transverse-momentum dependent densities
in the light-cone gauge:
definition, renormalization and evolution

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Prepared for the ECT Workshop "Orbital angular momentum of partons in hadrons", 9 - 13 Nov 2009, Trento, Italy* different approaches to define **tmd pdf**s are considered from the point of view of their renormalization-group properties

one-loop anomalous dimensions of **tmd pdf**s are presented and compared to each other

the arguments are given in favor of the "**pure light-like**" definition, and the use of the light-cone gauge

- **integrated parton densities:** definition; gauge invariance; RG properties
- **unintegrated (tmd) densities**: complete gauge invariance, extra divergences, RG properties
- **generalized definition** and cancelation of extra divergences; UV-evolution
- open problems

KINEMATICS

$$l^\mu = (l^+, l^-, \boldsymbol{l}_\perp)\ ,\ l^\pm = (l^0 \pm l^3)/\sqrt{2}\ ,\ l^2 = 2l^+l^- - \boldsymbol{l}_\perp^2$$

$$n^{*\mu} = \Omega(1, 1, \mathbf{0}_\perp)\ ,\quad n^\mu = \frac{1}{2\Omega}(1, -1, \mathbf{0}_\perp)\quad ,\quad n^{*+} = \sqrt{2}\Omega$$

$$n^{*-} = 0\ ,\ n^+ = 0\ ,\ n^- = \frac{1}{\sqrt{2}\Omega}\ ,\quad n^* n = 1\ ,\ (n^*)^2 = n^2 = 0$$

$$P^\mu = n^{*\mu} + \frac{M^2}{2} n^\mu\ ,\quad P^2 = M^2$$

$$q^\mu = -x_N n^{*\mu} + \frac{Q^2}{2x_N} n^\mu \quad \rightarrow \quad q^+ = -\sqrt{2} x_N \Omega\ ,\quad q^- = \frac{Q^2}{2\sqrt{2} x_N \Omega}$$

x_N — Nachtmann variable

$x_B = Q^2/2(Pq)$ — Bjorken variable

$$\sqrt{2}\Omega = P^+ \to x_B = \frac{x_N}{1 - \frac{M^2}{Q^2}x_N^2} = x_N + O\left(\frac{M^2}{Q^2}\right)$$

kinematical approximations are important!

`Collins, Rogers, Stasto: PRD (2008)`

$\to$ **fully unintegrated** parton correlation functions

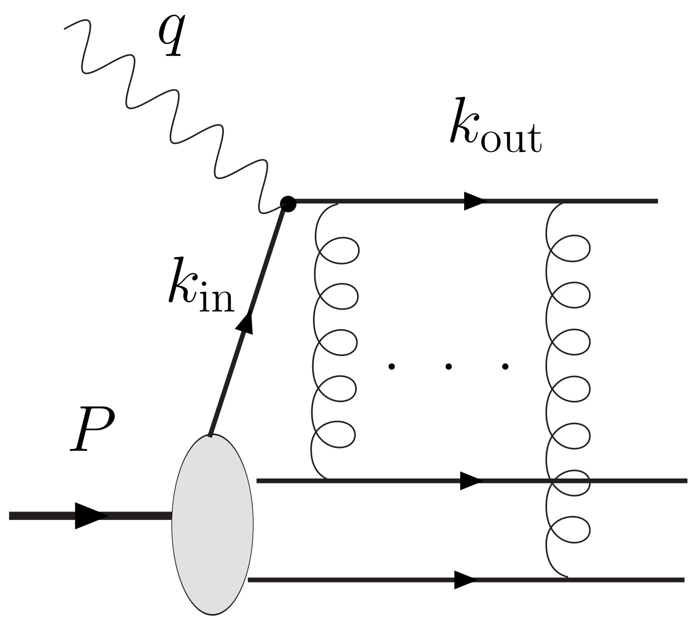
M^2/Q^2 corrections neglected

$$x_B \approx x_N$$

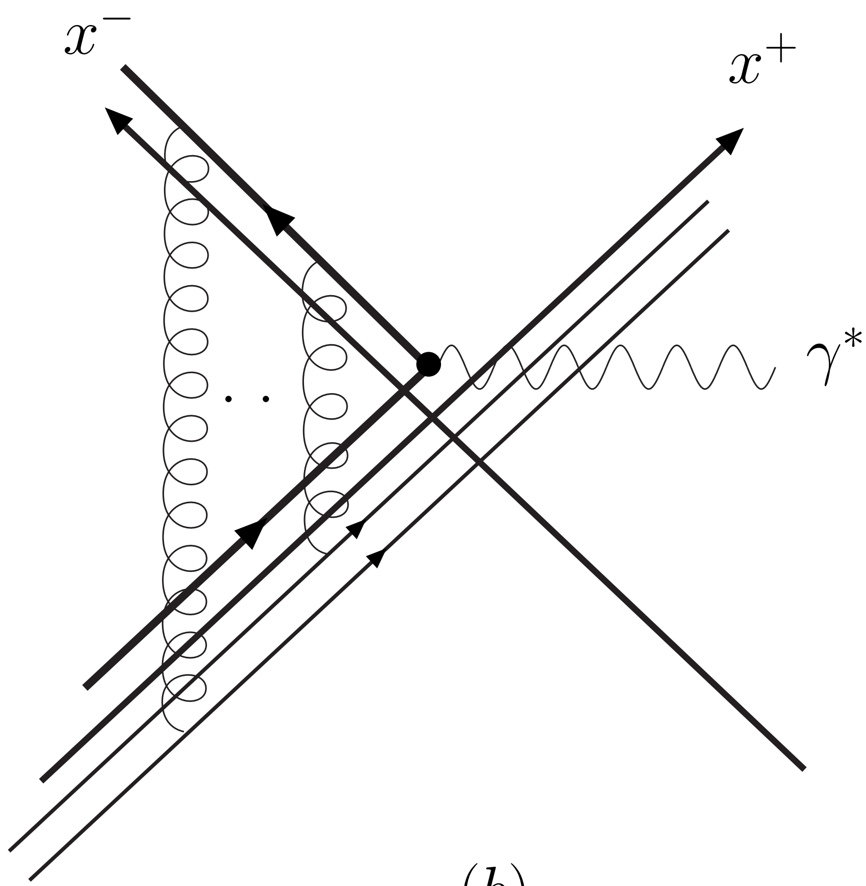
$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_\perp\right) \quad , \quad q^\mu = \left(-x_B P^+, \frac{Q^2}{2x_B P^+}, \mathbf{0}_\perp\right)$$

$$P^+ \sim E_P = \text{hadron energy}$$

$$s \sim \frac{Q^2}{x_B}$$



(a)



(b)

completely **gauge invariant** (quark) density:

$$Q_{i/h}(x,\mu) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, \mathrm{e}^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-,\mathbf{0}_\perp)[\xi^-,0^-]\gamma^+\psi_i(0^-,\mathbf{0}_\perp)|h(P)\rangle$$

renormalization group properties: **DGLAP**

$$\mu\frac{d}{d\mu}\mathcal{P}_{i/h}(x,\mu) = \sum_j \int_x^1 \frac{dz}{z}\, P_{ij}\left(\frac{x}{z}\right)\mathcal{P}_{j/h}(x,\mu)$$

gauge invariance is saved by the insertion of the **gauge link**

$$[y,x]_r = \mathcal{P}\exp\left[-ig\int_{\tau_1}^{\tau_2} d\tau r^\mu A_\mu^a(r\tau)t^a\right]$$

$$r^\mu\tau_1 = x\ ,\ r^\mu\tau_2 = y\ ,$$

distinguish between **longitudinal** $[\ ,\ ]_{[n,\,v,\,v_0]}$ and **transversal** $[\ ,\ ]_{[\boldsymbol{l}\,]}$ gauge links

SEMI – INCLUSIVE PROCESSES

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \to \mathbf{H_2}(\mathbf{P}') + \mathcal{X}$$

Soper: PRL 43 (1979) 1847

$$P_{a/A}(x, \mu) \to \mathcal{P}_{a/A}(x, \boldsymbol{k}, \mu, \zeta)$$

depends on how fast the hadrons is moving: ζ!

three different definitions of a **unintegrated quark distribution**:

A . *pure light-cone* $\mathcal{F}_{[n]}$: $\left[\, n^2 = 0 \;,\; n^+ = 0 \;,\; \boldsymbol{n}_\perp = 0 \right]$

$$\mathcal{F}_{[n]}\,(x, \boldsymbol{k}_\perp; \mu, \eta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \cdot \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger_n [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger_{\boldsymbol{l}} \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{\boldsymbol{l}} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_n\, \psi(0^-, \boldsymbol{0}_\perp)\, |\, h \rangle$$

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A . *pure light-cone* $\mathcal{F}_{[n]}$: $\left[\, n^2 = 0 \;,\; n^+ = 0 \;,\; \boldsymbol{n}_\perp = 0 \right]$

B . *off-light-cone* $\mathcal{F}_{[v]}$: $\left[\, v^2 > 0 \;,\; v^- \ll v^+ \;,\; \boldsymbol{v}_\perp = 0 \right] \;,\; \zeta = \frac{4(P\cdot v)^2}{v^2}$

$$\mathcal{F}_{[v]}\left(x, \boldsymbol{k}_\perp; \mu, \zeta\right) =$$

$$= \frac{1}{2}\int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]_v^\dagger [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]_{\boldsymbol{l}}^\dagger \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{\boldsymbol{l}} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_v\, \psi(0^-, \boldsymbol{0}_\perp)\,|\, h\rangle$$

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Γ. *direct link* $\mathcal{F}_{[v_0]}$: $\left[\, v_0^2 = \boldsymbol{v}_0^2 < 0 \;,\; v^+ = 0 \;,\; ,\; \zeta = \frac{4(P\cdot v_0)^2}{v_0^2} \right]$

$$\mathcal{F}_{[v_0]}\,(x, \boldsymbol{k}_\perp; \mu, \zeta_0) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, \gamma^+\, [\xi^-, \boldsymbol{\xi}_\perp; 0^-, \mathbf{0}_\perp]_{v_0}\, \psi(0^-, \mathbf{0}_\perp)\,|\, h \rangle$$

classification of **singularities** in the pure light-cone definition

1. $\sim \frac{1}{\varepsilon}$ poles, usual UV-singularities: removed by the standard $R-$operation and are controlled by renormalization-group evolution equations (DGLAP in integrated case)

2. pure rapidity divergences: give rise to logarithmic and double-logarithmic terms of the form $\sim \ln \eta \, , \, \ln^2 \eta$; have to be resumed

3. overlapping divergences: contain both UV and soft singularities simultaneously $\sim \frac{1}{\varepsilon} \ln \eta$; **highly undesirable**—depend on the parameters of the chosen gauge; prevents the removal of *all* UV-singularities by the standard $R-$procedure; a special *generalized* renormalization procedure is needed

approaches to avoid the problems

- definition **B**: in the **covariant gauges**, the gauge links shifted off the light-cone $v^2 > 0$, $v^+ \ll v^-$; or use the **non-light-like axial gauge** $(v \cdot A) = 0$, $v^2 > 0$ (Collins, Soper, Ji, Ma, Yuan);

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- definition **A** + **soft factor**: stay **on the light-cone**, but subtract soft factor R, which cancels the extra divergences: $\mathcal{F}_{[n]} \to \mathcal{F}_{[n]} \cdot R^{-1}$ (Collins, Hautmann):

approaches to avoid the problems

- definition **A** + **soft factor**: **direct regularization** of the light-cone singularities in the gluon propagator

$$\frac{1}{q^+} \to \frac{1}{[q^+](\eta)}$$

—generalized renormalization procedure; $\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta) \cdot R^{-1}(\eta)$; keeps the overlapping singularities under control and treats the extra term in the UV-divergent part by means of the *cusp anomalous dimension*— specific form of the gauge contour in the soft factor `(Ch., Stefanis)`

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- definition **A** + **soft factor**: **light-cone gauge** with the **Mandelstam-Leibbrandt** pole prescription

$$\frac{1}{q^+} \to \frac{1}{q^+ + i0q^-} \text{ or } \frac{q^-}{q^+q^- + i0}$$

—**overlapping singularities do not appear** at all (`Ch., Stefanis`)

renormalization group equations

$\mathbf{B}, \mathbf{\Gamma}$. *off-light-cone* and *direct link*

$$\mu \frac{d}{d\mu} \mathcal{F}_{[v\,,\,v_0]} = \gamma_0 \, \mathcal{F}_{[v\,,\,v_0]} \quad , \quad \gamma_0 = \frac{3}{4} \frac{\alpha_s C_{\rm F}}{\pi} + O(\alpha_s^2)$$

extraction of the **soft factor**

$$\mathcal{F}_{[\,v\,]} \to \mathcal{F}_{[\,v\,]} \cdot R_v^{-1}$$

$$\mu \frac{d}{d\mu} \left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right] = (\gamma_0 - \gamma_{\rm R}) \left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right]$$

$\gamma_{\rm R}$—anomalous dimension of the soft factor

A. *pure light-cone*

$$\mu \frac{d}{d\mu} \mathcal{F}_{[\,n\,]} = (\gamma_0 - \gamma_{\rm cusp})\, \mathcal{F}_{[\,n\,]}$$

generalized renormalization “restores” the anomalous dimension

$$\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta) \cdot R^{-1}(\eta)$$

$$\mu \frac{d}{d\mu} \left[\mathcal{F}_{[\,n\,]} \cdot R_n^{-1}\right] = \gamma_0 \left[\mathcal{F}_{[\,n\,]} \cdot R_n^{-1}\right]$$

A. *pure light-cone* with Mandelstam-Leibbrandt prescription

$$\mu\frac{d}{d\mu}\left[\mathcal{F}^{\rm ML}_{[\,n\,]}\cdot R_n^{-1}\right] = \mu\frac{d}{d\mu}\,\mathcal{F}^{\rm ML}_{[\,n\,]} =$$

$$\gamma_0\left[\mathcal{F}^{\rm ML}_{[\,n\,]}\cdot R_n^{-1}\right] = \gamma_0\,\mathcal{F}^{\rm ML}_{[\,n\,]}$$

anomalous dimension without light-cone artifacts from the very beginning!

open problems

- proof of **factorization**: all-order factorization (in a covariant gauge) is studied by `Ji, Ma, Yuan` using the definition $\mathbf{B}$ (off-the-light-cone gauge links); no explicit proof of a factorization theorem for definition $\mathbf{A}$ is known at present.

open problems

- proof of **factorization**
- relationship between unintegrated **tmd pdf**s and integrated distribution functions:

  definition $\mathbf{A}$ reproduces (at least, in principle) the DGLAP evolution after integration:

$$\int d^2\boldsymbol{k}_\perp \, \mathcal{F}_{[\,n\,]}(x, \boldsymbol{k}_\perp, \mu) = F_{[\,n\,]}(x, \mu)$$

$$\mu \frac{d}{d\mu} \, F_{[\,n\,]} = \mathcal{K}_{\mathrm{DGLAP}} \otimes F_{[\,n\,]}$$

open problems

- proof of **factorization**
- relationship between unintegrated **tmd pdf**s and integrated distribution functions

  definition $\mathbf{B}$ fails to reproduce the DGLAP evolution after integration:

$$\int d^2\boldsymbol{k}_\perp\, \mathcal{F}_{[\,v\,]}(x,\boldsymbol{k}_\perp,\mu) = F_{[\,v\,]}(x,\mu)$$

$$\mu\frac{d}{d\mu}\, F_{[\,v\,]} = \mathcal{K}_v \otimes F_{[\,v\,]} \quad , \quad \mathcal{K}_v \neq \mathcal{K}_{\mathrm{DGLAP}}$$

more deep old-fashioned

quantum field-theoretic analysis

of **tmd** definitions is needed!

I.Ch., N. Stefanis:

Phys. Rev. D77 (2008) 094001

Nucl. Phys. B802 (2008) 146

Phys. Rev. D80 (2009) 054008

arXiv:0910.3108 (hep-ph)

arXiv:0911.1031 (hep-ph)

appendices

source of extra divergences: pole in the gluon propagator

$$D^{\mu\nu}_{\rm LC}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

q^--independent pole prescriptions:

$$d^{\mu\nu}_{\rm PV}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

$$d^{\mu\nu}_{\rm Adv/Ret}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

Mandelstam-Leibbrandt pole prescriptions:

$$\frac{1}{[q^+]_{\rm ML}} = \begin{cases} \frac{1}{q^+ + i0q^-} \\ \\ \frac{q^-}{q^+q^- + i0} \end{cases}$$

FACTORIZATION at small $\boldsymbol{P}_\perp$ **SIDIS**

Ji, Ma, Yuan: PRD (2005)

small $\boldsymbol{P}_\perp$ — $\boldsymbol{P}_\perp \sim \Lambda_{\rm QCD}$

moderate Q^2 — $Q^2 \sim 10^2\ {\rm GeV}^2$

$$F(x_B, z_h, \boldsymbol{P}_{h\perp}, Q^2) = \sum_i e_i^2 \cdot$$

$$\cdot H(Q^2, \mu^2, \rho) \otimes \mathcal{F}_D(x_B, \boldsymbol{k}_\perp, \mu^2, x_B\zeta, \rho) \otimes \mathcal{F}_F(z_h, \boldsymbol{q}_\perp, \mu^2, \hat{\zeta}/z_h, \rho) \otimes S(\boldsymbol{l}_\perp^2, \mu^2, \rho)$$

$$\zeta^2 x_B^2 = \frac{\hat{\zeta}^2}{z_h^2} = Q^2 \rho$$

μ = renormalization (collinear factorization) scale

ρ = rapidity cutoff

a set of **evolution equations** for unintegrated densities

- **UV-evolution** (in the integrated case—DGLAP)

$$\mu\frac{d}{d\mu}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm UV}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

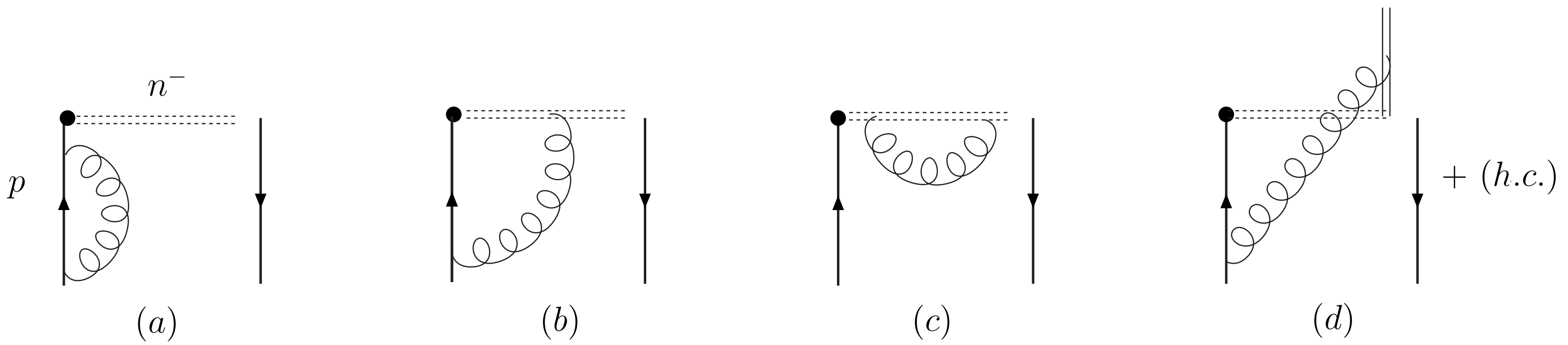
- **rapidity evolution** (Collins-Soper equation) (no correspondence in the integrated case!)

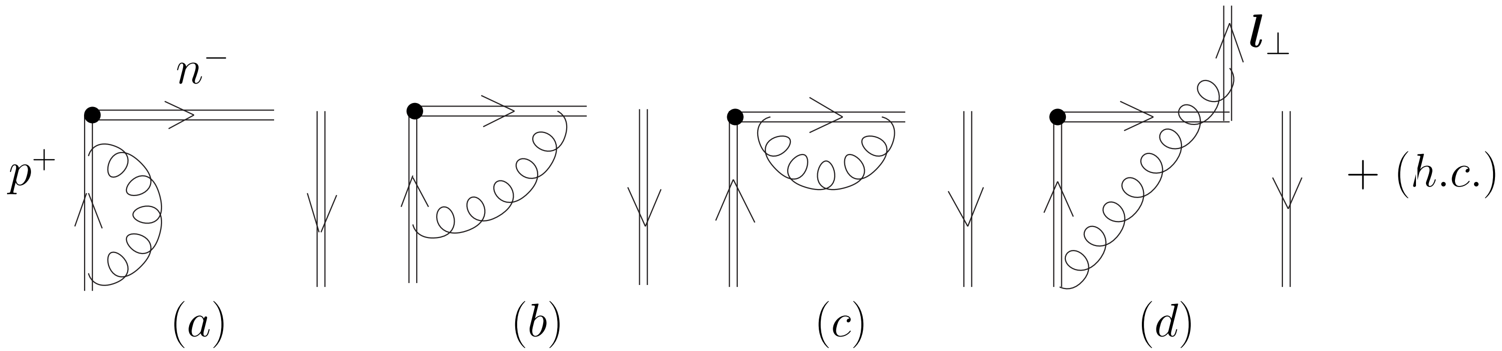
$$\zeta\frac{d}{d\zeta}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm CS}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

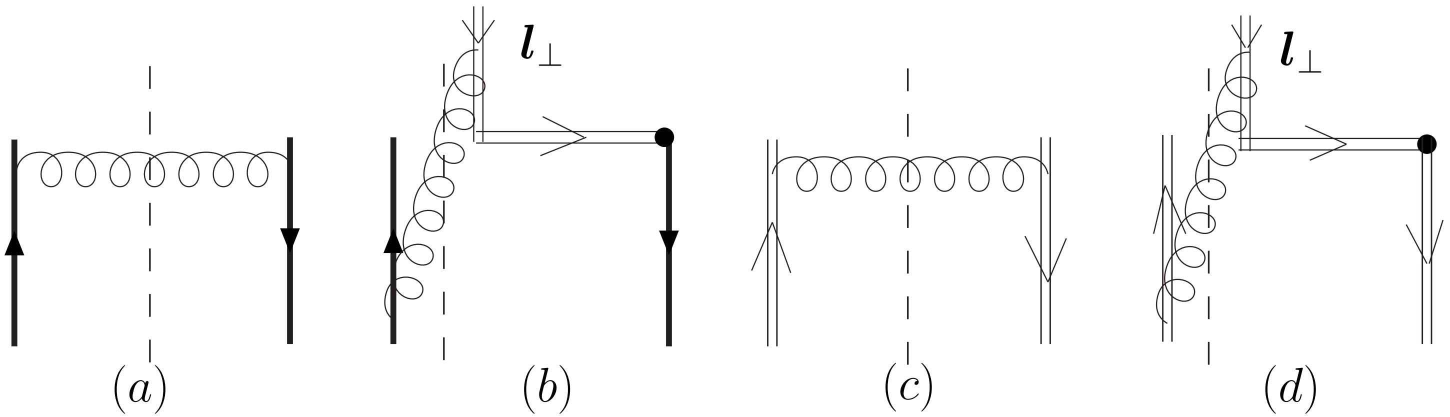
- **BFKL evolution** (relation to the Collins-Soper evolution is not known!)

$$x\frac{d}{dx}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm BFKL}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

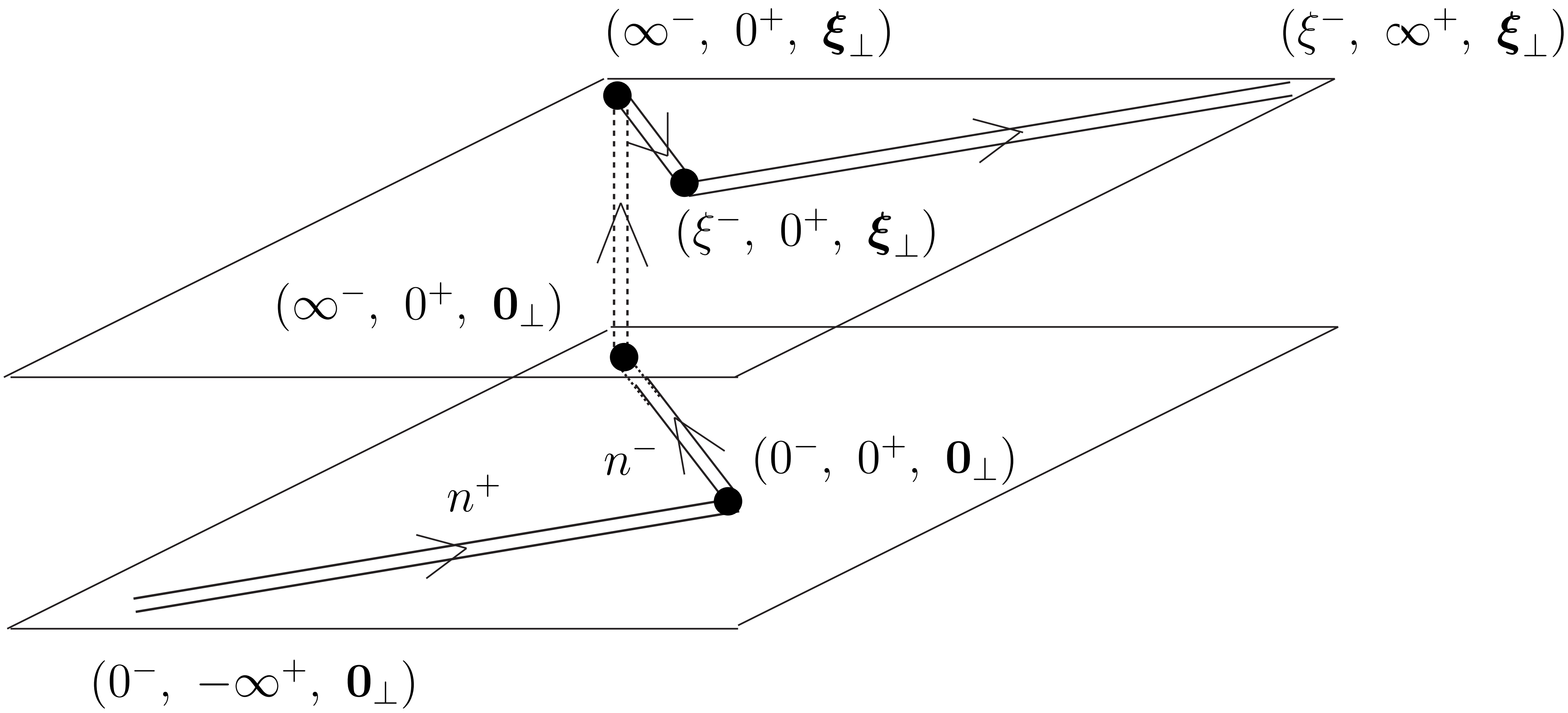
calculation of the **one-gluon** diagrams







integration contour for the soft factor



renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle (`Korchemsky, Radyushkin`)

$$Z_\chi = \left[\langle 0 | \mathcal{P} \exp \left[ig \int_\chi d\zeta^\mu \ \hat{A}^a_\mu(\zeta) \right] | 0 \rangle \right]^{-1}$$

generalized renormalization:

$$\mathcal{O}_{\rm ren}(\chi, ...) = Z_\chi Z_{\rm R} \ \mathcal{O}(\chi, ...)$$

generalized definition of TMD PDF:

$$\mathcal{F}\left(x, \boldsymbol{k}_\perp; \mu, \eta\right) =$$
$$\frac{1}{2}\int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp\cdot\xi_\perp} \langle h|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)[\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger$$
$$\times[\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger \gamma^+ [\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \mathbf{0}_\perp][\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp]$$
$$\times\psi(0^-, \mathbf{0}_\perp)|P\rangle \left[\Phi(p^+, n^-|0^-, \mathbf{0}_\perp)\Phi^\dagger(p^+, n^-|\xi^-, \boldsymbol{\xi}_\perp)\right]^{-1}$$

soft factor:

$$\Phi(p^+, n^-|0) = \left\langle 0 \middle| \mathcal{P}\exp\left[ig\int_{\mathcal{C}_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\zeta)\right] \middle| 0 \right\rangle$$

$$\Phi^\dagger(p^+, n^-|\xi) = \left\langle 0 \middle| \mathcal{P}\exp\left[-ig\int_{\mathcal{C}'_{\mathrm{cusp}}} d\zeta^\mu\, t^a A^a_\mu(\xi + \zeta)\right] \middle| 0 \right\rangle$$