General method(s) of solution of SDEs in Minkowski space

Vladimír Šauli
CFT PIST Lisbon, OTF UJF Rez
sauli@ujf.cas.cz

QCD TNT, Trento, Sept, 2009

Outline

- Motivation and introduction
- Semiperturbative methods based on KLR
- Standard philosophy versus direct Minkowski calculation
- Minkowski solution for CSB QED2+1, E-M contour continuation
- DCSB in 3+1 Minkowski space, Technicolors, QCD

Motivation and Introduction (M+I)

- - In perturbation theory (PT) Wick rotation is OK. Euclidean and real Lorentz (Minkowski) space results are related and relation is well established.
- In momentum space GFs have real branch points as a consequence of matching poles in convolution of distributions. The location of branch points corresponds with physics– with opening threshold productions.
- Real and imaginary parts are uniquely* related- dispersion relation , unitarity relations..

*up to scheme dependence of GFs

DCS and confinement is beyond access of PT

Hadrons $P^2 > 0$, first principle calculation: GFs of confined fields are basic element to build hadronic wave functions. Nonperturbative solutions in Minkowski space is welcome.

- Standard philosophy for nonperturbative method:

- SDEs are selfconsistent- what enters SDEs is output as well. GFs are momentum dependent and thouse they appear inside the integrals. Analyticity is not guaranted, but it is assumed or even defined!

- The philosophy is to start with Euclidean space, and analytically continue Euclidean results to Minkowski space (no matter what the singularities are, even if their structure would prevent original Minkowski–> Euclidean)

Standard philosophy: Start with Euclidean metric $\delta_{\mu\nu}$; $p_E = (p_1, ...p_N)\,; p_E^2 = p_1^2 + ..p_N^2,$

make continuation $p_E^2, ... \to -p_E^2, ...$

and believe that GFs stemming from Euclidean generating functional are well defined as the Minkowski spacelike GFs $p_0 < |\vec{p}|$, so after continuation we get all Minkowski result $p_0 > |\vec{p}|$.

Numerical study of strong coupling QFTs: Branch points others then real ones were identified in various approximation of SDEs

P. Maris QED3 (1995), P. Maris & H.A. Holties , IJMP, (1992)- ladder QCD

Philosophy II.

Solve SDEs system directly in Minkowski space.

- Problem 1: small problem- possible singularities of GFs- poles, branch points

- Problem 2:. big problem- singularities in SDEs kernels stemming from integral measure because of Minkowski metric–>

  very often very badly defined numerical problem

- Type A,B solutions

- A. Solutions constrained by analyticity assumptions, WR allowed and well defined. 1+2 are elegantly solved, but assumption must be checked afterwords (it does not work for strong QFT)

- B. No assumptions needed. 1. problem -solvable 2. Unknown in general, but simple models are solvable

A:Semiperturbative methods based on KLR

Assuming perturbative anlyticity of Green's function,e.g. Propagators satisfy Khallen-Lehman representation (KLR), (n-vertices- more complicated forms),

KLR:

$$G^{KLR}(k) = \int_0^\infty dM^2 \rho(M^2) D(k^2, M^2)$$

$$D(k, M^2) = \frac{1}{k^2 - M^2 + i\varepsilon}$$

with property

$$< D >= \int_{-\infty}^{\infty} dk^2 D(k) = -i\pi$$

Independently on the details of the models the momentum SDEs turns to "regular" equation for Lehmann weight. Using the Feynman tricks we always get for the inverse propagator the dispersion relations,

$$G^{-1} = polyn. + \int_{T}^{\infty} dM^2 \sigma(M^2) D(k^2, M^2)$$

It works! SDE are solved above threshold by comparing Re and Im parts of propagators. but It does not work for strong coupling theory. KLR and DR must be valid everywhere.

Underthreshold and spacelike check of An.assumption

σ, rho are solutions above T by the construction. Use them independently and compare obtained GFs bellow T. If they disagree then neither of these is solution of original momentum SDE.

Very likely toady known theories with dynamical symmetry breaking and confinement do not give GFs possessing KLR and DR. Until 2009 all checked solutions of SDEs based on KLR (i.g. gauge technique, CSB solutions) exhibit discrepancies between $G(\sigma)$ and $G(\rho)$ Checked cases: gauge technique based solution by Cornwall, Papavasilliou 2009, Sauli 2009, CBS by Sauli,Bicudo,Adam. In best, KLR based solution can serve as a gross approximation of true solution.

$C - R_+$ analyticity is too strong assumption for QCD gluon and quark propagators.

QED2+1

B- model

Example of numerically soluble theory in M Space: ladder fermion SDE in QED2+1

- problem 1- is completely absent since no pole on the real p^2 axis exists
- problem 2- Clearly absent in $A = 1$ approximation

$$S = S_s(p^2)\ \not{p} \quad + \quad S_v(p^2)$$

$$S_s = \frac{B(p)}{A^2(p)p^2 - B^2(p)} \quad ; \quad S_v = \frac{A(p)}{A^2(p)p^2 - B^2(p)}$$

ladder SDE in Landau gauge:

$$\Sigma_A = \frac{ie^2}{p^2}\int \frac{d^3k}{(2\pi)^3} S_v(k^2)G(q^2)\left[2(q^2-p^2-k^2)+\frac{1}{2}(k^2+p^2)-\frac{k^2-p^2}{2q^2}\right]$$

$$\Sigma_B = -ie^2 2\int \frac{d^3k}{(2\pi)^3} S_s(k^2)G(q^2)$$

$$q = k-p\,;A=1-\Sigma_A\,;B=m+\Sigma_B;G=\frac{1}{q^2}.$$

Numerically advantage- the arguments of propagators are identical with the integral variables,i.e. k^2, p^2

Solution for timelike region p^2.

$p^\mu = (p, 0, 0)$

the most general loop integral in timelike Minkowski subspace reads

$$
\begin{aligned}
I[U, V; N, p^2) &= i \int \frac{d^3k}{(2\pi)^3} U(k^2) V(q^2) (k.q)^N = \\
&- \frac{i}{2\pi^2} \frac{1}{4\sqrt{p^2}} \int_0^\infty dk^2 \int_{(k-p)^2}^{(k+p)^2} dq^2 U(k^2) V(q^2) (\frac{1}{2}(q^2 + p^2 - p^2))^N \\
&+ \frac{i}{2\pi^2} \frac{1}{2\sqrt{p^2}} \int_{-\infty}^{\infty} dk^2 \int_{-\infty}^{\infty} dq^2 U(k^2) V(q^2) (\frac{1}{2}(q^2 + p^2 - p^2))^N \\
k &= \sqrt{k^2}\, ;\; p = \sqrt{p^2}
\end{aligned}
$$

Note $< 1/q^2 >= 0$ and q^2 integration can be performed, leading to

$$B(p^2) = m + \frac{ie^2}{4\pi^2} \int dk^2 S_s(k^2) ln \frac{|k-p|}{(k+p)}$$

The propagator functions are complex

$$\begin{aligned} S_s &= \frac{B(p)}{A^2(p)p^2 - B^2(p)} = \frac{R_B c_1 + \Gamma_B c_2}{c_1^2 + c_2^2} + i \frac{\Gamma_B c_1 - R_B c_2}{c_1^2 + c_2^2}\,; \\ S_v &= \frac{R_A c_1 + \Gamma_A c_2}{c_1^2 + c_2^2} + i \frac{\Gamma_A c_1 - R_A c_2}{c_1^2 + c_2^2} \\ c_1 &= (R_A^2 - \Gamma_A^2) p^2 - (R_B^2 - \Gamma^2) \\ c_2 &= (R_A \Gamma_A p^2 - R_B \Gamma^2) \end{aligned}$$

R, Γ are Re an Im parts of proper GFs $B = R_B + i\Gamma_B\,; B = R_B + i\Gamma_B$.

QED2+1 versus QED3

Stress k, p are timelike fourmomenta, solutions in spacelike and timelike regions have decoupled.

S_s is complex function in Minkowski space

$$S_s = \frac{B(p)}{A^2(p)p^2 - B^2(p)}$$

Compare to Euclidean partner (note $p_E^2 = -p^2$)

$$S_s = \frac{B(p_E)}{A^2(E)p_E^2 + B^2(p_E)}$$

Euclidean ladder SDE:

$$B_E(p_E^2) = m - \frac{e^2}{4\pi^2} \int_0^\infty dk_E^2 S_s(p_E) ln \frac{|k-p|}{(k+p)}$$

Minkowskian ladder SDE for timelike p:

$$B(p^2) = m + \frac{\mathrm{i}e^2}{4\pi^2} \int_0^\infty dk^2 S_s(k) ln \frac{|k-p|}{(k+p)}$$

QED2+1 versus QED3

Recall known:E space: Appelquist1988, Burden1992, Gusynin1995, Bashir, Maris,... M space:V.S.& B. Zoltan, arXiv:0901.0110

B_E is real function in Euclidean space, while B is complex, we have DCS for any e (topologically massive theory, B is known to be nontrivial when $m = 0$) in both cases.

Two main messages:

1. Euclidean SDE is formally derivable by using the contour:

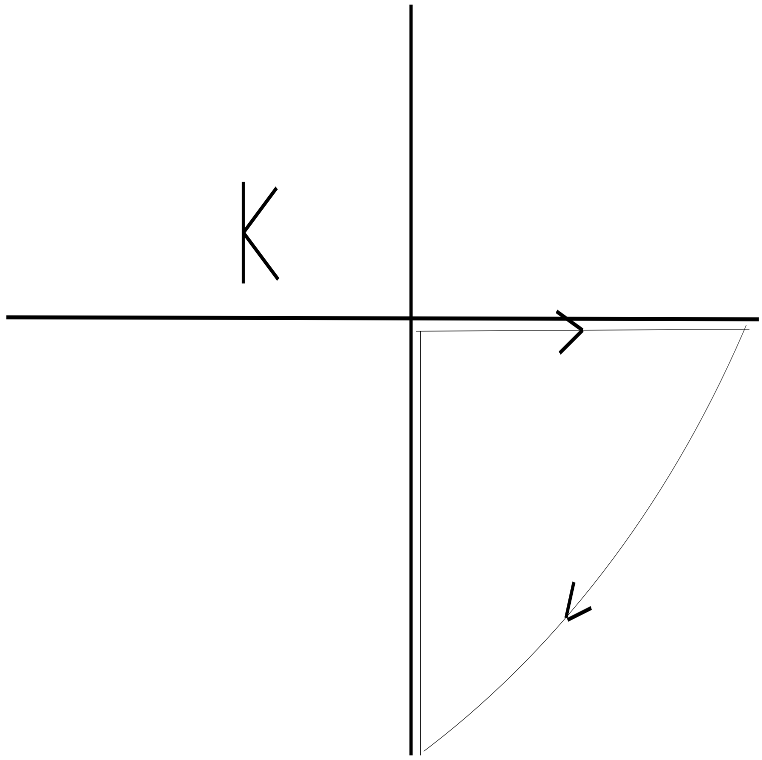

Figure 1: Minkowski –> Euclidean continuation for ladder QED2=1

WITHOUT further deforming contour in fig. 1 and considering contributions from complex branch points observed in P. Maris 1995 , however timelike Minkowski solution is not! analytical continuation of the Euclidean one (apparently, M would be non-holomorphic here $p^2 = 0$), WR is invalid.

Numerical results QED2+1

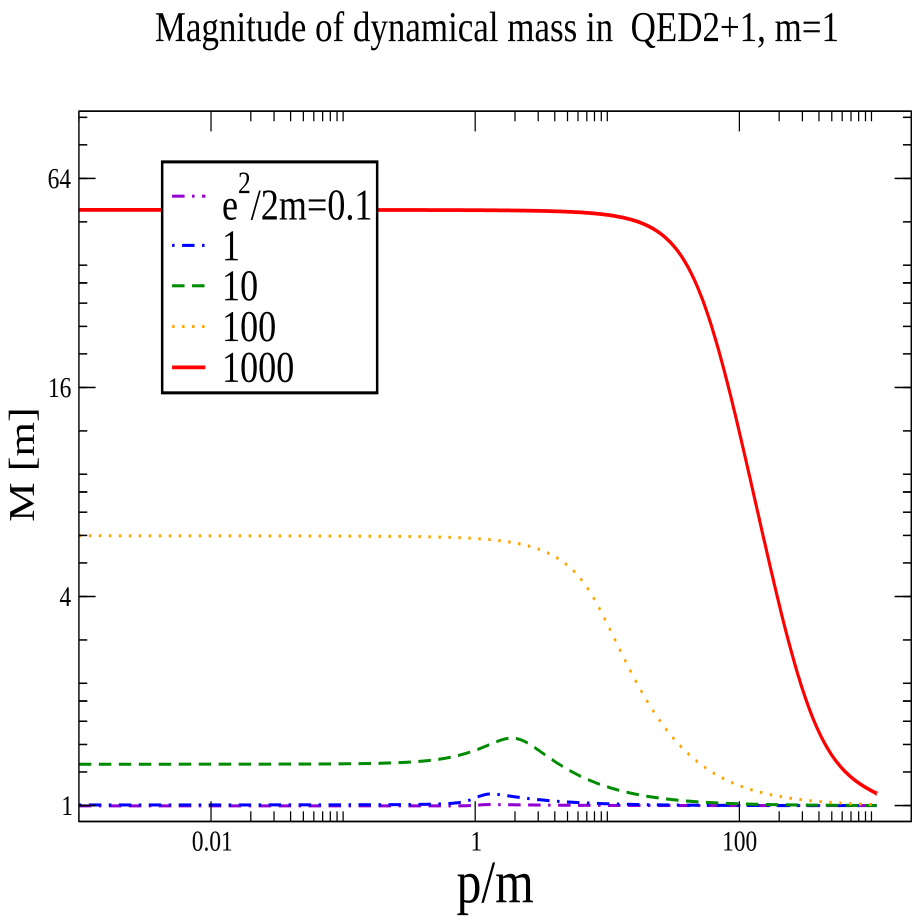

Figure 2: **Abs. value of electron mass function.**

Numerical results QED2+1

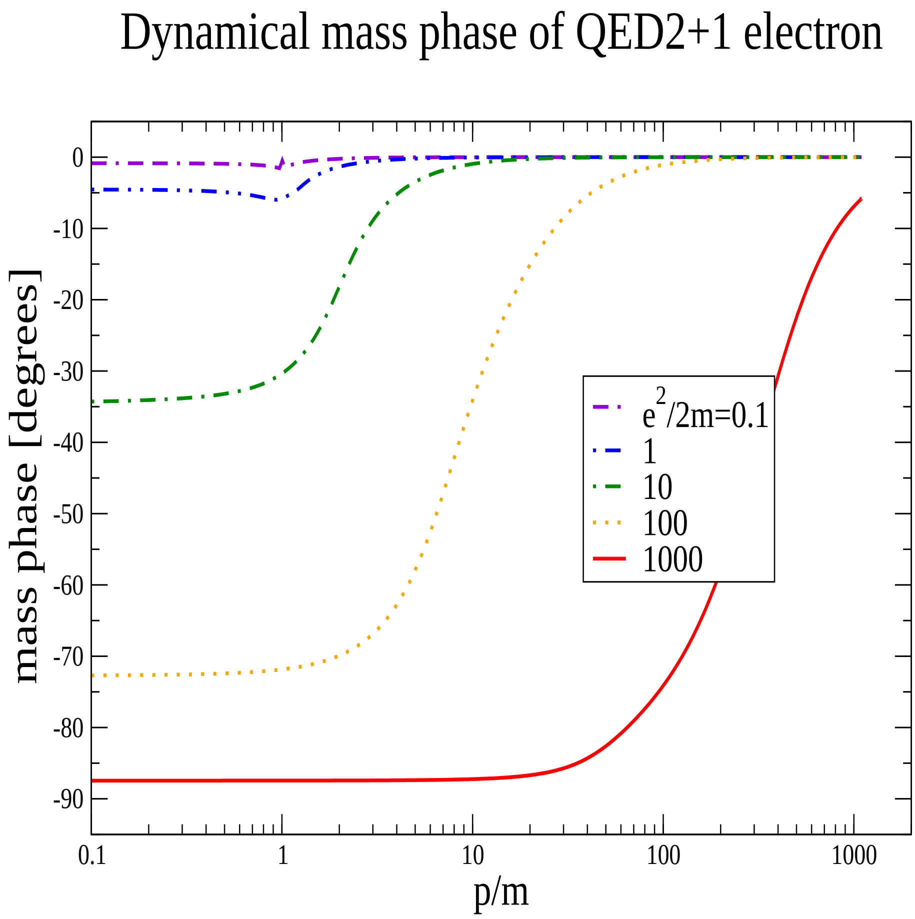

Figure 3: **Argument of dynamical electron mass in QED2+1**

$$\phi_{M_{DCS}} = 88^o$$

QED2+1

2. B is complex for all p^2 with large non-zero Im –> absence of real pole in GFs = confinement of given degrees of freedom

Spacelike Minkowski solution:

- Direct Minkowski calculation for spacelike momenta- not numerically well defined problem (more in 4D)

- Continuation of known timelike solution is possible (sometimes)

$$B(-p^2) = m + \frac{ie^2}{4\pi^2\sqrt{p^2}} \int_0^\infty dk^2 S_s(k) arcsin \frac{2kp}{(k^2+p^2)}$$

where S_s is timelike.

DCS in 3+1 Minkowski space, Technicolors, QCD

First recall: spacelike physics still not reasonably accessible unless WR is not performed in practice.

Typical integral in four dimension:

$$B(p) \simeq i \int \frac{d^4k}{(2\pi)^4} G(k^2) S(q^2)$$

Lorentz invariance dictates $B(p^2)$ for any p^μ, $p^\mu = (0,0,0,p)$ ->

$$for\ p^2 < 0,$$

$$B(p) \simeq \int_{-1}^{1} dz \left(\int_0^\infty dk^2 \int\limits_{k^2+p^2}^{sgnz\infty} dq^2 + \int_{-\infty}^{0} dk^2 \int\limits_{k^2+p^2+2\sqrt{-p^2}\sqrt{-k^2}z}^{sgnz\infty} dq^2 \right)$$

$$\frac{1}{z^3}\frac{1}{k^2+a/z^2} I G(k^2) S(q^2)$$

$$I = \frac{(q^2-k^2-p^2)^2}{8(\sqrt{-p^2})^3}\ ; a = \frac{(q^2-k^2-p^2)^2}{2(\sqrt{-p^2}^2)}$$

Study of analytical continuation of Euclidean results P.Maris 1995 shows up branch points in complex plane, positions and number of singularities depends on details of interaction. Interaction strong enough can lead to the absence of real branch point in the quark propagator.

Large N_f QCD with CBS

Simplest model in MS. Effective running coupling is constant at low and intermediate q^2 , it vanishes for large q^2 in accordance with asymptotic freedom. Λ_{QCD} becomes effective cutoff, and being close to the critical coupling value we get $m << \Lambda$, i.e. near criticality Technicolor like scenario of DCSB. Effective running coupling and techniquark masses are necessarily real function as they are obtained in Euclidean space.

Simplest model in MS: SDE with running charge:

$$for|q^2| < \Lambda^2 : \alpha_{TC}(q^2) = Const$$
$$for|q^2| > \Lambda^2 \alpha_{TC}(q^2) = 0$$

Const -large enough to establish DCSB

-Timelike region of p^2

Ladder SDE in in Landau gauge Z=1 approx in MS becomes:

$$M(p^2) = m + i const \int \frac{d^4k}{(2\pi)^4} S_s(k) \frac{\alpha_{TC}(q^2)}{q^2}$$

arguments of GFs=integral variables

$$M(p^2) = \frac{iconst}{16p^2\pi^3}\int_0^\infty dk^2 \left[\int\limits_{(k+p)^2}^{\infty} dq^2 \sqrt{\Delta(q^2,k^2,p^2)} + \int\limits_{-\infty}^{(k-p)^2} dq^2 \sqrt{\Delta(q^2,k^2,p^2)} \right.$$

$$\left. + \int_{-\infty}^{0} dk^2 \int_{-\infty}^{\infty} dq^2 \sqrt{\Delta(q^2,k^2,p^2)} \right] S_s(k^2) \frac{\alpha_{TC}(q^2)}{q^2} \; ;$$

$$\Delta(a,b,c) = a^2 + b^2 + c^2 - 2(ab + ac + bc)$$

Not similarly to QED2+1, coupled with SDEs for spacelike arguments, still numerically not well defined problem, factorization of infinities which are not truly presented in the solution.

Assuming continuation in fig1. (performed only for internal momenta here) is a reasonable approximation we get

$$
\begin{aligned}
M(p^2) &= \frac{iconst}{16p^2\pi^3}\int_0^{\infty} dk^2 \left[\int\limits_{(k+p)^2}^{\infty} dq^2 \sqrt{\Delta(q^2,k^2,p^2)} + \int\limits_{-\infty}^{(k-p)^2} dq^2 \sqrt{\Delta(q^2,k^2,p^2)} \right. \\
&\left. - \int_{-\infty}^{\infty} dq^2 \sqrt{\Delta(q^2,-k^2,p^2)} \right] S_s(k^2) \frac{\alpha_{TC}(q^2)}{q^2} \; .
\end{aligned}
$$

Numerical Results, large N_f QCD

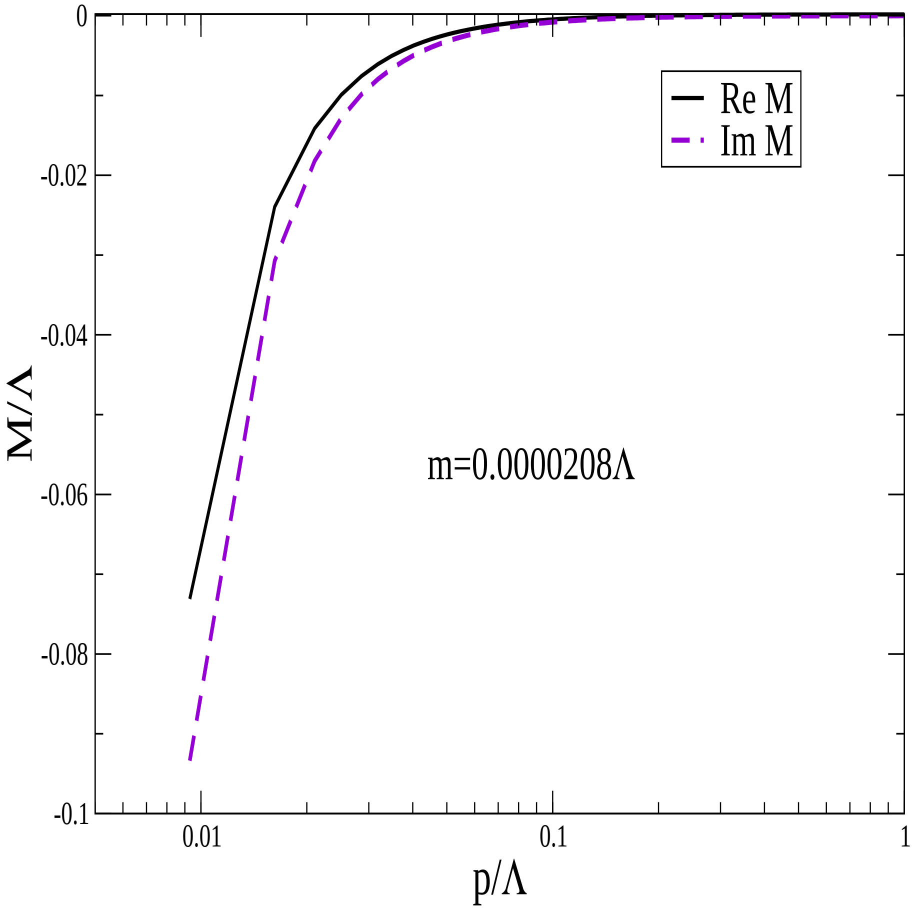

Figure 4: Re an Im parts of M

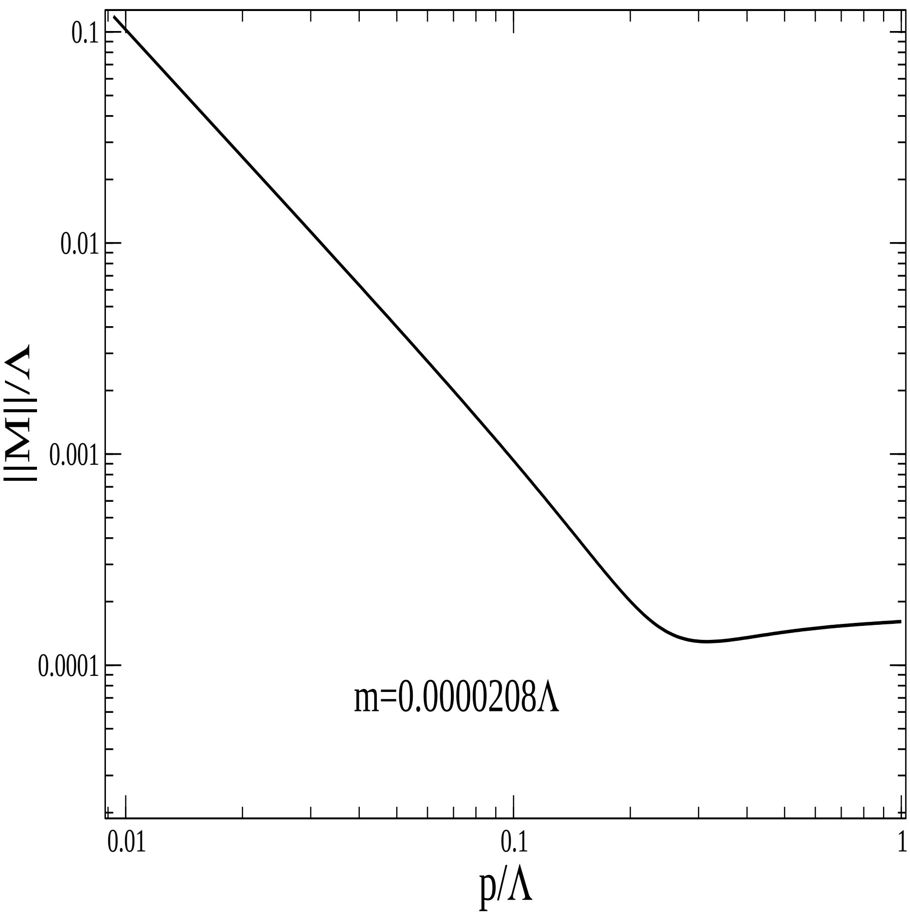

Figure 5: Absolute value

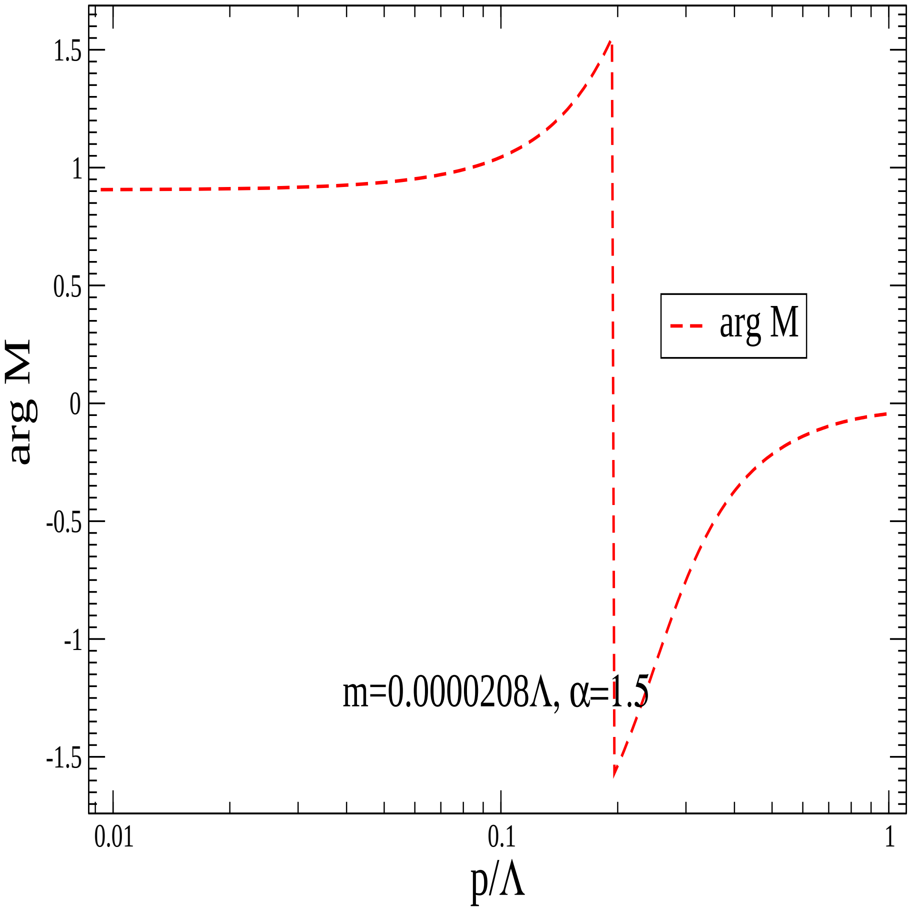

Figure 6: Dynamical fermion mass in 3+1 MS -phase ϕ, $M = ||M||e^{i\phi}$

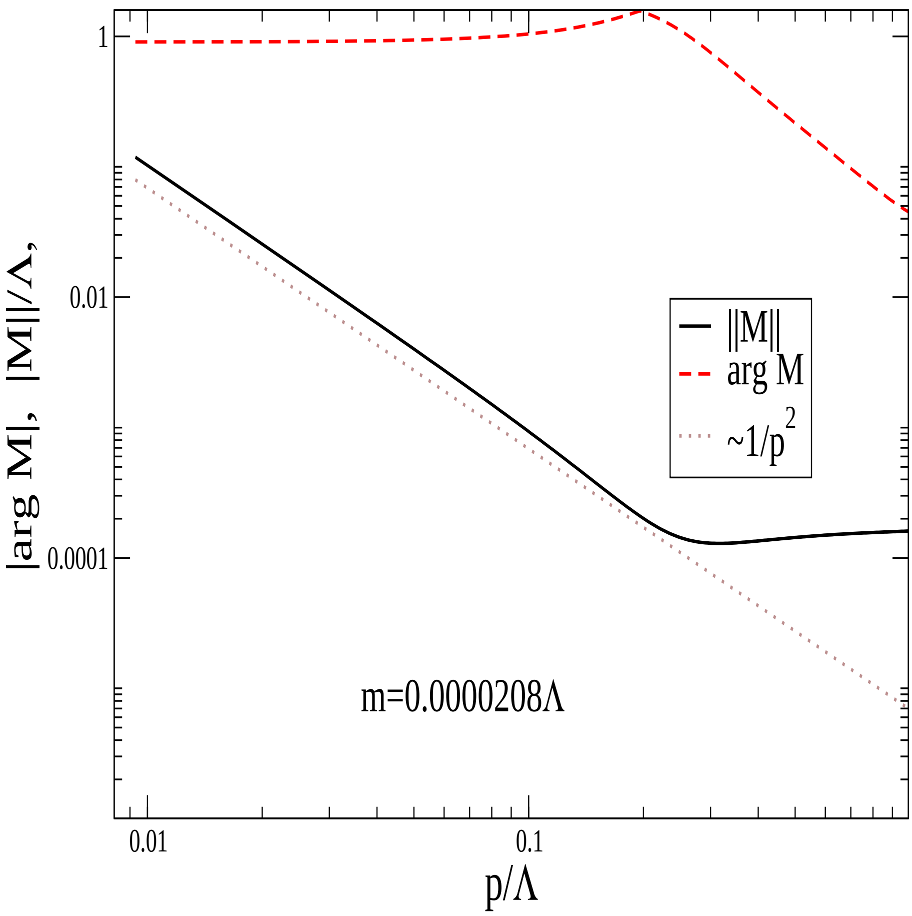

Figure 7: Dynamical fermion mass in 3+1 Minkowski space

DCS in 3+1 Minkowski space, Technicolors, QCD

GFS of confined objects-techniquarks are complex function without real pole (i.e. no free particle in in- and out- states) and without real branch points. Following LSZ-reduction formula for S-matrix such objects does not appear for asymptotically large times as a physical states. Quarks (techniquarks) are permanently confined (independently on the details how they are confined).

Constituent techniquark mass is not well defined quantity, $x \rightarrow \infty ReM(x), ImM(x) \rightarrow \infty$.

remark: $m \neq 0$ but renormalization undone, since Λ is used also for S!, R is admitable when spacelike physics is known!:(remains to be done

Conclusion:

- Two examples of model with DCSB solved in Minkowski.

- QED2+1 and large N_f QCD ladder SDE for fermions solved

- propagators exhibit confinement through the absence of real pole, no real particles on their mass shells.

- (dis)conection of Euclidean and Minkowskian world exhibited

- Minkowski low energy QCD-recently unknown, low q^2 running coupling, perhaps complex variable, is recently undetermined.