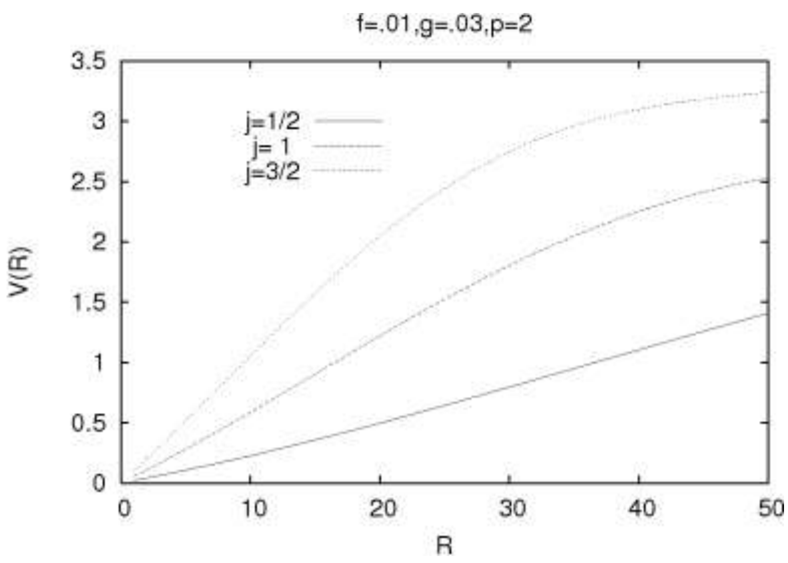
- G_2

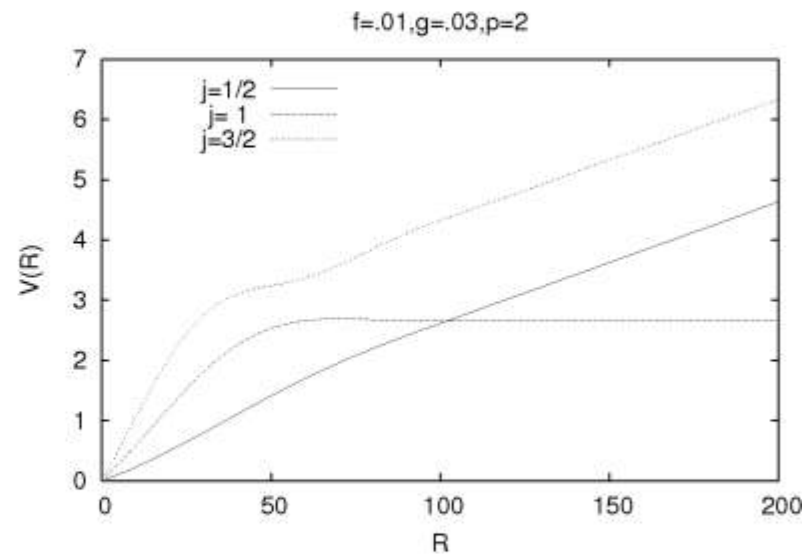


- Leads to (approximate) Casimir scaling at intermediate distances and N-ality dependence at large distances.

$$V_j(R) \approx \sigma_j R$$

$$\sigma_j \approx \begin{cases} \dfrac{p_0 + p_1}{6} \dfrac{L_v^2}{2\mu} j(j+1) & \dots \quad R \ll L_v \sim \sqrt{A_v} \\ -\ln\left[(1-p_1) + p_1 G_j(2\pi)\right] & \dots \quad R \gg L_v \end{cases}$$





Casimir scaling in G_2 gauge theory

- Specific prediction of the model: **Casimir scaling of string tensions** of higher-representation potentials **even for the** (centerless) **G_2 gauge theory**.

- Can be – and had to be – tested in numerical simulations of the G_2 lattice gauge theory.
  - Lipták, ŠO, arXiv:0807.1390

- A straightforward task, but not cheap: simulations more CPU time consuming, determination of potentials requires all the machinery developed in the past for calculating potentials (anisotropic lattices, ground-state overlap enhancement, smearing) plus some information from group theory (thanks to colleagues from mathematics departments all over the world!).

G_2 – some group-theory wisdom

- G_2 is the smallest among the exceptional Lie groups G_2, F_4, E_6, E_7, and E_8. It has a **trivial center**, its universal covering group is the group itself, and contains the group SU(3) as a subgroup.
- G_2 has rank 2, 14 generators, and the fundamental representation is 7-dimensional. It is a subgroup of the rank 3 group SO(7) which has 21 generators.

$$U_{ab}U_{ac} = \delta_{bc}$$

$$T_{abc} = T_{def}U_{da}U_{eb}U_{fc}, \qquad T_{abc} \text{ is totally antisymmetric}$$

$$T_{127} = T_{154} = T_{163} = T_{235} = T_{264} = T_{374} = T_{576} = 1$$

- With respect to the SU(3) subgroup:

$$\text{fundamental rep.: } \{7\} \rightarrow \{3\} \oplus \{\bar{3}\} \oplus \{1\}$$

$$\text{adjoint rep.: } \{14\} \rightarrow \{8\} \oplus \{3\} \oplus \{\bar{3}\}$$

- 3 G_2 gluons can **screen** a G_2 quark:

$$\{7\} \oplus \{14\} \otimes \{14\} \otimes \{14\} = \{1\} \oplus \ldots$$

- G_2 irreducible representations $\{D\}$ are labeled by two Dynkin coefficients $[\lambda_1, \lambda_2]$, the dimension of the representation is given by:

$$D = \frac{9}{40}(\ell_1^2 - \ell_2^2)(\ell_2^2 - \ell_3^2)(\ell_3^2 - \ell_1^2)$$

$$\ell_1 = \frac{1}{3}(1+\lambda_1) \qquad \ell_2 = \frac{1}{3}(4+\lambda_1+3\lambda_2) \qquad \ell_3 = \frac{1}{3}(5+2\lambda_1+3\lambda_2)$$

- The ratio of quadratic Casimir operators:

$$d_{\{D\}} \equiv \frac{C_{\{D\}}}{C_F} = \frac{1}{4}\left(\ell_1^2 + \ell_2^2 + \ell_3^2 - \frac{14}{3}\right)$$

- The adjoint-representation matrix, corresponding to an element g of G_2, can be constructed from the fundamental-representation matrix in the usual way:

$$\mathbb{D}^A_{ab}(g) = 2\,\mathrm{Tr}\,\left[\mathbb{D}^F(g)^\dagger\, t_a\, \mathbb{D}^F(g)\, t_b\right]$$

- Using tensor decompositions of different products of rep's, traces of higher-rep. matrices can be expressed through traces of the F- and A-representation matrices, e.g.:

$$7 \otimes 7 = 1 \oplus 7 \oplus 14 \oplus 27$$

$$\mathrm{Tr}\,\mathbb{D}^{\{27\}} = -1 - \mathrm{Tr}\,\mathbb{D}^A - \mathrm{Tr}\,\mathbb{D}^F + (\mathrm{Tr}\,\mathbb{D}^F)^2$$

G_2 on a lattice

- **Wilson action** on anisotropic $L^3 \times (2L)$ lattice:

$$S = -\frac{\beta}{7}\left\{\xi_0 \sum_{x,i>0} \text{Re Tr}\left[P_{i0}(x)\right] - \frac{1}{\xi_0}\sum_{x,i>j>0} \text{Re Tr}\left[P_{ij}(x)\right]\right\}$$

ξ_0 tuned so that $\xi = a_s^{\text{phys}}/a_t^{\text{phys}} = 2$

- Klassen, arXiv:hep-lat/9803010

- Most results for $14^3 \times 28$ at β = 9.5, 9.6, and 9.7.
- Complex parametrization of G_2 matrices based on explicit separation of the SU(3) subgroup and the G_2/SU(3) coset group.
  - Macfarlane, IJMP A, 2002
  - Pepe, Wiese, arXiv:hep-lat/0510013
- To increase overlap of the trial quark-antiquark state with the ground state: construct Wilson loops from smeared (spatial) links – **stout smearing**.
  - Morningstar, Peardon, arXiv:hep-lat/0311018

G_2 static potentials

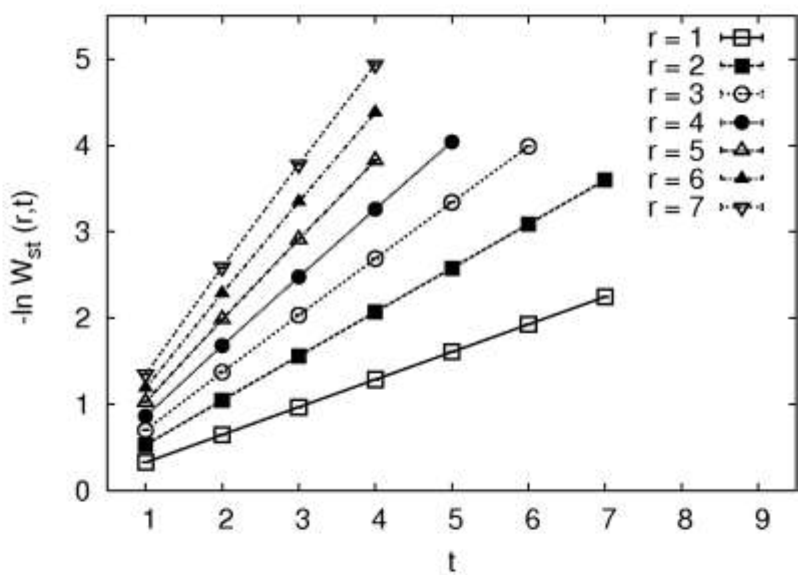
- Standard procedure:

$$W_{\{D\}}(r,t) = c_{\{D\}}(r) \exp\left[-V_{\{D\}}(r)\, t\right] \ldots \text{at large } t$$

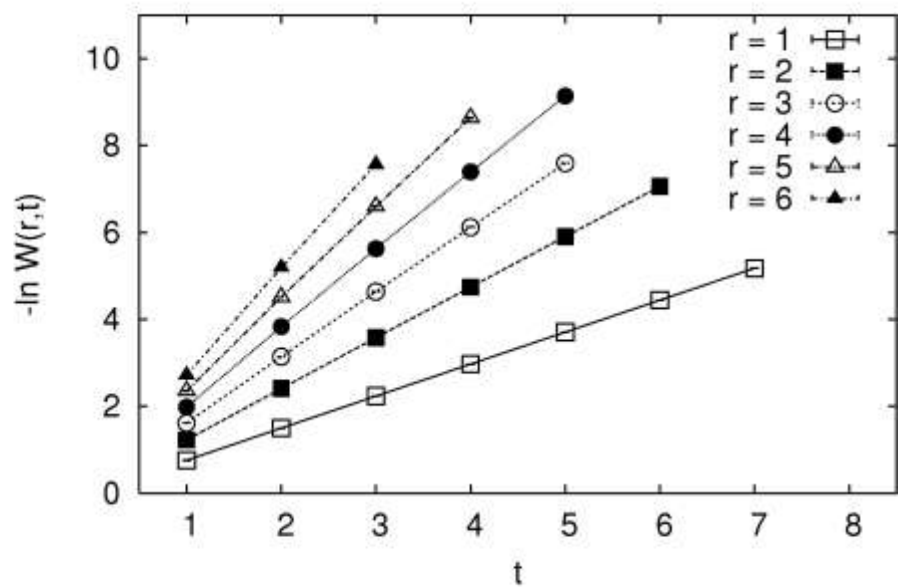
$$\text{fit } -\ln W_{\{D\}}(r,t) = C_{\{D\}} + V_{\{D\}}(r)\cdot t \text{ for } t \in (t_{\min}, t_{\max})$$

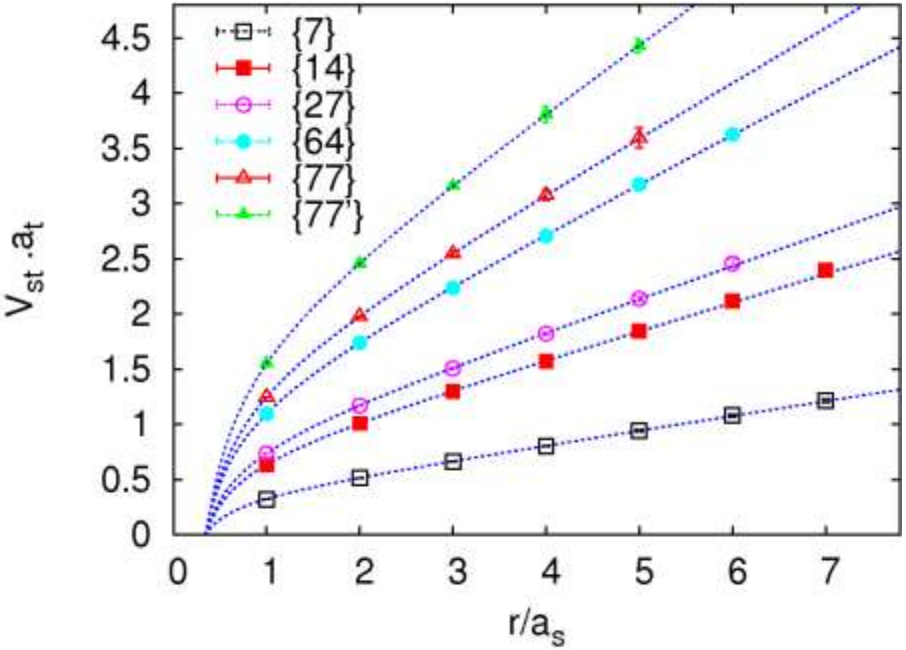
$$V_{\{D\}}(r) = c_{\{D\}} - \frac{\alpha_{\{D\}}}{r} + \sigma_{\{D\}}\, r$$

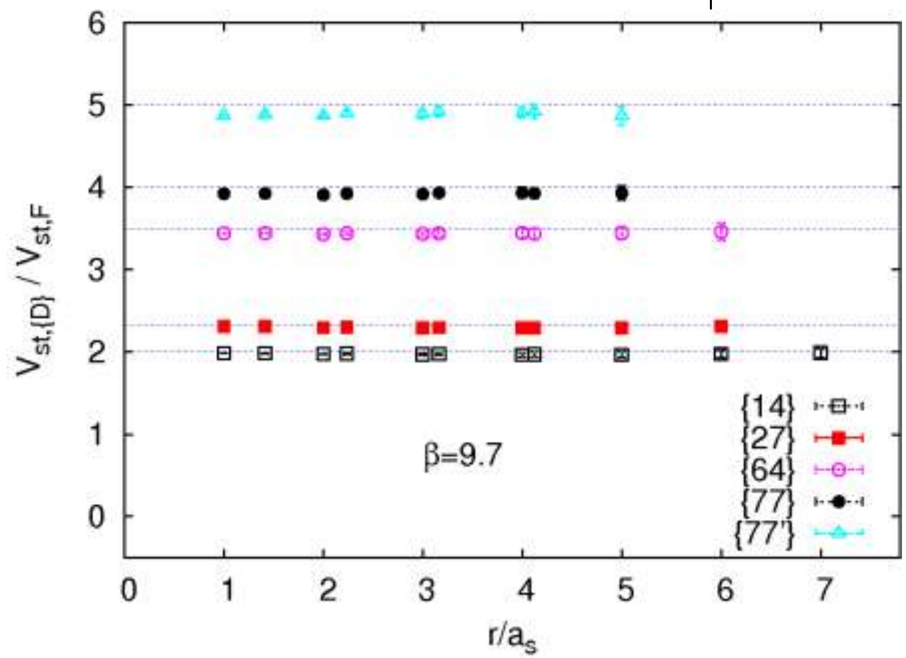
fundamental rep.



{27}

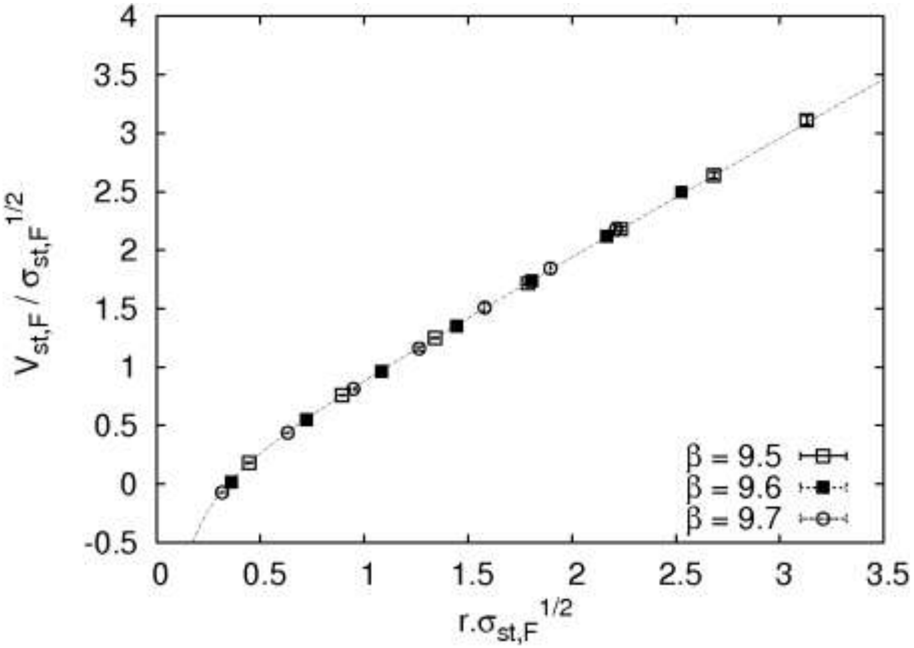


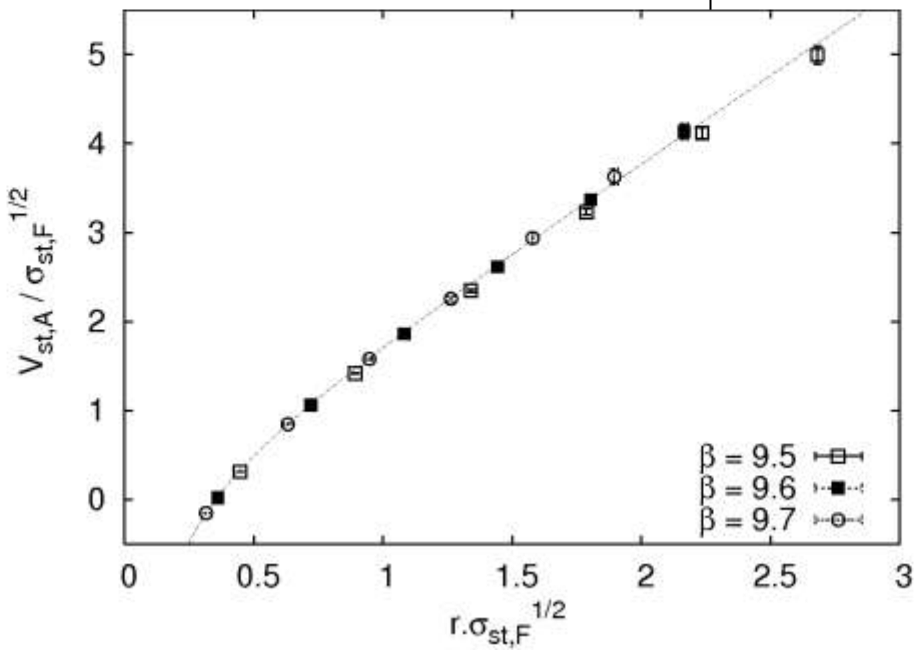




β	A	{27}	{64}	{77}	{77′}
9.5	1.88(4)	2.15(5)	3.1(1)	—	—
9.6	1.94(4)	2.24(6)	3.35(8)	3.8(2)	4.6(2)
9.7	1.96(6)	2.28(7)	3.5(1)	4.0(2)	4.9(2)
CS	2.0	2.333	3.5	4.0	5.0

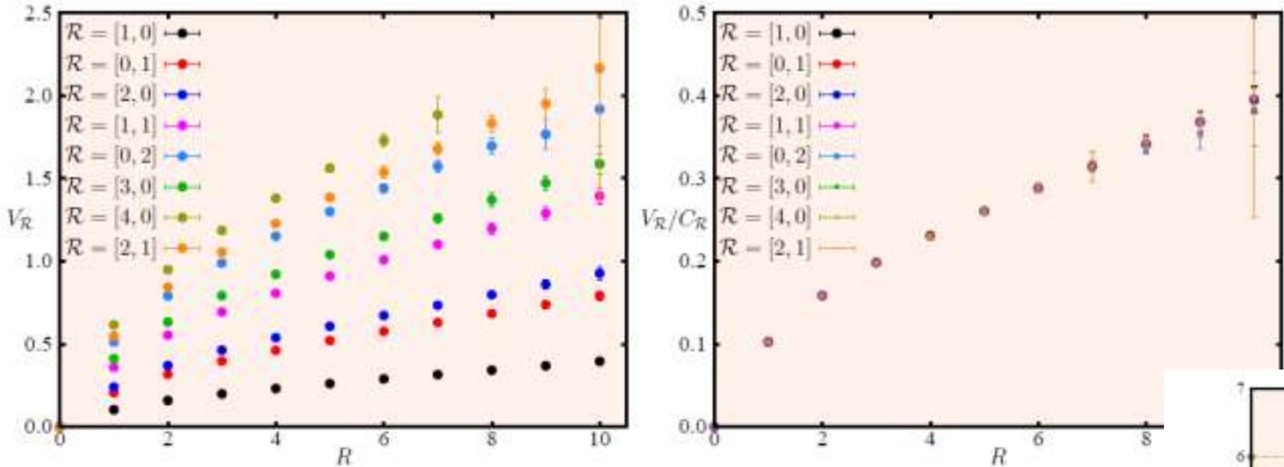
Ratios $\sigma_{\{D\}}/\sigma_F$ for different representations



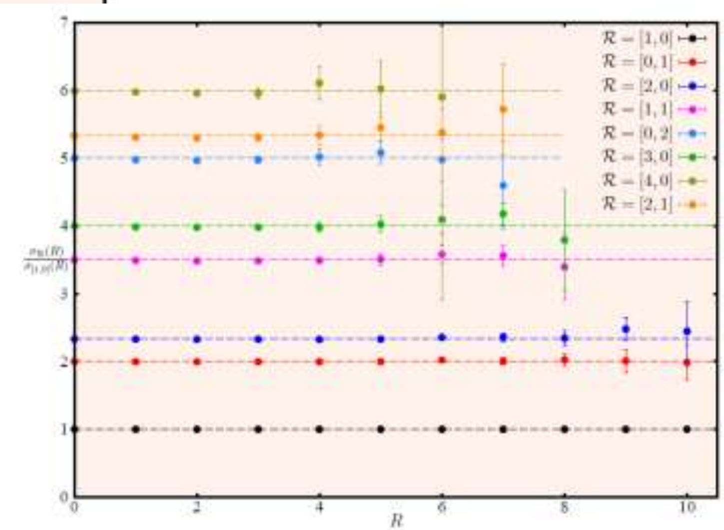


- This result supports the model based on the following elements:
  - QCD vacuum containing domain structures.
  - Color magnetic fields fluctuating almost independently in each domain.
  - Fields fulfilling the constraint that the total flux through the domain corresponds to an element of the gauge group center.

Casimir scaling – (2+1) G_2 YM theory



Static quark anti-quark potential for different representations with $\beta = 40$ on a 28^3 lattice unscaled *(left panel)* and scaled with the corresponding Casimir ratio $C_{\mathcal{R}}$ *(right panel)*.



- Wellegehausen, Wipf, Wozar, talk in St. Goar (1.9.2009)

Can we derive (at least some) elements of the picture from first principles?

- At large distance scales one expects:

$$\Psi_0^{\text{eff}}[A] \approx \exp\left[-\mu \int d^dx \; F_{ij}^a(x) F_{ij}^a(x)\right]$$

  - Halpern (1979), Greensite (1979)
  - Greensite, Iwasaki (1989)
  - Kawamura, Maeda, Sakamoto (1997)
  - Karabali, Kim, Nair (1998)

- Property of **dimensional reduction**: Computation of a spacelike loop in d+1 dimensions reduces to the calculation of a Wilson loop in Yang-Mills theory in d Euclidean dimensions.

$$W(C) = \langle \text{Tr}[U(C)] \rangle^{D=3+1} = \left\langle \Psi_0^{(3)} | \text{Tr}[U(C)] | \Psi_0^{(3)} \right\rangle$$

$$\sim \langle \text{Tr}[U(C)] \rangle^{D=2+1} = \left\langle \Psi_0^{(2)} | \text{Tr}[U(C)] | \Psi_0^{(2)} \right\rangle$$

$$\sim \langle \text{Tr}[U(C)] \rangle^{D=1+1} \quad \ldots$$ area law! (+ Casimir scaling)

Suggestion for an approximate vacuum wavefunctional

$$\Psi_0[A] = \exp\left[-\frac{1}{2}\int d^2x d^2y\; B^a(x)\left(\frac{1}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}}\right)^{ab}_{xy} B^b(y)\right]$$

$B^a(x) = F^a_{12}(x)$ ... the color magnetic field strength

$\mathcal{D}_k[A]$ ... the covariant derivative in the adjoint representation

$\mathcal{D}^2 = \mathcal{D}_k \cdot \mathcal{D}_k$ ... the covariant laplacian in the adjoint representation

$$(-\mathcal{D}^2)^{ab}_{xy} = \sum_{k=1}^{2}\left[2\delta^{ab}\delta_{xy} - \mathcal{U}^{ab}_k(x)\delta_{y,x+\hat{k}} - \mathcal{U}^{\dagger ba}_k(x-\hat{k})\delta_{y,x-\hat{k}}\right]$$

$$\mathcal{U}^{ab}_k(x) = \frac{1}{2}\mathrm{Tr}\left[\sigma^a U_k(x)\sigma^b U^\dagger_k(x)\right]$$

λ_0 ... the lowest eigenvalue of $(-\mathcal{D}^2)$

m ... a constant proportional to $g^2 \sim 1/\beta$

- Greensite, ŠO, arXiv:0707.2860 [hep-lat]

Free-field limit (g→0)

$$\Psi_0[A] = \exp\left\{ -\frac{1}{2} \int d^2x d^2y \, [\nabla \times A^a(x)] \left(\frac{\delta^{ab}}{\sqrt{-\nabla^2}} \right)_{xy} [\nabla \times A^b(y)] \right\}$$

For $g \to 0$:

- $m \sim g^2 \to 0$,
- $\mathcal{D}^2 \to \nabla^2, \qquad \lambda_0 \to 0$,

$\Psi_0[A]$ becomes the well-known vacuum wavefunctional of ED.

Zero-mode, strong-field limit

- D. Diakonov (private communication to JG)

- Let's assume we keep only the zero-mode of the A-field, i.e. fields constant in space, varying in time. The lagrangian is

$$\mathcal{L} = \tfrac{1}{2}V\left[\sum_{k=1}^{2}\partial_t\vec{A}_k\cdot\partial_t\vec{A}_k - g^2(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)\right]$$

  and the hamiltonian operator

$$\hat{\mathcal{H}} = -\frac{1}{2V}\sum_{k=1}^{2}\frac{\partial^2}{\partial\vec{A}_k\cdot\partial\vec{A}_k} + \frac{1}{2}g^2V(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)$$

- The ground-state solution of the YM Schrödinger equation, up to 1/V corrections:

$$\Psi_0 = \exp[-VR_0] = \exp\left[-\frac{1}{2}gV\frac{(\vec{A}_1\times\vec{A}_2)\cdot(\vec{A}_1\times\vec{A}_2)}{\sqrt{\vec{A}_1\cdot\vec{A}_1+\vec{A}_2\cdot\vec{A}_2}}\right]$$

Dimensional reduction and confinement

- **What about confinement with such a vacuum state?**
- Define "slow" and "fast" components using a mode-number cutoff:

$$(-\mathcal{D}^2)^{ab}_{xy}\,\varphi^b_n(y) = \lambda_n\,\varphi^a_n(x)$$

$$B^a(x) = \sum_{n=0}^{\infty} b_n\,\varphi^a_n(x)$$

$$B^a_{\text{slow}}(x) = \sum_{n=0}^{n_{\max}} b_n\,\varphi^a_n(x), \qquad \lambda_{n_{\max}} - \lambda_0 \ll m^2$$

- Then:

$$\int d^2x d^2y\; B^a_{\text{slow}}(x) \left(\frac{1}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}}\right)^{ab}_{xy} B^b_{\text{slow}}(y)$$

$$\approx \frac{1}{m}\int d^2x\; B^a_{\text{slow}}(x)\; B^a_{\text{slow}}(x)$$

- Effectively for "slow" components

$$|\Psi_0|^2 \approx \exp\left[-\frac{1}{m}\int d^2x\; B^a_{\text{slow}}(x)\; B^a_{\text{slow}}(x)\right]$$

we then get the probability distribution of a 2D YM theory and can compute the string tension analytically (in lattice units):

$$\sigma(\beta) = \frac{3m(\beta)}{4\beta}$$

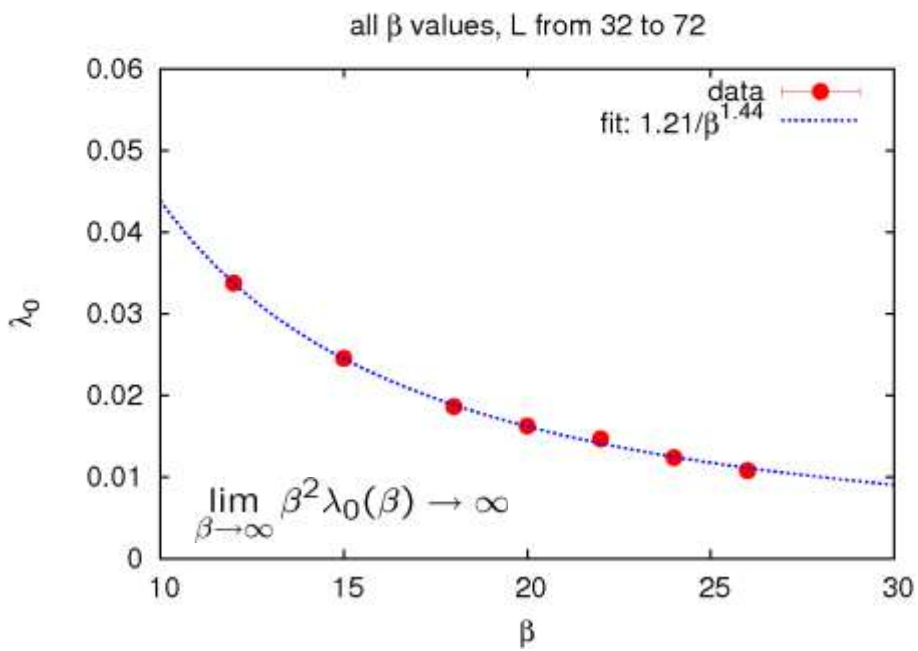
- **Non-zero value of *m* implies non-zero string tension σ and confinement!**
- Let's revert the logic: to get σ with the right scaling behavior ~ $1/\beta^2$, we need to choose

$$m(\beta) = \tfrac{4}{3}\beta\sigma(\beta) \sim \beta^{-1} \sim g^2$$

Why $m_0^2 = -\lambda_0 + m^2$?

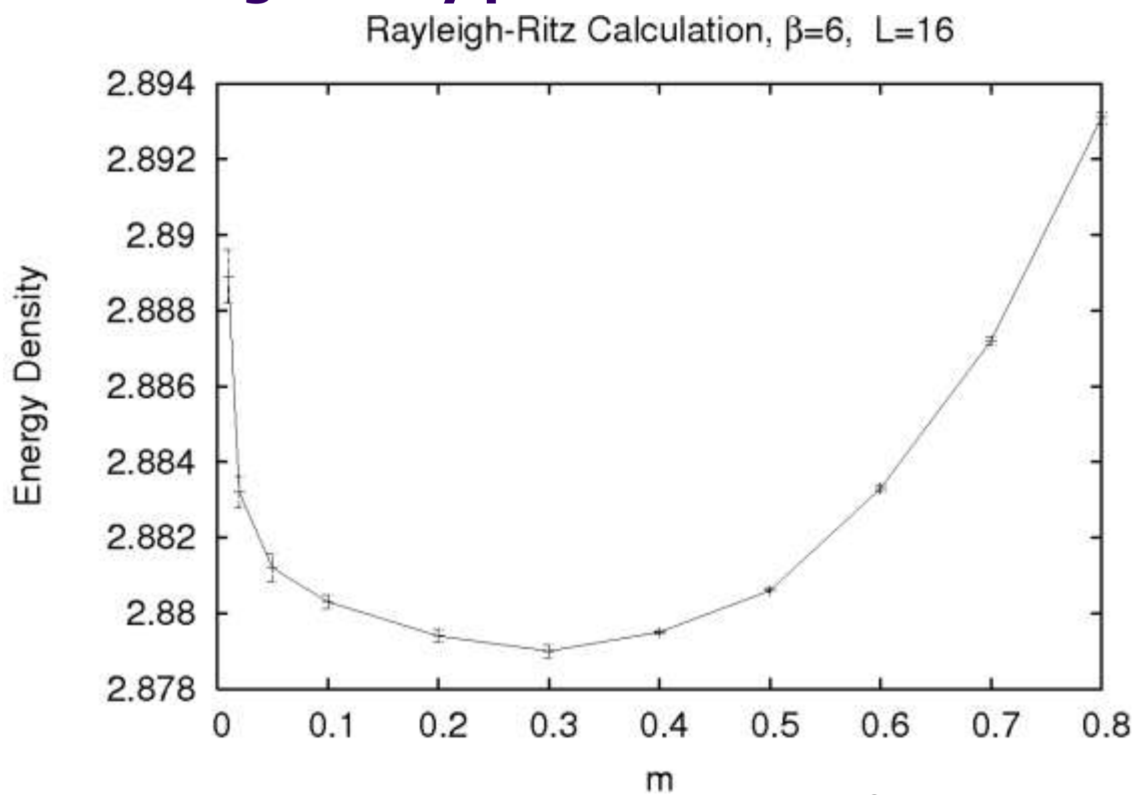
$$\Psi_0[A] = \exp\left[-\frac{1}{2}\int d^2x d^2y\, B^a(x)\left(\frac{1}{\sqrt{-\mathcal{D}^2 + m_0^2}}\right)^{ab}_{xy} B^b(y)\right]$$

Samuel (1997)



Why not to choose $(-\mathcal{D}^2 - \lambda_0) + \lambda_0$?

Non-zero m is energetically preferred



- **Non-abelian case:** Minimum at non-zero m^2 (~ 0.3), though a higher value (~ 0.5) would be required to get the right string tension.
- Could (and should) be improved!

Calculation of the mass gap

- To extract the mass gap, one would like to compute

$$\mathcal{G}(x-y) = \langle (B^a B^a)_x (B^b B^b)_y \rangle - \langle (B^a B^a)_x \rangle^2$$

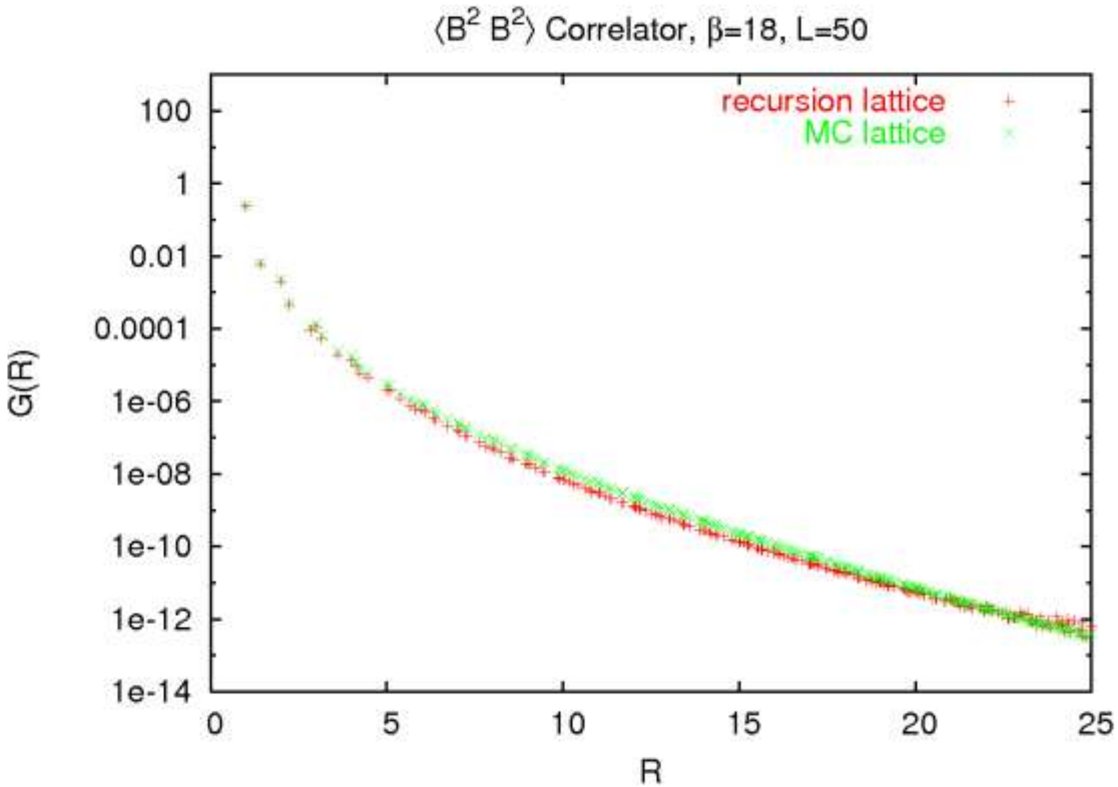
in the probability distribution:

$$P[A] = |\Psi_0[A]|^2 = \mathcal{N} \exp\left[-\int d^2x d^2y \; B^a(x) K^{ab}_{xy}[A] B^b(y) \right]$$

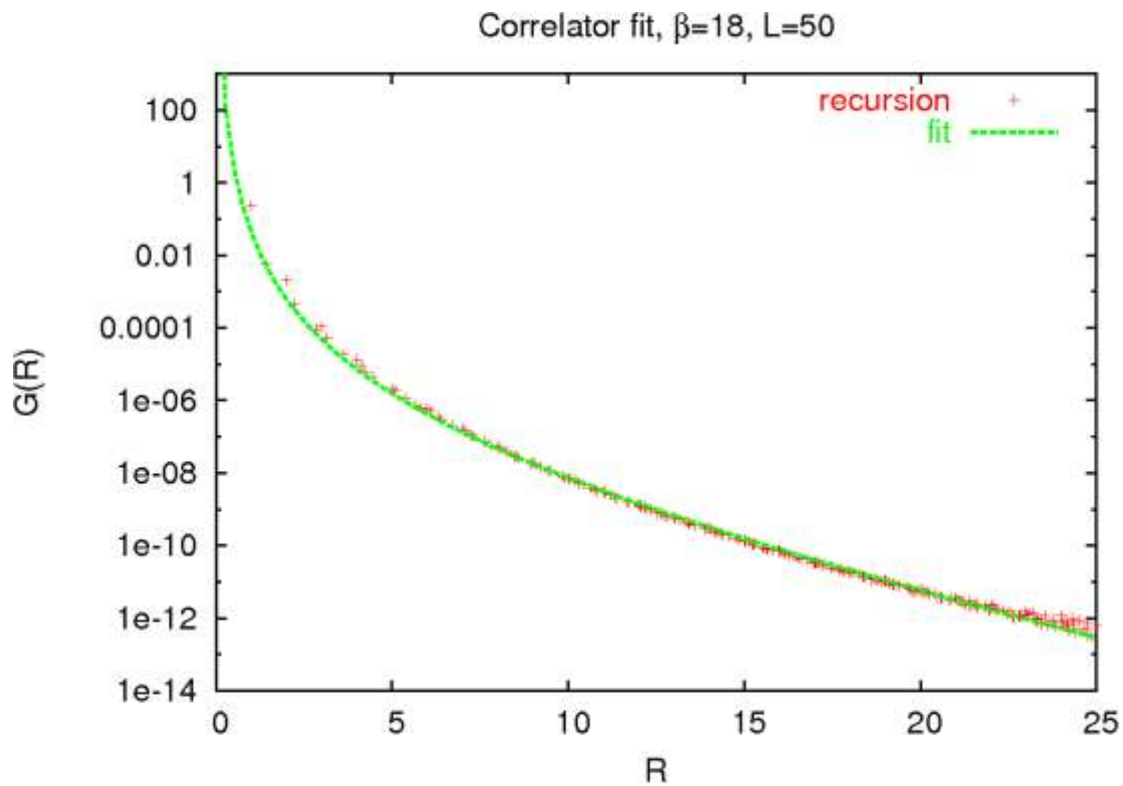
$$K^{ab}_{xy}[A] = \left(\frac{1}{\sqrt{-\mathcal{D}^2[A] - \lambda_0 + m^2}} \right)^{ab}_{xy}$$

- Looks hopeless, K[A] is highly non-local, not even known for arbitrary fields.
- But if - after choosing a gauge - K[A] does not vary a lot among thermalized configurations … then something can be done.

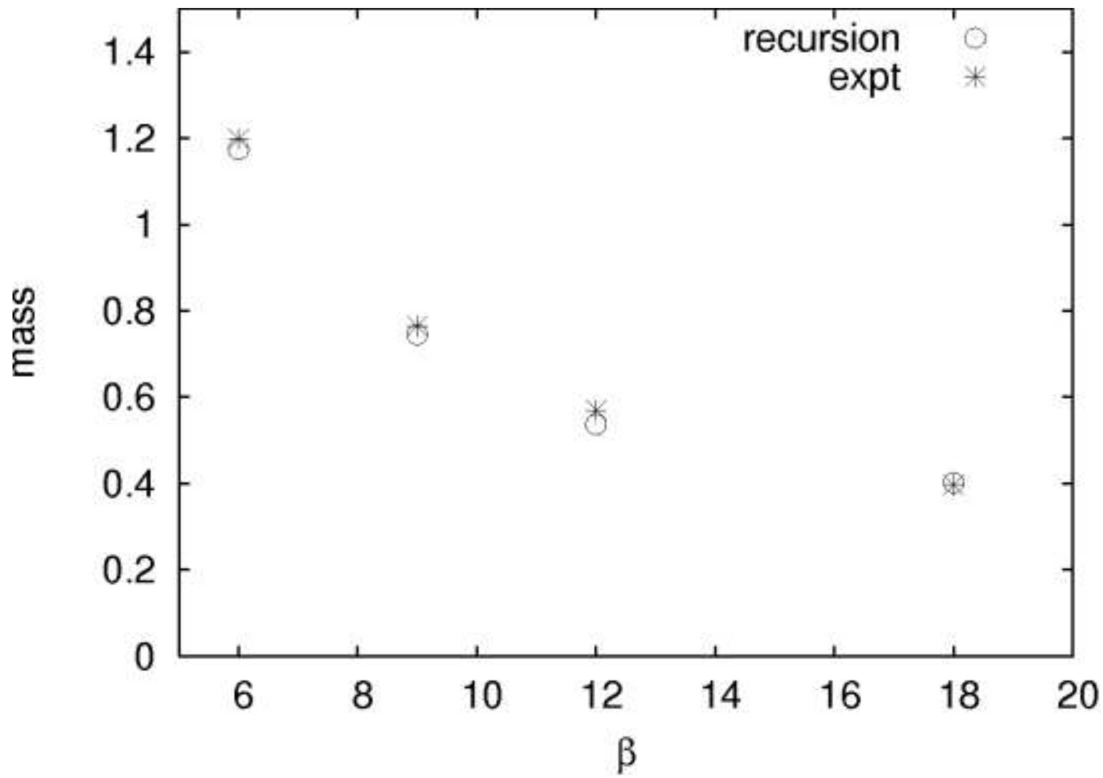
Mass gap



⟨B² B²⟩ Correlator, β=18, L=50
recursion lattice
MC lattice
G(R)
R
100
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
0
5
10
15
20
25



$$G_0(x) = \delta^{ab}\delta^{ba}\left[\left(\sqrt{-\nabla^2+\mu^2}\right)_{xy}\right]^2 \sim (1+\mu R)^2\,\frac{e^{-2\mu R}}{R^6}, \qquad M = 2\mu$$



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N-ality

- **Dimensional reduction form** at large distances implies area law for large Wilson loops, but also **Casimir scaling** of higher-representation Wilson loops.
- How does **Casimir scaling** turn into **N-ality dependence**, how does **color screening** enter the game?
- A possibility: Necessity to introduce additional term(s), e.g. a gauge-invariant gluon-mass term
  - Cornwall, arXiv:hep-th/0702054

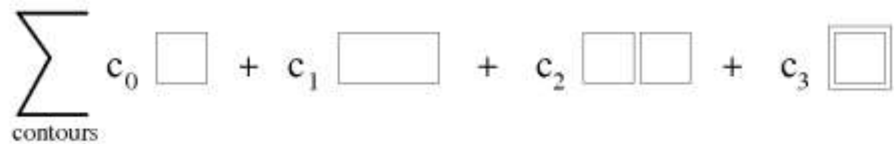
  … but color screening may be contained!

- Strong-coupling:

$$\Psi_0[U] = \exp(R[U])$$

  - Greensite (1980)

$$R[U] \;=$$



$$+ \;\text{larger contours}$$

$$\Psi_0[U] = \exp\left[-\frac{2}{\beta}\int d^2x \;\left(a\kappa_0 B^2 - a^3\kappa_2 B(-\mathcal{D}^2)B + \ldots\right)\right]$$

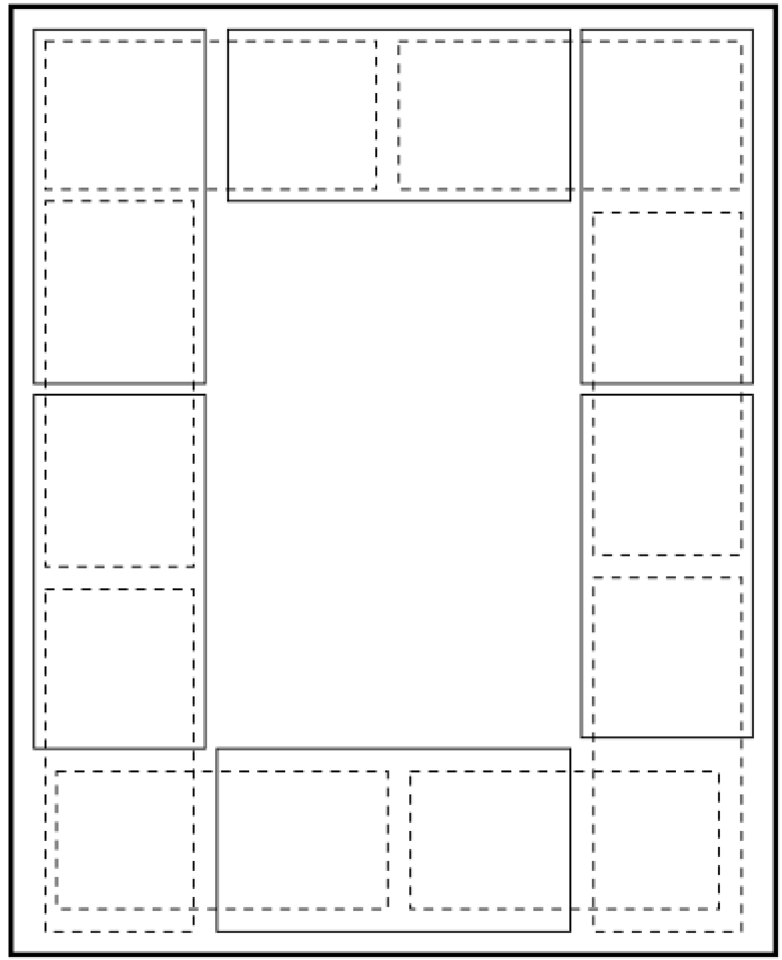
$$\kappa_0 = \tfrac{1}{2}c_0 + 2(c_1 + c_2 + c_3), \qquad \kappa_2 = \tfrac{1}{4}c_1$$

$$c_0 = \mathcal{O}(\beta^2), \qquad c_1, c_2, c_3 = \mathcal{O}(\beta^4)$$

- Guo, Chen, Li (1994)

$$\frac{1}{\sqrt{-\mathcal{D}^2 - \lambda_0 + m^2}} = \frac{1}{m}\left(1 - \frac{-\mathcal{D}^2 - \lambda_0}{2m^2} + \ldots\right)$$

$$|\Psi_0|^2 = \exp\left[-\frac{1}{m}\int d^2x \;\left(B_{\text{slow}}B_{\text{slow}} - B_{\text{slow}}\frac{-\mathcal{D}^2 - \lambda_0}{2m^2}B_{\text{slow}} + \ldots\right)\right]$$



contributes to $W_{\text{adj}}[C]$ by $\left(\frac{c_1}{2}\right)^{P(C)-4}$

Summary

- Casimir scaling is a natural outcome of a picture of the QCD vacuum as a medium in which color magnetic fields fluctuate almost independently and are only weakly constrained that the total flux through a cross-section of the domain corresponds to an element of the gauge group center.
- Lattice data show that string tensions of higher-representation potentials at intermediate distances satisfy Casimir scaling to surprising precision for SU(2), SU(3), and even G_2 gauge theory.
- Some properties of the model are (or may be) contained in our simple approximate form of the confining YM vacuum wavefunctional in 2+1 dimensions. Its properties:
  - It is a solution of the YM Schrödinger equation in the weak-coupling limit, …
  - … and also in the zero-mode, strong-field limit.
  - Dimensional reduction works: There is confinement (non-zero string tension) if the free mass parameter m is larger than 0, …
  - … and $m > 0$ seems energetically preferred.
  - If the free parameter m is adjusted to give the correct string tension at the given coupling, then the correct value of the mass gap is also obtained.
  - Gives the right ghost propagator and color Coulomb potential (cf. Jeff Greensite's talk).
  - **But questions remain:** how to improve the Ansatz; N-ality dependence at large distances; generalization to D=3+1; …

Európska únia
Európsky fond regionálneho rozvoja

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