Chiral symmetry breaking and the Banks–Casher relation on the lattice

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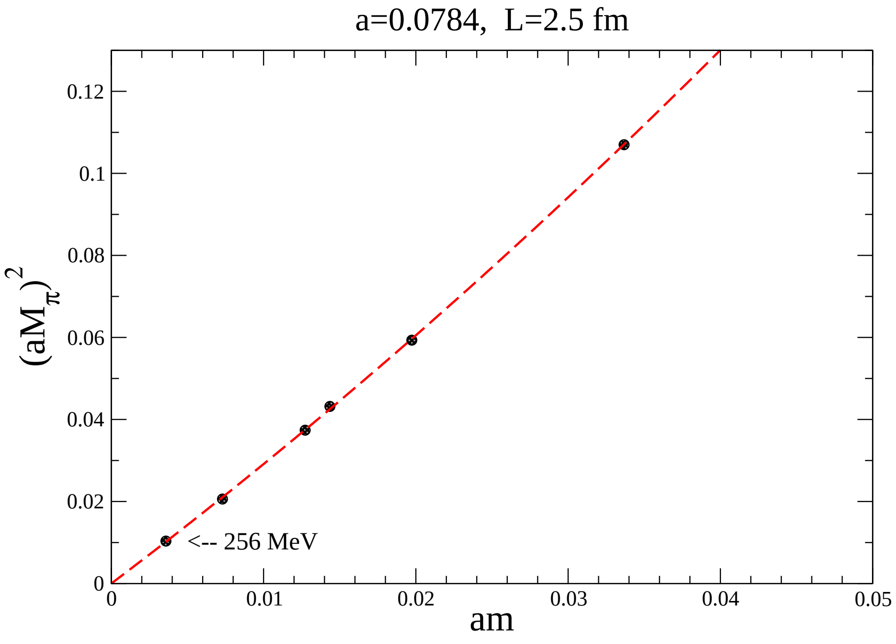
Based on: L. G. and M. Lüscher JHEP 0903 (2009) 013 [arXiv:0812.3638]

QCD Green's Functions, Confinement and Phenomenology - September 8th 2009

Outline

- Banks-Casher relation
- Renormalization of the spectral density
- Spectral density in ChPT
- Spectral projectors and extraction of Σ
- Exploratory numerical study
- Conclusions and Outlook

Pion Mass with $O(a)$-improved Wilson quarks [Del Debbio et al. 06-08]



$$(aM_\pi)^2 = 2.80(3) \times am + 11.3(12) \times (am)^2 \qquad \chi^2/\text{dof} = 0.9$$

- A striking linearity observed: at the smallest masses non-linear correction is 1 - 3%. SSB as expected. Result confirmed by other collaborations

- It is time for a precise and quantitative study of QCD in the chiral regime

- Consequences of the SSB can be tested from first principles. The GMOR relation is maybe the simplest to start with

Banks-Casher relation [Banks, Casher 80; Leutwyler, Smilga 92; Shuryak, Verbaarschot 93]

- The spectral density of the Dirac operator (two degenerate flavours of mass m) is

$$\rho(\lambda, m) \equiv \frac{1}{V} \sum_k \langle \delta(\lambda - \lambda_k) \rangle$$

where $\langle \dots \rangle$ indicates the path-integral average, and for each gauge configuration

$$D\psi_k = i\bar{\lambda}_k \psi_k \, , \qquad \bar{\lambda}_k = \lambda_k + \mathcal{O}(a^2)$$

- $\rho(\lambda, m)$ has a well defined infinite volume limit, χ-sym. implies $\rho(\lambda, m) = \rho(-\lambda, m)$
- The chiral condensate is given by

$$\Sigma(m_v, m) \equiv -\langle \bar{\psi}\psi \rangle = \int_{-\infty}^{\infty} d\lambda \, \frac{\rho(\lambda, m)}{i\lambda + m_v} \, , \qquad \frac{\Sigma(0,0)}{\pi} = \lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda, m)$$

- A condensate is generated if eigenvalues condensate near zero with spacing $\propto 1/V$

- The resolvent

$$R(z,m) = \int_0^\infty d\lambda \frac{\rho(\lambda,m)}{(\lambda^2+m_v^2)^2(z-\lambda^2-m_v^2)}$$

is an analytic function of z with a cut on the real axis for $z \geq m_v^2$

- Spectral density proportional to the discontinuity across the cut, uniquely determined from $R(z)$

- For $|z| < m_v^2$ one can expand in power series

$$R(z,m) = \sum_{k=0}^{\infty} m_k z^k\ , \qquad m_k = -\int_0^\infty d\lambda \frac{\rho(\lambda,m)}{(\lambda^2+m_v^2)^{k+3}}$$

- It is enough to study the renormalization and continuum limit of the moments m_k

Renormalization and continuum limit of the spectral density (II) [Del Debbio, LG, Lüscher, Petronzio, Tantalo 06]]

- The moments m_k can be written as

$$m_k = a^{4n-4} \sum_{x_1,\dots,x_{n-1}} \langle P_{12}(x_1) P_{23}(x_2) \dots P_{n1}(0) \rangle \ , \qquad n = 2k+6 \ , \qquad P_{ij} = \bar{\psi}_i \gamma_5 \psi_j$$

  if a sufficient number of valence quarks are added to the theory

- An overall factor $(1/Z_m)^n$ renormalizes the correlation function when the pseudoscalar densities are inserted at physical distance

- At short distances ($x_1 \to x_2$) the flavour structure implies

$$P_{12}(x_1) P_{23}(x_2) \sim C(x_1 - x_2) S_{13}(x_1)$$

  where by power counting $C(x)$ diverges like $|x|^{-3}$

- This singularity is integrable. Analogous argument for all other short-distance singularities. No extra contact terms are needed to renormalize m_k

Renormalization and continuum limit of the spectral density (III) [Del Debbio, LG, Lüscher, Petronzio, Tantalo 06]

- Once the the gauge coupling and the mass are renormalized, the resolvent

$$\hat{R}(z,m) = Z_m^{-6} R\left(\frac{z}{Z_m^2}, \frac{m}{Z_m}\right)$$

is ultraviolet finite. In particular no extra UV power divergences need to be subtracted

- The renormalized density reads

$$\hat{\rho}(\lambda,m) = Z_m^{-1} \rho\left(\frac{\lambda}{Z_m}, \frac{m}{Z_m}\right)$$

- For Wilson fermions a gap for the operator D_{m_v} not guaranteed, a more involved derivation needed [LG, Lüscher 08]

Spectral density in the chiral effective theory (I) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- Standard chiral effective theory supplemented with a valence quark

$$SU(2)_L \otimes SU(2)_R \to SU(2)_V \quad \Longrightarrow \quad SU(3|1)_L \otimes SU(3|1)_R \to SU(3|1)_V$$

- The unitary fields are given by

$$U = \exp\left\{\frac{2i}{F}\Phi\right\}, \qquad \Phi = \sum_a \phi^a T^a$$

and the leading order Lagrangian is [$M = \text{diag}(m, m, m_v, m_v)$]

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left\{\text{Str}\left[\partial_\mu U^\dagger \partial_\mu U\right] - \frac{2\Sigma}{F^2}\,\text{Str}\left[MU^\dagger + M^\dagger U\right]\right\}$$

- At the NLO the relevant term is given by

$$\mathcal{L}^{(4)} = -\frac{4\Sigma^2}{F^4} L_6\, \text{Str}\left[U^\dagger M + M^\dagger U\right] \text{Str}\left[U^\dagger M + M^\dagger U\right] + \ldots$$

Spectral density in the chiral effective theory (II) [Smilga, Stern 93; Osborn et al. 99; L.G., Lüscher 08]

- If the condensate is analytically continued in the valence quark mass

$$\Sigma(z,m) \equiv -\langle\bar\psi\psi\rangle = \int_{-\infty}^{\infty} d\lambda\, \frac{\rho(\lambda,m)}{i\lambda+z}$$

then the spectral density can be computed as

$$2\pi\rho(\lambda,m) = \lim_{\epsilon\to 0}\Big[\Sigma(i\lambda+\epsilon,m)+\Sigma(-i\lambda+\epsilon,m)\Big]$$

- In the infinite volume limit an at the NLO in ChPT

$$\begin{aligned}\rho^{\rm nlo}(\lambda,m) \quad &= \quad \frac{\Sigma}{\pi}\Big\{1+\frac{m\Sigma}{(4\pi)^2F^4}\Big[\,(3\,\bar l_6-1)+2\frac{\lambda}{m}\arctan\Big(\frac{\lambda}{m}\Big)-\pi\frac{|\lambda|}{m}\\ &- \quad 2\,\ln\left(\frac{\Sigma\sqrt{\lambda^2+m^2}}{F^2\mu^2}\right)-\ln\left(\frac{2\Sigma|\lambda|}{F^2\mu^2}\right)\Big]\Big\}\end{aligned}$$

where $\bar l_6 = (1024\pi^2)\hat L_6/3+\ldots$

Extracting Σ from the spectral density

- The dynamical properties of the theory are encoded in $\rho(\lambda, m)$
- The power divergences in the chiral condensate

$$\Sigma(z, m) \equiv \int_{-\infty}^{\infty} d\lambda \, \frac{\rho(\lambda, m)}{i\lambda + z}$$

  do not originate from the spectral density itself but from the particular integral taken

- Note that if $\rho(\lambda, m)$ is extracted from

$$2\pi\rho(\lambda, m) = \lim_{\epsilon \to 0} \left[\Sigma(i\lambda + \epsilon, m) + \Sigma(-i\lambda + \epsilon, m)\right]$$

  the divergences cancel out on the r.h.s.

- By choosing a different probe function, it is possible to extract Σ from integrals of the spectral density which are not plagued by power divergences [L.G., S. Necco 07]

Spectral Projectors in QCD

- The average number of eigenstates of D with $|\lambda| < \Lambda$

$$\nu(\Lambda, m) = V \int_{-\Lambda}^{\Lambda} d\lambda \rho(\lambda, m)$$

  is maybe the simplest integral to consider

- In the infinite volume limit it scales with V if SSB is at work

- The continuum limit can be taken for any value of Λ and m, i.e.

$$\nu_{\mathrm{R}}(M_{\mathrm{R}}, m_{\mathrm{R}}) = \nu(M, m)$$

  even without chiral symmetry implemented

- $O(a)$ improvement automatic

Spectral Projectors in ChPT

- In the infinite volume limit

$$\nu^{\rm nlo}(\Lambda,m) = \frac{2\Lambda\Sigma V}{\pi}\left\{1+\frac{\Sigma\, m}{(4\pi)^2F^4}\left[3\,\bar{l}_6 - 3\ln\left(\frac{\Sigma\Lambda}{F^2\mu^2}\right) - \ln(2) - \frac{\pi}{2}\frac{m}{\Lambda} + O\left(\frac{m^2}{\Lambda^2}\right)\right]\right\}$$

  is clearly sensitive to the infinite volume chiral condensate

- NLO corrections to leading behaviour (vanishing for $m \to 0$) expected rather small for $\Lambda = 50$–100 MeV and $m \leq 20$ MeV [$\overline{\rm MS}$ @ 2 GeV]. No chiral logs $\propto \ln(m)$

- Finite volume effects: a fraction of a percent at NLO ChPT in the p-regime

- When $\Lambda\Sigma V$ is not very large, threshold effects can be sizeable. They can be quantified in ChPT

Spectral Projectors: numerical implementation

- The quantity $\nu(\Lambda, m)$ can be computed as

$$\nu(\Lambda, m) = \langle \mathcal{O} \rangle \; , \qquad \mathcal{O}[U] = \mathrm{Tr} P_M$$

$$P_M = \theta(M^2 - D_m^\dagger D_m) \; , \qquad M^2 = \Lambda^2 + m^2$$

- To avoid the (unnecessary) computation of $O(V)$ eigenvalues

$$\nu(\Lambda, m) = \langle \hat{\mathcal{O}} \rangle \; , \qquad \hat{\mathcal{O}}[U, \eta] = (\eta, P_M \eta)$$

where η are Gaussian random sources

- The cost scales with V rather than V^2, and the variance scales like $V^{-1/2}$.
Infinite volume and continuum limit extrapolation feasible

Coordinated Lattice Simulations (CLS) [https://twiki.cern.ch/twiki/bin/view/CLS/WebHome]

Several European groups coordinate their efforts to generate configuration ensembles with $N_\mathrm{f} = 2$ dynamical quarks degenerate in mass

Gluon action: Wilson
Quark action: $O(a)$-improved Wilson
Algorithm : DD-HMC

List of runs used in this work:

β	V	a[fm]	L[fm]	Masses
5.3	48×24^3	0.08	1.9	6
5.3	64×32^3	0.08	2.5	6

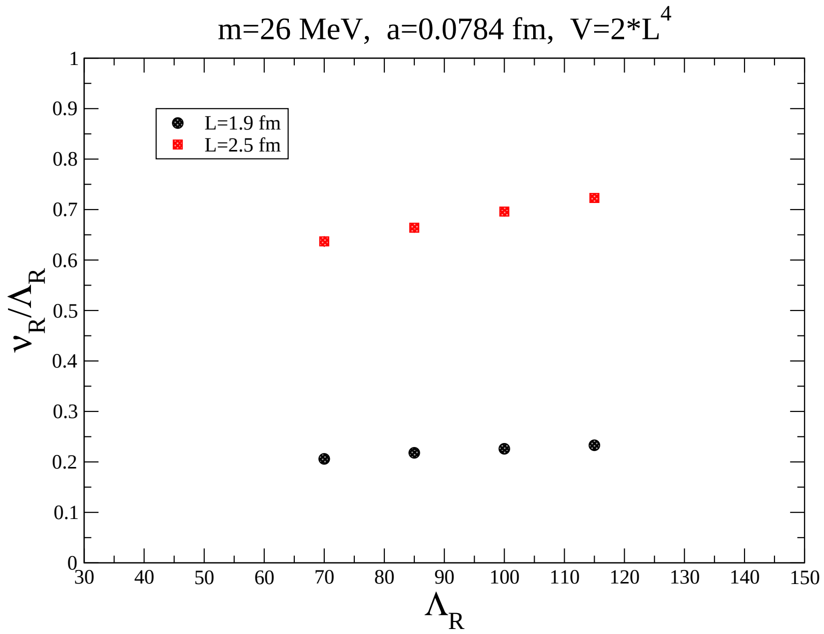
Groups:
Berlin (U. Wolff)
CERN (L. G., M. Lüscher)
DESY-Zeuthen (R. Sommer)
Madrid (C. Pena)
Mainz (H. Wittig)
Rome (R. Petronzio)
Valencia (P. Hernández)

Physics:
Spontaneous symmetry breaking
Fundamental parameters
Light-light mesons
Heavy-light mesons
Baryons
Weak matrix elements
.....

Numerical computation: parameters and scaling with the volume

Lattice details:

- ∗ Action: $O(a)$-improved Wilson
- ∗ $a = 0.0784$ fm
- ∗ $V = 2L \times L^3$, $L = 1.9, 2.5$ fm
- ∗ $m_R^{\overline{\rm MS}}(2\,{\rm GeV}) = 12.8, 26.5, 45.8$ MeV
- ∗ $\Lambda_{\rm R} = 70, 85, 100, 115$ MeV



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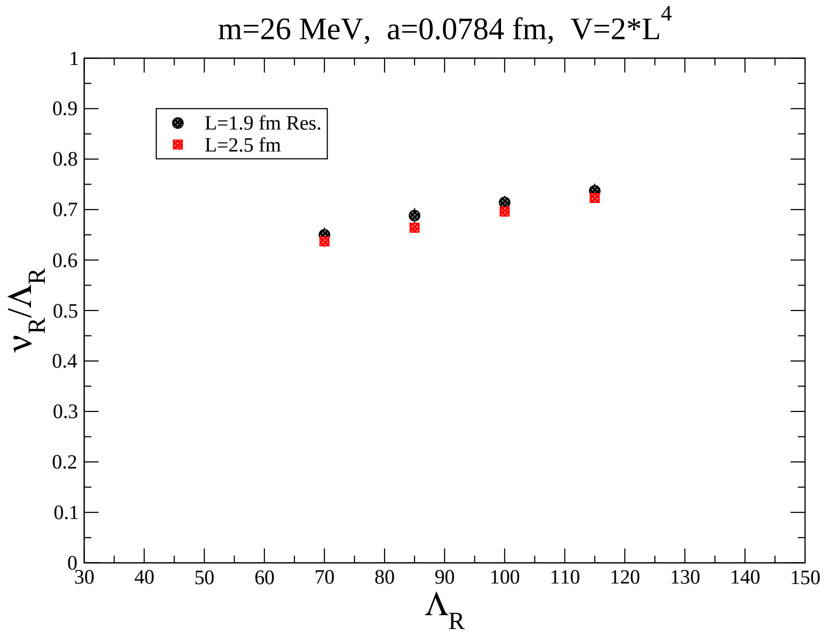

- Scaling is $\propto V$ as expected in presence of SSB.
  Sub-leading effects smaller than statistical errors of $\sim 1.5\%$
- NLO ChPT predicts deviations to be

$$1 - \frac{\nu_{\rm R}^V}{\nu_{\rm R}^\infty} \propto e^{-M_\Lambda L/2}\,, \qquad M_\Lambda^2 = \Lambda \frac{M_\pi^2}{m}$$

a fraction of a percent for our parameter choice

Lattice details:

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$* \; a = 0.0784$ fm

$* \; V = 2L \times L^3$, $L = 1.9, 2.5$ fm

$* \; m_R^{\overline{\rm MS}}(2\,{\rm GeV}) = 12.8, 26.5, 45.8$ MeV

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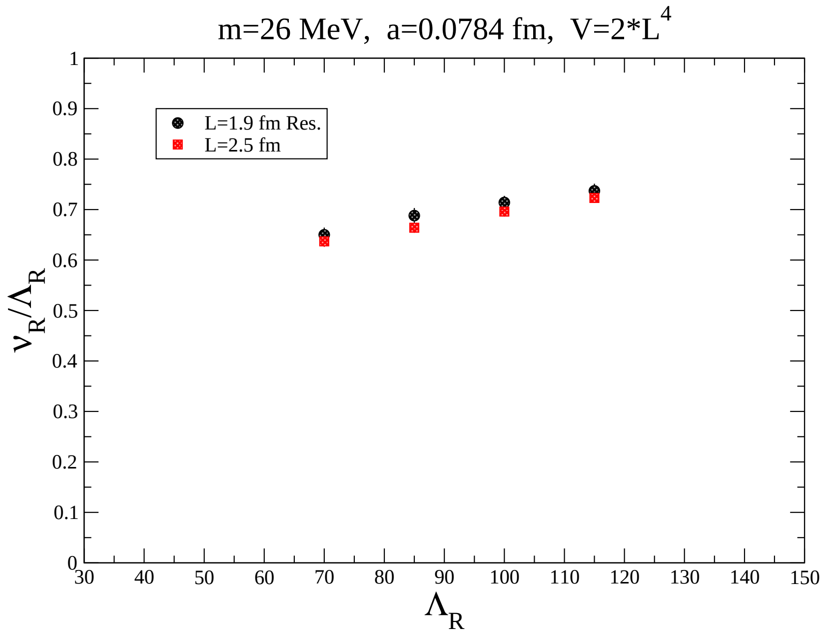
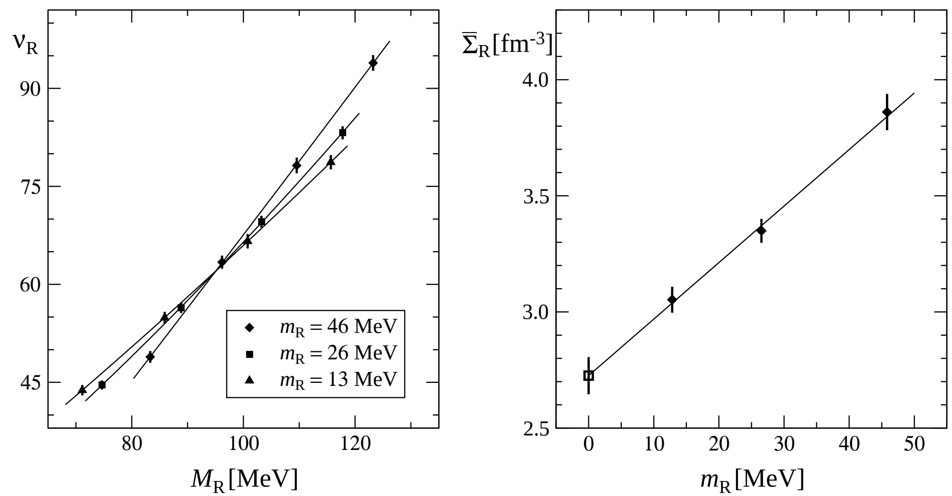


- Scaling is $\propto V$ as expected in presence of SSB.
- Eigenvalues condensate near zero with spacing $\propto 1/V$ even without chiral symmetry!
- One my speculate that SSB is an effect of the condensation

Numerical computation: quark mass dependence



- An effective condensate can defined as

$$\overline{\Sigma}_R = \frac{\pi}{2V}\sqrt{1-\left(\frac{m_R}{M_R}\right)^2}\,\frac{\partial}{\partial M_R}\nu_R(M_R, m_R)$$

with the prefactor chosen so that $\overline{\Sigma}_R$ coincides with Σ to LO

- Extrapolation to the chiral limit mild. At NLO no chiral logs $\propto \ln(m)$

Numerical computation: summary of the results for $L = 2.5$ fm and $M_{\rm R} = 95$ MeV

- The simulation results for $\overline{\Sigma}_{\rm R}$ at $M_{\rm R} = 95$ MeV are

$m_{\rm R}$[MeV]	$\overline{\Sigma}_{\rm R}^{1/3}$[MeV]
$12.8(2)(3)$	$286(2)(4)$
$26.5(2)(6)$	$295(2)(4)$
$45.8(3)(11)$	$310(2)(4)$

and a linear extrapolation to the chiral limit yields the estimate

$$\Sigma^{1/3} = 276(3)(4)(5)\text{MeV}$$

- To be compared, for instance, with the value extracted from the GMOR by the ETMC

$$\Sigma_{\rm eff}^{1/3} = 267(2)(10) \qquad \text{[C. Urbach @ Lattice 2008 for the ETM Coll.]}$$

or from fixed topology simulations by JLQCD

$$\Sigma_{\rm eff}^{1/3} = 243(4)(0)\ \text{MeV} \qquad \text{[Fukaya et al. 09]}$$

Conclusions and outlook

- Simulations with dynamical fermions are feasible in interesting ranges of parameters

$$a = 0.04 - 0.08 \text{ fm}\,, \qquad M_\pi = 200 - 500 \text{ MeV}$$

thanks mainly to a breakthrough in algorithms and also to the Moore's law. The chiral regime of QCD is accesible to simulations

- Spectral density of the Dirac operator renormalizable, i.e. universal continuum limit

- Spectral projectors open a new perspective to study the chiral regime of QCD:
    - $*$ Chiral condensate without power divergences
    - $*$ Topological susceptibility without power divergences
    - $*$ Ward Identities . . .

- The moderate computational cost allows for the infinite volume and continuum limits

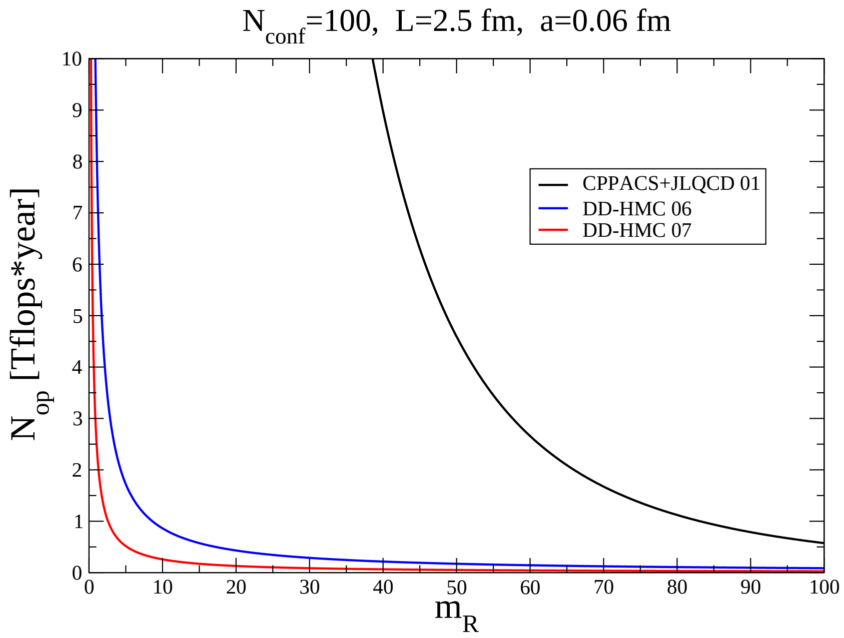
- More simulations (running) needed to estimate the systematics on Σ with confidence

A huge effort (and progress) in the community to simulate light dynamical quarks

Groups	N_f	Action	$a[\mathrm{fm}]$	$L[\mathrm{fm}]$	LM_π	$M_\pi[\mathrm{MeV}]$
CERN-TOV (CLS)	2	$O(a)$-imp. Wilson	≥ 0.04	≤ 2.5	≥ 3.2	≥ 260
ETM	2	Twist-mass	≥ 0.07	≤ 2.8	≥ 3.7	≥ 270
JLQCD	2	Neuberger	0.12	1.9	≥ 2.7	≥ 280
QCDSF+UKQCD	2	$O(a)$-imp. Wilson	≥ 0.07	≤ 2.5	≥ 4.2	≥ 340
BMW	2+1	Stout-link Wilson	≥ 0.07	≤ 4.0	≥ 4.0	≥ 200
PACS-CS	2+1	$O(a)$-imp. Wilson	0.09	2.9	≥ 2.3	≥ 160
RBC/UKQCD	2+1	Domain Walls	0.11	2.7	≥ 4.5	≥ 330

- Simulations with light dynamical quarks are clearly feasible

- Large amount of theoretical and numerical work also with rooted staggered fermions. See reviews by Creutz, Kronfeld and Golterman



Breakthrough in algorithms:

∗ Multiple step size integration
[Sexton, Weingarten 92]

∗ Frequency splitting of the fermion force in molecular dynamics
[Hasenbusch 01; Lüscher 03-04; Urbach et al. 06]

∗ Deflation of low fermion modes
[Lüscher 07]

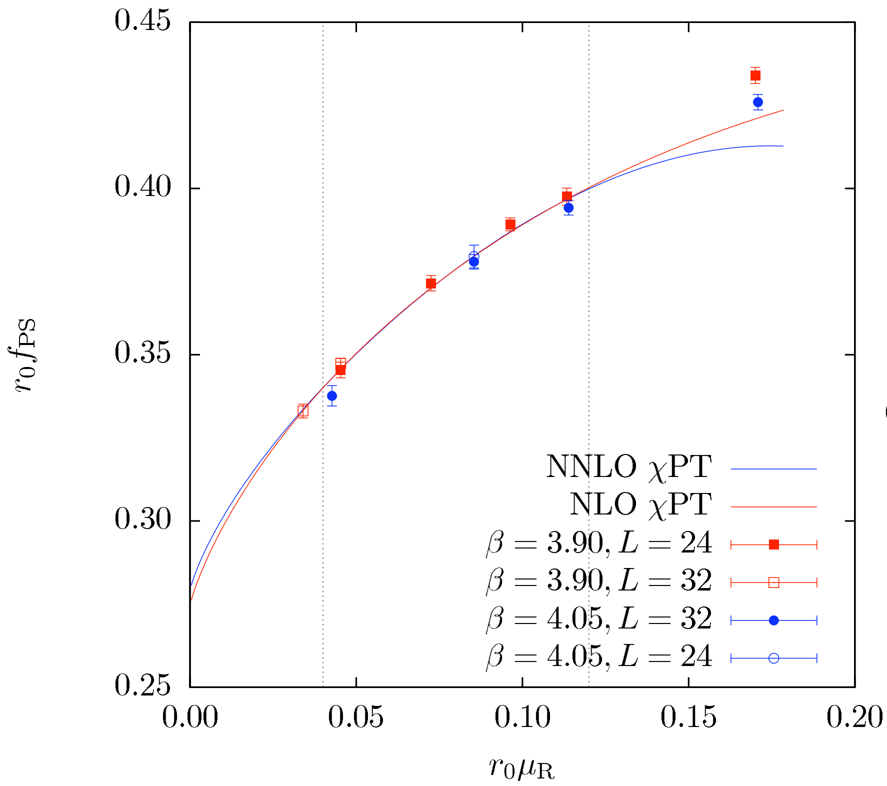
- A very crude cost formula (still large uncertainties) for the DD-HMC algorithm ($N_{\rm f} = 2$)
[Lüscher 03-07; Del Debbio et al. 06-07]

$$N_{\rm op} \sim k \left(\frac{\#\text{confs}}{100}\right)\left(\frac{20\,\text{MeV}}{m_R}\right)\left(\frac{L}{3\,\text{fm}}\right)^5\left(\frac{0.1\ \text{fm}}{a}\right)^6\ \text{Tflops} \times \text{year}$$

with $k = 0.010$–0.015. Similar formulas apply to the other recent algorithms

- Simulations feasible in the interesting range of parameters with computers of sustained power of 5–10 Tflops. Collaborations tend to be of tens of peoples

Pion decay constant with twisted mass quarks [C. Urbach @ Lattice 2008 for the ETM Coll.]



- constant continuum extrapolation
- red: $\beta = 3.90$
- blue: $\beta = 4.05$

overall χ^2:

- NLO fit: $\chi^2/\mathrm{dof} = 21/19$
- NNLO fit: $\chi^2/\mathrm{dof} = 19/19$
- NNLO, extended fit-range $\chi^2/\mathrm{dof} = 50/23$

- The lightest pion is 265 MeV. Similar results by the other collaborations
- For an updated discussion on F_K/F_π: L. Lellouch plenary talk @ Lattice 2008