Algebraic Geometry and Lattice Landau Gauge Fixing

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Plan of The Talk:

1. Motivation

2. Lattice Landau gauge for compact U(1)

   Groebner basis – Method and Results

   Numerical Algebraic Geometry (Polynomial Homotopy)

3. Results

4. Conclusion

Motivation

• Field theories on the lattice – extremely successful non-perturbative method

• Systems of non-linear equations/minimization of multivariate functions

• Many, if not all, CAN BE VIEWED having polynomial-like non-linearity

• Can use Algebraic Geometry methods !

Lattice Landau Gauge for Compact U(1)

Usually, Landau gauge fixing on the lattice is done via minimizing a gauge-fixing functional,

$$F(\theta) = \sum_{i,\mu} (1 - \cos(\phi_{i,\mu} + \theta_{i+\hat{\mu}} - \theta_i)) = \sum_{i,\mu} (1 - ReTr\, \Omega_{i+\hat{\mu}} U_{i,\mu} \Omega_i^{\dagger})$$

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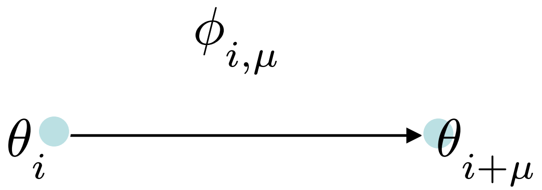
On lattice, gauge fields lie on the link variables, for i-th link in μ -th direction,

$$U_{i,\mu} = e^{i\phi_{i,\mu}}$$

Where, $\phi_{i,\mu}$ is the link varible



$\phi_{i,\mu}$ corresponds to A_μ in continuum case.

And its gauge transformation is $\phi^g_{i,\mu} = \phi_{i,\mu} + \theta_{i+\hat{\mu}} - \theta_i$

$$\theta_i \in (-\pi, \pi]$$

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$$\phi^g_{i,\mu} = (\phi_{i,\mu} + \theta_{i+\hat{\mu}} - \theta_i) \bmod 2\pi \in (-\pi, \pi]$$

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Random phase XY model, Kuramoto Model

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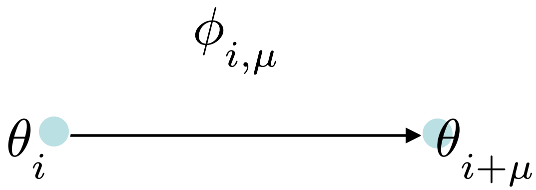

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First, we will work with the trivial orbit case i.e. $\phi_i = 0$, in 1-d.

Using trigonometric relations, the gauge-fixing equations are:

$$f_i(\theta) = \frac{\partial F}{\partial \theta_i} = \cos\theta_i(\sin\theta_{i+1} + \sin\theta_{i-1}) - \sin\theta_i(\cos\theta_{i+1} + \cos\theta_{i-1}),$$

$$i = 1, \ldots, n$$

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By taking $\cos\theta_i = c_i$ and $\sin\theta_i = s_i$

$$f_i(c,s) = c_i(s_{i+1} + s_{i-1}) - s_i(c_{i+1} + c_{i-1}) = 0$$

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And I introduce additional constraint equations,

$$g_i(c,s) = \mathrm{s}_i^{\,2} + c_i^{\,2} - 1 = 0, i = 1, \ldots, n$$

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$$g_i(c, s) = \mathrm{s}_i^{\,2} + c_i^{\,2} - 1 = 0, i = 1, \ldots, n$$

e.g. for the $n = 3$ lattice sites case in 1-d, the new gauge-fixing equations are

$$f_1(c,s) = -c_2 s_1 - c_3 s_1 + c_1 s_2 - c_1 s_3$$
$$f_2(c,s) = c_2 s_1 - c_1 s_2 - c_3 s_2 + c_2 s_3$$
$$f_3(c,s) = -c_3 s_1 + c_3 s_2 - c_1 s_3 - c_2 s_3$$
$$g_1(c,s) = c_1^2 + s_1^2 - 1$$
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So, we have now $2n$ polynomial equations for $2n$ variables c_is and s_is, instead of n trigonometric equations for n no. of θ variables.

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Can use two methods:

1. Groebner basis technique
2. Numerical Algebraic Geometry

Groebner Basis

- Very roughly speaking, one can obtain another system of polynomial equations by performing a finite set of operations on the original system (the Buchberger algorithm)

- The new system is ‘easier’ to solve (how? A. Will be clear soon)

- The new system has the same solutions as the original

- The new system is called the Groebner basis

- Packages like Singular, COCOA, McCAULEY2, Maple, Mathematica, etc.

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Mathematica-interface of Singular STRINGVACUA by Oxford-Durham group

How is it helpful ?!!

Mathematica gives for the $n = 3$ case,

$$-s_3 + s_3^2 = 0,$$
$$c_3 s_3 = 0,$$
$$-1 + c_3^2 + s_3^2 = 0,$$
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And solve all equations by back-substitutions.

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There are 16 solutions.

Numerical Algebraic Geometry/ Homotopy Continuation Method

Thanks to Applied Mathematicians !!!

Jan Verschelde, A Sommese, CW Wampler, TY Lee, etc.

Basic strategy:

1. Estimate the number of solutions of the given system
2. Take another 'simple' system that has the same number of solutions, and solve it.
3. 'Track' these solutions towards the given system using a homotopy

- There are well-written packages for these calculations, e.g. PHCPack, HOM4PS2, PHoM, Bertini
- All are free !!!

Estimate of the no. of Solutions

For a given system of algebraic equations known to have finite no. of solutions, there exists an upper bound for its number of solutions:

Bernshtein/Bezout bound: Let $f_i(x_1, x_2, , x_k) = 0$, $i = 1, , k$ algebraic equations and their maximum degrees d_i. Then this system of equations can have at most $\prod_{i=1}^{k} d_i$ no. of complex and real solutions.

In the $n = 3$ case, there are $k = 6$ equations all of which have degree $d_i = 2$ and so they can have at most $2^6 = 64$.

Take another system whose solutions are known and the upper bound for the no. of solutions is the same as the one for the above system, e.g.

$$\vec{g}(c,s) = \begin{pmatrix} c_1^2 - 1 \\ c_2^2 - 1 \\ c_3^2 - 1 \\ s_1^2 - 1 \\ s_2^2 - 1 \\ s_3^2 - 1 \end{pmatrix}$$

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Then take the homotopy of these two,

$$\vec{H}((c,s),t) = (1-t)\vec{f}(c,s) + e^{i\gamma} t \vec{g}(c,s) = 0$$

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For $t = 1$, i.e. $H((c,s),t) = \vec{g}(c,s)$, the solutions are known.

Start from $t = 1$ and go towards $t = 0$ for each solution and by predictor-corrector method, check if the path is smooth i.e. there is a solution for $\vec{f}(c,s)$.

Groebner Basis	Numerical Algebraic Geometry
1. Exact solutions	Numerical, but ALL solutions/extrema
2. Exponential space complexity	No such scaling problems
3. Highly sequential	‘Embarrassingly’ parallelizable
4. Non-integer coefficients a problem	Any floating point coefficients are fine

In particular, for the 2-dimensional lattice, the 3x3 lattice case was intractable for the Groebner basis technique, but it took only 1 hour with the NAG.

Applications?

Many !!! In lattice field theories, statistical mechanics, complex systems, theoretical chemistry etc.

ALL Gribov copies !

Let's check the Neuberger zero:

H Neuberger: The gauge-fixed partition function on the lattice is zero and so the expectation values of a gauge-fixed observable is 0/0 !

Martin Schaden: For a compact gauge group, the gauge-fixing partition function computes the Euler character of the group manifold at each site. And with the Poincare-Hopf theorem,

$$Z_{GF} = (\chi(G))^n = \sum_{\text{Gribov copies}} \text{sgn}(\det M_{FP})$$

For compact U(1) and for SU(N), the Euler char. is 0 !

Trivial Orbit (Classical XY model), 3x3 lattice:

CBB	$AllSol.$	$\mathrm{Re}\,al$	$NonSingular$	$Singular$
262144	10738	2968	1816	1152

Singular solutions = Gribov horizon
Real solutions = No. of Gribov copies

For the 1816 nonsingular solutions,

i	0	1	2	3	4	5	6	7	8	9
$K_i =$ no. of sol. with i neg. evalues	2	18	216	342	330	330	342	216	18	2

i.e., 1st, …, 9th , 10th Gribov region. Thus,

$$\sum_{\text{Gribov copies}} \mathrm{sgn}(\det M_{FP}) = 0$$

Random Orbit, 3x3 lattice:

CBB	$AllSol.$	$\mathrm{Re}\,al$	$NonSingular$	$Singular$
262144	20558	2480	2480	0

Singular solutions = Gribov horizon
Real solutions = No. of Gribov copies

For the 2480 nonsingular solutions,

i	0	1	2	3	4	5	6	7	8	9
K_i = no. of sol. with i neg. evalues	2	58	202	402	576	576	402	202	58	2

i.e., 1st, …, 9th , 10th Gribov region. Thus,

$$\sum_{\text{Gribov copies}} \mathrm{sgn}(\det M_{FP}) = 0$$

Other examples - work in progress…

• Solving classical field equations for various models on the lattice, e.g., compact QED/Instantons, Abelian Higgs model, XY model, Heisenberg model, SU(2) Yang-Mills theory etc. !!!

• Parameterization of SU(N) (with Jon-Ivar Skullerud)

• Complex Systems

• Modified Lattice Landau Gauge (von Smekal, Mehta, Sternbeck, Williams 2007)

Conclusions

• Many problems in lattice field theories are non-linear with polynomial-like non-linearity

• Computational and Numerical Algebraic Geometry can be of great help – can replace the existing numerical methods on the lattice at least in lower dimensional models

• NAG is worth-attempting, due to its efficiency, in many of the statistical mechanics and lattice field theory problems to solve the corresponding non-linear equations