THE TRANSVERSE ANGULAR MOMENTUM SUM RULE AND COMMENTS ON ORBITAL ANGULAR MOMENTUM

Elliot Leader

Imperial College London

partly done in collaboration with

Ben Bakker and Larry Trueman

Outline of talk

1) Reminder why problem is non-trivial in traditional approach

2) The incorrect result

3) New ultra-simple derivation of correct result

4) Comments on orbital angular momentum

Background to the study—or why did we bother to work like slaves for several months?

Shore and White's, at first sight, surprising claim about the axial anomaly.

Background to the study—or why did we bother to work like slaves for several months?

Shore and White's, at first sight, surprising claim about the axial anomaly.

Based on a classical paper of Jaffe and Manohar who stressed the subtleties and warned that 'a careful limiting procedure has to be introduced'.

Background to the study—or why did we bother to work like slaves for several months?

Shore and White's, at first sight, surprising claim about the axial anomaly.

Based on a classical paper of Jaffe and Manohar who stressed the subtleties and warned that 'a careful limiting procedure has to be introduced'.

Despite all the care, there are flaws. With the J-M result one cannot have a sum rule for a transversely polarized nucleon.

With the correct version one can!

What is the aim???

We consider a nucleon with 4-momentum p^μ and covariant spin vector S corresponding to some specification of its spin state e.g. helicity, transversity or spin along the Z-axis i.e. a nucleon in state $|p, S\rangle$.

What is the aim???

We consider a nucleon with 4-momentum p^μ and covariant spin vector S corresponding to some specification of its spin state e.g. helicity, transversity or spin along the Z-axis i.e. a nucleon in state $|p, \mathsf{S}\rangle$.

We require an expression for the expectation value of the angular momentum in this state i.e. for $\langle p, \mathsf{S}|\boldsymbol{J}|p, \mathsf{S}\rangle$

i.e. we require an expression in terms of p and S. This can then be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

Why is this useful?

Why is this useful?

When we introduce a model of the nucleon it can be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

Why is this useful?

When we introduce a model of the nucleon it can be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

But models (e.g. the parton model) are only valid in certain kinematic regimes e.g for partons in the ‘infinite momentum frame’, which, in practice, means a frame where $E \gg M$ for the nucleon.

Why is this useful?

When we introduce a model of the nucleon it can be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

But models (e.g. the parton model) are only valid in certain kinematic regimes e.g for partons in the ‘infinite momentum frame’, which, in practice, means a frame where $E \gg M$ for the nucleon.

So we need an expression for the matrix element valid in any frame!

The traditional approach:

In every field theory there is an expression for the angular momentum density operator. The angular momentum operator $\boldsymbol{J}$ is then an integral over all space of this density.

The traditional approach:

In every field theory there is an expression for the angular momentum density operator. The angular momentum operator $\boldsymbol{J}$ is then an integral over all space of this density.

Typically the angular momentum density involves the energy-momentum tensor density $T^{\mu\nu}(x)$ in the form e.g.

$$\boldsymbol{J}_z = \boldsymbol{J}^3 = \int dV \, [x T^{02}(x) - y T^{01}(x)]$$

Consider the expectation value of the first term in the expression for the angular momentum tensor:

Consider the expectation value of the first term in the expression for the angular momentum tensor:

$$\langle p, \mathsf{S}| \int dV x T^{02}(\boldsymbol{x})|p, \mathsf{S}\rangle = \int dV x \langle p, \mathsf{S}|T^{02}(\boldsymbol{x})|p, \mathsf{S}\rangle$$

Consider the expectation value of the first term in the expression for the angular momentum tensor:

$$\langle p, S| \int dV x T^{02}(\boldsymbol{x}) |p, S\rangle = \int dV x \langle p, S| T^{02}(\boldsymbol{x}) |p, S\rangle$$

$$= \int dV x \langle p, S| e^{i\boldsymbol{P}.\boldsymbol{x}} T^{02}(0) e^{-i\boldsymbol{P}.\boldsymbol{x}} |p, S\rangle$$

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dV x\langle p, \mathsf{S}|T^{02}(0)|p, \mathsf{S}\rangle$$

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dVx\langle p, \mathsf{S}|T^{02}(0)|p, \mathsf{S}\rangle$$

The matrix element is independent of x so we are faced with $\int dVx = \infty$? or $= 0$? Totally ambiguous!

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dVx\langle p,\mathsf{S}|T^{02}(0)|p,\mathsf{S}\rangle$$

The matrix element is independent of x so we are faced with $\int dVx = \infty$? or $= 0$? Totally ambiguous!

The problem is an old one: In ordinary QM plane wave states give infinities

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dVx \langle p, \mathsf{S}|T^{02}(0)|p, \mathsf{S}\rangle$$

The matrix element is independent of x so we are faced with $\int dVx = \infty$? or $= 0$? Totally ambiguous!

The problem is an old one: In ordinary QM plane wave states give infinities

The solution is an old one: Build a wave packet, a superposition of physical plane wave states

In QM we use

$$\Psi_{\boldsymbol{p}_0}(\boldsymbol{x}) = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, e^{i\boldsymbol{p}.\boldsymbol{x}}$$

where $\psi(\boldsymbol{p}_0 - \boldsymbol{p})$ is peaked at $\boldsymbol{p} = \boldsymbol{p}_0$

We then calculate some physical quantity and at the end take the limit of a very sharp wave packet

In field theory we do essentially the same and build a physical wave packet state:

$$|\Psi(\boldsymbol{p}_0)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}\rangle$$

then an expectation value in the state $|\Psi(\boldsymbol{p}_0)\rangle$ will involve non-diagonal matrix elements

$$\langle \boldsymbol{p}'|\boldsymbol{J}|\boldsymbol{p}\rangle$$

What about the spin??? J-M use

$$|\Psi(\boldsymbol{p}_0, S)\rangle = \int d^3\boldsymbol{p}\,\psi(\boldsymbol{p}_0 - \boldsymbol{p})\,|\boldsymbol{p}, S\rangle$$

i.e. with a fixed S on both sides of the equation.

What about the spin??? J-M use

$$|\Psi(\boldsymbol{p}_0, S)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, S\rangle$$

i.e. with a fixed S on both sides of the equation.

They do this to simplify things so that the expectation value only involves

$$\langle \boldsymbol{p}', S|\boldsymbol{J}|\boldsymbol{p}, S\rangle$$

What about the spin??? J-M use

$$|\Psi(\boldsymbol{p}_0, S)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, S\rangle$$

i.e. with a fixed S on both sides of the equation.

They do this to simplify things so that the expectation value only involves

$$\langle \boldsymbol{p}', S|\boldsymbol{J}|\boldsymbol{p}, S\rangle$$

i.e. is at least diagonal in S—important for them because they try to write down the most general form for this matrix element

What about the spin??? J-M use

$$|\Psi(\boldsymbol{p}_0, S)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, S\rangle$$

i.e. with a fixed S on both sides of the equation.

They do this to simplify things so that the expectation value only involves

$$\langle \boldsymbol{p}', S|\boldsymbol{J}|\boldsymbol{p}, S\rangle$$

i.e. is at least diagonal in S—important for them because they try to write down the most general form for this matrix element

But this is incorrect. The wave packet is not physical. Recall that for a physical nucleon

$$p.S = 0$$

Thus if $\boldsymbol{p}$ is to vary freely in the wave packet integration S cannot remain fixed. — Point 1

Thus if $\boldsymbol{p}$ is to vary freely in the wave packet integration S cannot remain fixed. — Point 1

The second difficulty is the general form written down for the matrix element. The Lorentz structure assumed is not correct for non-diagonal matrix elements.

Thus if $\boldsymbol{p}$ is to vary freely in the wave packet integration S cannot remain fixed. — Point 1

The second difficulty is the general form written down for the matrix element. The Lorentz structure assumed is not correct for non-diagonal matrix elements.

To see this think of electromagnetic form factors:

$$\langle p', S | j^{\mu}_{em} | p, S \rangle$$

We cannot say: this transforms like a 4-vector, therefore we can express it terms of vectors built from p, p', S

Thus if $\boldsymbol{p}$ is to vary freely in the wave packet integration S cannot remain fixed. — Point 1

The second difficulty is the general form written down for the matrix element. The Lorentz structure assumed is not correct for non-diagonal matrix elements.

To see this think of electromagnetic form factors:

$$\langle p', S | j^{\mu}_{em} | p, S \rangle$$

We cannot say: this transforms like a 4-vector, therefore we can express it terms of vectors built from p, p', S

We have to first factor out the Dirac spinors

$$\bar{u}(p', S)[\gamma^{\mu} F_1 + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} F_2] u(p, S)$$

This is the second problem—-Point 2

Now Jaffe and Manohar are generally very careful, but nonetheless there are errors in their derivation. They end up with the following expression for the matrix elements of the angular momentum operator:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i - \frac{3p_0 + m}{p_0 + m}(\boldsymbol{p}.\boldsymbol{s})\boldsymbol{p}_i\right]$$

where $p^\mu = (p^0, \boldsymbol{p})$ and $\boldsymbol{s}_i$ are the components of the rest frame spin vector.

Now Jaffe and Manohar are generally very careful, but nonetheless there are errors in their derivation. They end up with the following expression for the matrix elements of the angular momentum operator:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i - \frac{3p_0 + m}{p_0 + m}(\boldsymbol{p}.\boldsymbol{s})\boldsymbol{p}_i\right]$$

where $p^\mu = (p^0, \boldsymbol{p})$ and $\boldsymbol{s}_i$ are the components of the rest frame spin vector.

Recall that the parton picture is supposed to be valid when the nucleon is viewed in a frame where it is moving very fast. In other words to derive a sum rule involving partons we must take the limit

$$p^0 \to \infty$$

if we consider longitudinal spin i.e $\boldsymbol{p} // \boldsymbol{s}$ one obtains:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{2} \boldsymbol{s}_i$$

and there is no problem.

if we consider longitudinal spin i.e $\boldsymbol{p}//\boldsymbol{s}$ one obtains:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{2}\boldsymbol{s}_i$$

and there is no problem.

But for transverse polarization one gets:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i\right]$$

which → ∞ as $p_0 \to \infty$, so no sum rule is possible.

Dear Larry, Elliot, Bernard,

Better late than never. Aneesh and I finally found ourselves in the same place with the time to review the issues you raised by email and in your recent paper. We agree that there is an error in our eq. (6.9). It came from treating the quantity u(p', s)u(p, s) with insufficient care.

Dear Larry, Elliot, Bernard,

Better late than never. Aneesh and I finally found ourselves in the same place with the time to review the issues you raised by email and in your recent paper. We agree that there is an error in our eq. (6.9). It came from treating the quantity u(p', s)u(p, s) with insufficient care.

Thanks for taking care and finding this mistake. It's good to get it cleared up.

I have to add that I found your paper rather difficult to read. There is quite a bit of stuff that gets in the way of the relatively simple error. For example I'm not sure why you have to go through wave packets any more than we did. We made a rather straightforward error of replacing u(p', s)u(p, s) by unity. Isn't it rather easy to set straight? All the best, Bob and Aneesh

Correcting the traditional approach

Correcting the traditional approach

Point 1: $\acute{a}$ la BLT, sandwich $\boldsymbol{J}$ between physical wave packet states

$$|\Psi(\boldsymbol{p}_0, \boldsymbol{s})\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, \boldsymbol{s}\rangle$$

where $\boldsymbol{s}$ is the spin vector in the rest frame.

Correcting the traditional approach

Point 1: $\acute{a}$ la BLT, sandwich $\boldsymbol{J}$ between physical wave packet states

$$|\Psi(\boldsymbol{p}_0, \boldsymbol{s})\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, \boldsymbol{s}\rangle$$

where $\boldsymbol{s}$ is the spin vector in the rest frame.

Note that the covariant spin vector, for spin quantized along the Z axis, is then

$$S^\mu = \left(\frac{\boldsymbol{p}.\boldsymbol{s}}{m}, \boldsymbol{s} + \frac{\boldsymbol{p}.\boldsymbol{s}}{m(p_0 + m)}\boldsymbol{p}\right)$$

Thus S varies as we integrate over $\boldsymbol{p}$

Point 2: Take care to utilize Dirac spinors when writing down general structure of

$$\langle \boldsymbol{p}', \boldsymbol{s} | \boldsymbol{J} | \boldsymbol{p}, \boldsymbol{s} \rangle$$

Point 2: Take care to utilize Dirac spinors when writing down general structure of

$$\langle \boldsymbol{p}', \boldsymbol{s} | \boldsymbol{J} | \boldsymbol{p}, \boldsymbol{s} \rangle$$

- Long, involved calculation!

Point 2: Take care to utilize Dirac spinors when writing down general structure of

$$\langle \boldsymbol{p}', \boldsymbol{s} | \boldsymbol{J} | \boldsymbol{p}, \boldsymbol{s} \rangle$$

- Long, involved calculation!

- Need to study narrow wave packet and limit as it approaches plane wave

Point 2: Take care to utilize Dirac spinors when writing down general structure of

$$\langle \boldsymbol{p}', \boldsymbol{s}|\boldsymbol{J}|\boldsymbol{p}, \boldsymbol{s}\rangle$$

- Long, involved calculation!

- Need to study narrow wave packet and limit as it approaches plane wave

Result: For general polarization state of nucleon BLT differs from J-M. Details later.

We have left out for later discussion a term

$$i(\boldsymbol{p} \times \boldsymbol{\nabla}_{\boldsymbol{p}})_i \delta^3(\boldsymbol{p}' - \boldsymbol{p})$$

which describes the angular momentum of the nucleon as a whole about the origin.

A NEW APPROACH

A NEW APPROACH

- It is simple.

- It is short

- It works for any spin. Previous methods only work for spin 1/2.

A NEW APPROACH

- It is simple.

- It is short

- It works for any spin. Previous methods only work for spin 1/2.

We know how rotations affect states. If $|\boldsymbol{p}, m\rangle$ is a state with momentum $\boldsymbol{p}$ and spin projection m in the rest frame of the particle, and if $\hat{R}_i(\beta)$ is the operator for a rotation β about the axis i, then

$$\hat{R}_i(\beta)|\boldsymbol{p}, m\rangle = |\boldsymbol{R}_i(\beta)\boldsymbol{p}, m'\rangle\, D^s_{m'm}[R_i(\beta)]$$

But rotations are generated by the angular momentum operators! i.e.

$$\hat{R}_i(\beta) = e^{-i\beta \boldsymbol{J}_i}$$

But rotations are generated by the angular momentum operators! i.e.

$$\hat{R}_i(\beta) = e^{-i\beta \boldsymbol{J}_i}$$

so that

$$\boldsymbol{J}_i = i\frac{d}{d\beta}\hat{R}_i(\beta)\Big|_{\beta=0}$$

But rotations are generated by the angular momentum operators! i.e.

$$\widehat{R}_i(\beta) = e^{-i\beta \boldsymbol{J}_i}$$

so that

$$\boldsymbol{J}_i = i\frac{d}{d\beta}\widehat{R}_i(\beta)\Big|_{\beta=0}$$

From the above we know what the matrix element of $\widehat{R}_i(\beta)$ looks like. So we simply differentiate, multiply by i, and put $\beta = 0$.

Thus we have

$$\langle \boldsymbol{p}', m' | \boldsymbol{J}_i | \boldsymbol{p}, m \rangle = i \frac{\partial}{\partial \beta} \langle \boldsymbol{p}', m' | R_i(\beta) | \boldsymbol{p}, m \rangle |_{\beta = 0}$$

Thus we have

$$\langle \boldsymbol{p}', m' | \boldsymbol{J}_i | \boldsymbol{p}, m \rangle = i \frac{\partial}{\partial \beta} \langle \boldsymbol{p}', m' | R_i(\beta) | \boldsymbol{p}, m \rangle |_{\beta=0}$$

$$= \; i \frac{\partial}{\partial \beta} \left[\langle \boldsymbol{p}', m' | R_i(\beta) \boldsymbol{p}, n \rangle \, D^s_{nm}[R_i(\beta)] \right]_{\beta=0}$$

Thus we have

$$\langle \boldsymbol{p}', m' | \boldsymbol{J}_i | \boldsymbol{p}, m \rangle = i \frac{\partial}{\partial \beta} \langle \boldsymbol{p}', m' | R_i(\beta) | \boldsymbol{p}, m \rangle |_{\beta=0}$$

$$= \quad i \frac{\partial}{\partial \beta} \left[\langle \boldsymbol{p}', m' | R_i(\beta) \boldsymbol{p}, n \rangle \, D^s_{nm}[R_i(\beta)] \right]_{\beta=0}$$

One technical point: you have to know that the derivative of the rotation matrix for spin s at $\beta = 0$ is just the spin matrix for that spin. (more correctly: the matrix generator of rotations for that spin) e.g. for spin 1/2 just $\sigma_i/2$.

COMPARISON OF RESULTS

For the expectation values we find, for any spin configuration (longitudinal, transverse etc) the remarkably simple result (suppressing a delta-function term):

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle = \frac{1}{2} \boldsymbol{s}_i$$

COMPARISON OF RESULTS

For the expectation values we find, for any spin configuration (longitudinal, transverse etc) the remarkably simple result (suppressing a delta-function term):

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle = \frac{1}{2} \boldsymbol{s}_i$$

Recall that the JM result for longitudinal spin was precisely:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{2} \boldsymbol{s}_i$$

in complete agreement with our result.

But for transverse polarization JM had:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i\right]$$

which disagrees with our result and which, as we said, would imply no possibility of a transverse sum rule.

But for transverse polarization JM had:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{4mp_0} \left[(3p_0^2 - m^2) \boldsymbol{s}_i \right]$$

which disagrees with our result and which, as we said, would imply no possibility of a transverse sum rule.

With our correct result there is no fundamental distinction between the transverse and longitudinal cases.

SUM RULES

Expand nucleon state as superposition of n-parton Fock states.

$$\begin{aligned} |\boldsymbol{p}, m\rangle \simeq \sum_n \sum_{\{\sigma\}} \int d^3\boldsymbol{k}_1 \dots d^3\boldsymbol{k}_n \\ \psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, ...\boldsymbol{k}_n, \sigma_n) \\ \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 ... - \boldsymbol{k}_n) |\boldsymbol{k}_1, \sigma_1, ...\boldsymbol{k}_n, \sigma_n\rangle. \end{aligned}$$

SUM RULES

Expand nucleon state as superposition of n-parton Fock states.

$$\begin{aligned} |\boldsymbol{p}, m\rangle \simeq \sum_n \sum_{\{\sigma\}} \int d^3\boldsymbol{k}_1 \dots d^3\boldsymbol{k}_n \\ \psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n) \\ \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \dots - \boldsymbol{k}_n)|\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n\rangle. \end{aligned}$$

There are two independent cases:

(a)Longitudinal polarization i.e. $\boldsymbol{s}$ along OZ. The sum rule for $\boldsymbol{J}_z$ yields the well known result

$$1/2 = 1/2\,\Delta\Sigma + \Delta G + \langle L_z^q \rangle + \langle L_z^G \rangle$$

(b) Transverse polarization i.e. $\boldsymbol{s} \perp \boldsymbol{p}$. The sum rule for $\boldsymbol{J}_x$ or $\boldsymbol{J}_y$ yields a a new sum rule

$$1/2 = 1/2 \sum_{q,\bar{q}} \int dx\, \Delta_T q(x) + \sum_{q,\bar{q},G} \langle L_{\boldsymbol{s}_T} \rangle$$

(b) Transverse polarization i.e. $\boldsymbol{s} \perp \boldsymbol{p}$. The sum rule for $\boldsymbol{J}_x$ or $\boldsymbol{J}_y$ yields a a new sum rule

$$1/2 = 1/2 \sum_{q,\bar{q}} \int dx\, \Delta_T q(x) + \sum_{q,\bar{q},G} \langle L_{\boldsymbol{s}_T} \rangle$$

Here $L_{\boldsymbol{s}_T}$ is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$.

The structure functions $\Delta_T q(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon.

(b) Transverse polarization i.e. $\boldsymbol{s} \perp \boldsymbol{p}$. The sum rule for $\boldsymbol{J}_x$ or $\boldsymbol{J}_y$ yields a a new sum rule

$$1/2 = 1/2 \sum_{q,\,\bar{q}} \int dx\, \Delta_T q(x) + \sum_{q,\,\bar{q},\,G} \langle L_{\boldsymbol{s}_T} \rangle$$

Here $L_{\boldsymbol{s}_T}$ is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$.

The structure functions $\Delta_T q(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon.

As mentioned no such parton model sum rule is possible with the J-M formula because, as $p \to \infty$, for $i = x, y$ the matrix elements diverge.

A comment about the calculation and meaning of the orbital angular momentum.

For each parton labelled r there is an orbital term involving

$$i(\boldsymbol{k_r} \times \boldsymbol{\nabla}_{\boldsymbol{k_r}})_i \delta^3(\boldsymbol{p} - \boldsymbol{k_1} - \boldsymbol{k_2} \boldsymbol{....} - \boldsymbol{k_r} \boldsymbol{....} - \boldsymbol{k_n})$$

A comment about the calculation and meaning of the orbital angular momentum.

For each parton labelled r there is an orbital term involving

$$i(\boldsymbol{k_r} \times \boldsymbol{\nabla}_{\boldsymbol{k_r}})_i \delta^3(\boldsymbol{p} - \boldsymbol{k_1} - \boldsymbol{k_2}\boldsymbol{....} - \boldsymbol{k_r}\boldsymbol{....} - \boldsymbol{k_n})$$

Beautifully consistently can show that this leads to two terms. The first

$$i(\boldsymbol{p} \times \boldsymbol{\nabla}_{\boldsymbol{p}})_i \delta^3(\boldsymbol{p'} - \boldsymbol{p})$$

i.e the angular momentum of the nucleon as a whole about the origin. This cancels the same term on the LHS of the sum rule.

The second is

$$\delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p})\langle L_i\rangle_{m'\,m} \tag{1}$$

where $\langle L_i\rangle_{m'\,m}$ is the contribution from the internal angular momentum arising from the partons

$$\begin{aligned}\langle L_i\rangle_{m'\,m} \;=\; & \sum_n \sum_r \int d^3k_1 \ldots\ldots d^3k_n \psi^*_{\boldsymbol{p},m'}(\boldsymbol{k_1}, ...\boldsymbol{k}_n,) \\ & \{[-i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i]\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, ...\boldsymbol{k}_r, ...\boldsymbol{k}_n)\} \\ & \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - ... - \boldsymbol{k}_n)\end{aligned}$$

The second is

$$\delta^{(3)}(\boldsymbol{p'} - \boldsymbol{p})\langle L_i\rangle_{m'\,m} \tag{2}$$

where $\langle L_i\rangle_{m'\,m}$ is the contribution from the internal angular momentum arising from the partons

$$\begin{aligned}\langle L_i\rangle_{m'\,m} \;=\; & \sum_n \sum_r \int d^3k_1 \ldots\ldots d^3k_n \psi^*_{\boldsymbol{p},m'}(\boldsymbol{k_1}, ...\boldsymbol{k}_n,) \\ & \{[-i(\boldsymbol{k}_r \times \boldsymbol{\nabla}_{k_r})_i]\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, ...\boldsymbol{k}_r, ...\boldsymbol{k}_n)\} \\ & \delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 - ... - \boldsymbol{k}_n)\end{aligned}$$

This is the usual QM form for angular momentum!

It is absolutely crucial to note that the sum rule involves a SUM of Quark and Antiquark densities.

Not realizing this has led to some misunderstandings.

What some people call the TENSOR CHARGE of the NUCLEON is the difference between quark and antiquark contributions.

Thus the transverse spin sum rule, although it involves the transverse spin or transversity quark and antiquark densities, does NOT involve the nucleon's transversity. The Tensor Charge operator is NOT related to the angular momentum.

Comments on orbital angular momentum

Ji defines $J_{q,g}$ by

$$J_{q,g}\,2\boldsymbol{S} = \langle p, S|\hat{\boldsymbol{J}}_{q,g}|p, S\rangle$$

which is incorrect, but I don't think it is ever used.

The shortest and most direct way to obtain the correct expression for the expectation value of $\boldsymbol{J}$ is actually from consideration of the effect of rotations on a state vector.

Ji defines $J_{q,g}$ by

$$J_{q,g}\, 2\boldsymbol{S} = \langle p, S | \hat{\boldsymbol{J}}_{q,g} | p, S \rangle$$

which is incorrect, but I don’t think it is ever used.

The shortest and most direct way to obtain the correct expression for the expectation value of $\boldsymbol{J}$ is actually from consideration of the effect of rotations on a state vector.

But if one uses the approach via the energy momentum tensor $T^{\mu\nu}$ then to begin with the expectation value depends on some of the scalar functions appearing in the matrix element of $T^{\mu\nu}$. I’ll use Ji’s notation for these.

$$
\begin{aligned}
\langle \psi_{p,\boldsymbol{s}} | M^{ij} | \psi_{p,\boldsymbol{s}} \rangle &= \frac{A}{2M(p_0+M)} [p^i (\boldsymbol{p} \times \boldsymbol{s})^j - p^j (\boldsymbol{p} \times \boldsymbol{s})^i] \\
&+ \frac{1/2(A+B)}{M} \epsilon^{ij\alpha\beta} S_\alpha p_\beta
\end{aligned}
$$

$$\begin{aligned}\langle\psi_{p,\boldsymbol{s}}|M^{ij}|\psi_{p,\boldsymbol{s}}\rangle &= \frac{A}{2M(p_0+M)}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i] \\ &+ \frac{1/2(A+B)}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta\end{aligned}$$

$$J_{(i=x,y,z)} = \epsilon_{ijk}M^{jk}$$

$$\langle\psi_{p,\boldsymbol{s}}|M^{ij}|\psi_{p,\boldsymbol{s}}\rangle = \frac{A}{2M(p_0+M)}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i] + \frac{1/2(A+B)}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta$$

$$J_{(i=x,y,z)} = \epsilon_{ijk}M^{jk}$$

Energy of proton fixes $A=1$; Helicity of longitudinally polarized proton fixes $1/2(A+B)=1/2$

$$\begin{aligned}\langle\psi_{p,\boldsymbol{s}}|M^{ij}|\psi_{p,\boldsymbol{s}}\rangle &= \frac{A}{2M(p_0+M)}[p^i(\boldsymbol{p}\times\boldsymbol{s})^j - p^j(\boldsymbol{p}\times\boldsymbol{s})^i] \\ &+ \frac{1/2(A+B)}{M}\epsilon^{ij\alpha\beta}S_\alpha p_\beta\end{aligned}$$

$$J_{(i=x,y,z)} = \epsilon_{ijk}M^{jk}$$

Energy of proton fixes $A = 1$; Helicity of longitudinally polarized proton fixes $1/2(A+B) = 1/2$

$$\therefore \qquad B^{Ji}_{proton} = 0$$

Ji writes (where the sum is over quarks and antiquarks)

$$A = \sum_q A_q + A_g = 1 \qquad B = \sum_q B_q + B_g = 0$$

Ji writes (where the sum is over quarks and antiquarks)

$$A = \sum_q A_q + A_g = 1 \qquad B = \sum_q B_q + B_g = 0$$

$$\begin{aligned} \langle \psi_{p,\boldsymbol{s}} | M^{ij}_{q,g} | \psi_{p,\boldsymbol{s}} \rangle &= \frac{A_{q,g}}{2M(p_0 + M)} [p^i(\boldsymbol{p} \times \boldsymbol{s})^j - p^j(\boldsymbol{p} \times \boldsymbol{s})^i] \\ &+ \frac{1/2(A+B)_{q,g}}{M} \epsilon^{ij\alpha\beta} S_\alpha p_\beta \end{aligned}$$

Ji writes (where the sum is over quarks and antiquarks)

$$A = \sum_q A_q + A_g = 1 \qquad B = \sum_q B_q + B_g = 0$$

$$\begin{aligned} \langle \psi_{p,\boldsymbol{s}} | M^{ij}_{q,g} | \psi_{p,\boldsymbol{s}} \rangle &= \frac{A_{q,g}}{2M(p_0 + M)} [p^i (\boldsymbol{p} \times \boldsymbol{s})^j - p^j (\boldsymbol{p} \times \boldsymbol{s})^i] \\ &+ \frac{1/2(A+B)_{q,g}}{M} \epsilon^{ij\alpha\beta} S_\alpha p_\beta \end{aligned}$$

In terms of 2nd moment of off-diagonal PDFs:

Ji writes (where the sum is over quarks and antiquarks)

$$A = \sum_q A_q + A_g = 1 \qquad B = \sum_q B_q + B_g = 0$$

$$\begin{aligned}\langle \psi_{p,\boldsymbol{s}} | M^{ij}_{q,g} | \psi_{p,\boldsymbol{s}} \rangle &= \frac{A_{q,g}}{2M(p_0+M)} [p^i (\boldsymbol{p} \times \boldsymbol{s})^j - p^j (\boldsymbol{p} \times \boldsymbol{s})^i] \\ &+ \frac{1/2(A+B)_{q,g}}{M} \epsilon^{ij\alpha\beta} S_\alpha p_\beta\end{aligned}$$

In terms of 2nd moment of off-diagonal PDFs:

$$A_q(\Delta^2 = 0) = H_q^{(2)}(\Delta^2 = 0) \qquad B_q(\Delta^2 = 0) = E_q^{(2)}(\Delta^2 = 0)$$

Something interesting!

Longitudinal polarization: $\boldsymbol{s} = (0, 0, 1) \qquad \boldsymbol{p} = (0, 0, p)$

Something interesting!

Longitudinal polarization: $\boldsymbol{s} = (0, 0, 1) \qquad \boldsymbol{p} = (0, 0, p)$

What Ji calls J_q is really

$$\langle\psi|J_z^q|\psi\rangle \equiv J_q = 1/2(A + B)_q$$

Something interesting!

Longitudinal polarization: $\boldsymbol{s} = (0, 0, 1) \qquad \boldsymbol{p} = (0, 0, p)$

What Ji calls J_q is really

$$\langle \psi | J_z^q | \psi \rangle \equiv J_q = 1/2(A + B)_q$$

$$\frac{p_z^q}{p} = A_q$$

Something interesting!

Longitudinal polarization: $\boldsymbol{s} = (0,0,1) \qquad \boldsymbol{p} = (0,0,p)$

What Ji calls J_q is really

$$\langle\psi|J_z^q|\psi\rangle \equiv J_q = 1/2(A+B)_q$$

$$\frac{p_z^q}{p} = A_q$$

which agrees with Ji, but.......

Transverse polarization: $\boldsymbol{s} = (1, 0, 0) \qquad \boldsymbol{p} = (0, 0, p)$

Transverse polarization: $\boldsymbol{s} = (1, 0, 0) \qquad \boldsymbol{p} = (0, 0, p)$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}\left[(M - p_0)A_q + 2p_0\, 1/2(A + B)_q\right]$$

Transverse polarization: $\boldsymbol{s} = (1, 0, 0) \qquad \boldsymbol{p} = (0, 0, p)$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}\left[(M - p_0)A_q + 2p_0\, 1/2(A+B)_q\right]$$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}\left[M A_q + p_0\, B_q\right]$$

Transverse polarization: $\boldsymbol{s} = (1,0,0) \qquad \boldsymbol{p} = (0,0,p)$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}[(M-p_0)A_q + 2p_0\,1/2(A+B)_q]$$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}[MA_q + p_0\,B_q]$$

What happens as $p_0 \to \infty$???

Transverse polarization: $\boldsymbol{s} = (1,0,0) \qquad \boldsymbol{p} = (0,0,p)$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}\left[(M - p_0)A_q + 2p_0\, 1/2(A+B)_q\right]$$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}\left[M A_q + p_0\, B_q\right]$$

What happens as $p_0 \to \infty$???

Fock expansion in partonic wave functions shows that p_0 term can only come from orbital angular momentum

Transverse polarization: $\boldsymbol{s} = (1, 0, 0) \qquad \boldsymbol{p} = (0, 0, p)$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}[(M - p_0)A_q + 2p_0\, 1/2(A + B)_q]$$

$$\langle\psi|J_x^q|\psi\rangle = \frac{1}{2M}[MA_q + p_0\, B_q]$$

What happens as $p_0 \to \infty$???

Fock expansion in partonic wave functions shows that p_0 term can only come from orbital angular momentum

Of course J_x^{proton} is fine, since $\sum_q B_q = -B_g$.

Teryaev suggests that $\sum_q B_q = 0$

Teryaev suggests that $\sum_q B_q = 0$

Corresponds to: anomalous gravitomagnetic moment $= 0$

Teryaev suggests that $\sum_q B_q = 0$

Corresponds to: anomalous gravitomagnetic moment $= 0$

which he claims is related to Belinfante Invariance

Teryaev suggests that $\sum_q B_q = 0$

Corresponds to: anomalous gravitomagnetic moment $= 0$

which he claims is related to Belinfante Invariance

This would imply:

Teryaev suggests that $\sum_q B_q = 0$

Corresponds to: anomalous gravitomagnetic moment $= 0$

which he claims is related to Belinfante Invariance

This would imply:

$$\sum_q J_q = \frac{1}{2}\left(\frac{\sum_q p_z^q}{p}\right) \qquad J_g = \frac{1}{2}\left(\frac{p_z^g}{p}\right)$$

Teryaev suggests that $\sum_q B_q = 0$

Corresponds to: anomalous gravitomagnetic moment $= 0$

which he claims is related to Belinfante Invariance

This would imply:

$$\sum_q J_q = \frac{1}{2}\left(\frac{\sum_q p_z^q}{p}\right) \qquad J_g = \frac{1}{2}\left(\frac{p_z^g}{p}\right)$$

$$\therefore \qquad \left(\frac{\sum_q J_q}{J_g}\right) = \left(\frac{\sum_q p_z^q}{p_z^g}\right)$$

Consistent with known result that

Consistent with known result that

both $\left(\frac{\sum_q p_z^q}{p_z^g}\right)$ and $\left(\frac{\sum_q J_q}{J_g}\right) \overset{Q^2\to\infty}{\Longrightarrow} \left(\frac{3n_f}{16}\right)$

Consistent with known result that

both $\left(\frac{\sum_q p_z^q}{p_z^g}\right)$ and $\left(\frac{\sum_q J_q}{J_g}\right) \overset{Q^2\to\infty}{\Longrightarrow} \left(\frac{3n_f}{16}\right)$

Is the Teryaev argument convincing ?!?

Consistent with known result that

$$\text{both}\quad \left(\frac{\sum_q p_z^q}{p_z^g}\right) \quad \text{and} \quad \left(\frac{\sum_q J_q}{J_g}\right) \quad \overset{Q^2\to\infty}{\Longrightarrow} \left(\frac{3n_f}{16}\right)$$

Is the Teryaev argument convincing ?!?

I have no idea!

SUMMARY

- In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

SUMMARY

- In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

- Such matrix elements are divergent and ambiguous in the traditional approach and are incorrect in some classic papers

SUMMARY

- In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

- Such matrix elements are divergent and ambiguous in the traditional approach and are incorrect in some classic papers

- This can be handled using wave packets but the calculations are long and unwieldy

SUMMARY

- In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

- Such matrix elements are divergent and ambiguous in the traditional approach and are incorrect in some classic papers

- This can be handled using wave packets but the calculations are long and unwieldy

- Using our knowledge of how states transform under rotations leads quickly and relatively painlessly to correct results

SUMMARY

- In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

- Such matrix elements are divergent and ambiguous in the traditional approach and are incorrect in some classic papers

- This can be handled using wave packets but the calculations are long and unwieldy

- Using our knowledge of how states transform under rotations leads quickly and relatively painlessly to correct results

- The great success of the correct approach is that it allows derivation of a sum rule also for transversely polarized nucleons

- For a transversely polarized nucleon, the quark and gluon orbital momentum separately grow with $\boldsymbol{p}$, but their sum remains finite as $\boldsymbol{p} \rightarrow \infty$.