Relativistic 3-body boundstate calculations in Bethe-Salpeter and lightfront formalism

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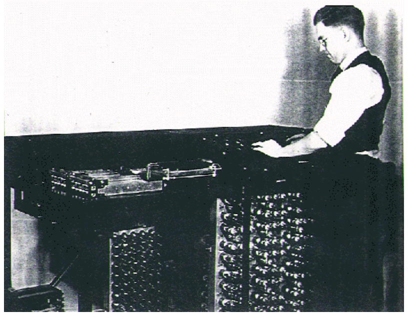
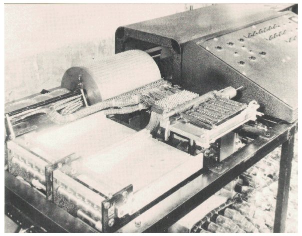
“Home of the first electronic digital computer, called the ABC, invented by Drs. Atanasoff and Berry, 1939 – 1942”

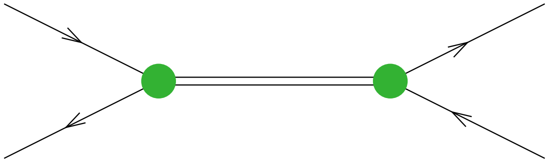




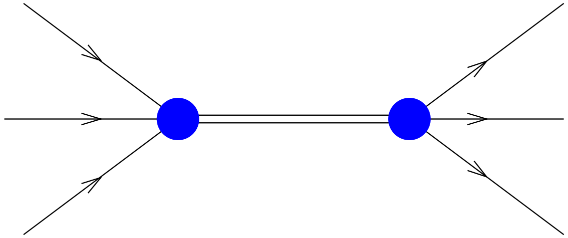
V. Karmanov and P.M.,
Few Body Syst. **46**, 95 (2009)
arXiv:0811.1100 [hep-ph]

Bound states in field theory

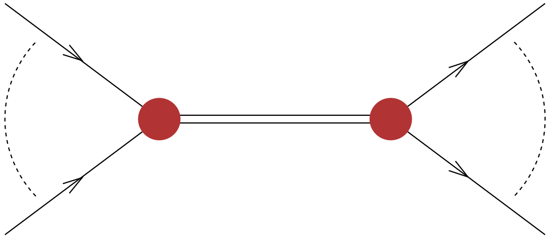
- n-particle bound state
  $\Longleftrightarrow$ pole in $2n$-point Green functions



$$G^{(4)} \sim \frac{\Gamma(p_1, p_2; P)\bar{\Gamma}(k_1, k_2; P)}{P^2 + M^2_{\text{meson}}}$$

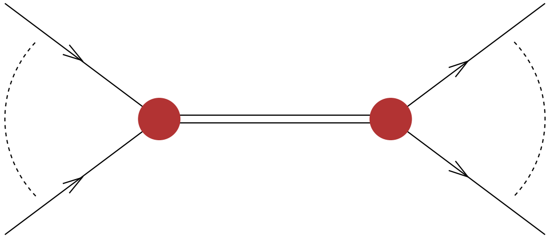


$$G^{(6)} \sim \frac{\Gamma(p_1, p_2, p_3; P)\bar{\Gamma}(k_1, k_2, k_3; P)}{P^2 + M^2_{\text{baryon}}}$$



$$G^{(2n)} \sim \frac{\Gamma^{(n)}(p_1, \ldots, p_3; P)\bar{\Gamma}^{(n)}(k_1, \ldots, k_3; P)}{P^2 + M^2}$$

Bound states in field theory



$$G^{(2n)} \sim \frac{\Gamma^{(n)}(p_1,\ldots,p_3;P)\bar{\Gamma}^{(n)}(k_1,\ldots,k_3;P)}{P^2+M^2}$$

- bound state amplitude Γ
  - solution of homogeneous bound state equation

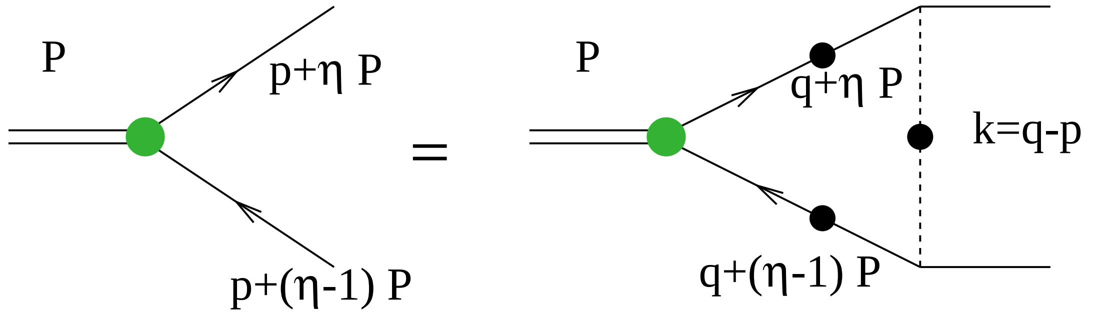
$$\Gamma^{(n)}(p_1,\ldots,p_n;P) = \int \frac{d^4k_1}{(2\pi)^4}\ldots\int \frac{d^4k_n}{(2\pi)^4}\,\delta^4(P-\Sigma k_i)$$

$$K(p_1,\ldots,p_n;k_1,\ldots,k_n;P)\,\Delta(k_1)\ldots\Delta(k_n)\,\Gamma^{(n)}(k_1,\ldots,k_n;P)$$

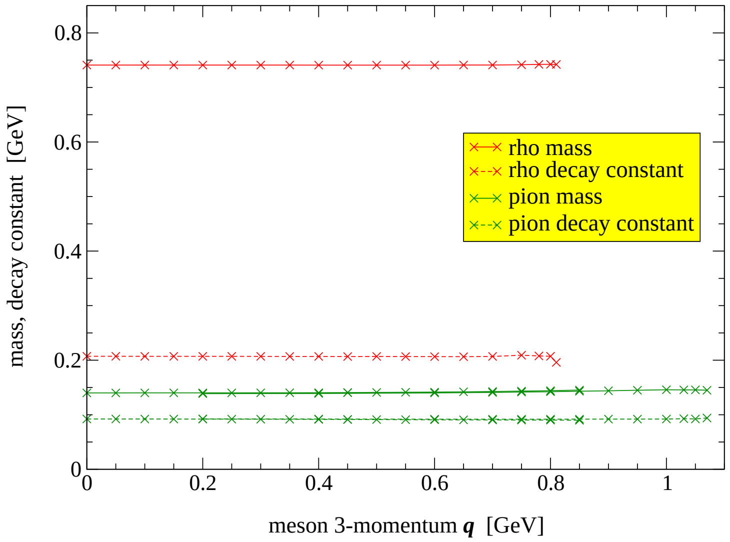
    - bound state is on-shell
    - K is n-to-n particle scattering kernel
  - satisfies canonical normalisation condition

Euclidean Bethe–Salpeter approach

- Bethe–Salpeter equation in ladder truncation



- Defined at at $P^2 = -M^2$
- Independent of
  - momentum partitioning η
  - overall momentum frame, commonly solved in “rest-frame” $P = (iM, 0, 0, 0)$



P.M. & P. Tandy, nucl-th/0511017

Two-body bound states in scalar field theory

- Scalar field theory w. interaction $g\,\phi^2\,\psi$
- Constituent scalar particles w. mass m (field ϕ)
- Scalar exchange particle w. mass μ (field ψ)
- Limit $\mu \to 0$: Wick–Cutkosky model
- Propagators (bare)

$$\Delta(p) = \frac{1}{p^2 + m^2} \qquad\qquad D(k) = \frac{1}{k^2 + \mu^2}$$

- Bethe–Salpeter eqn in ladder truncation for scalar bound state $\Gamma(p_1, p_2; P) = \Gamma(p^2, p \cdot P; P^2 = -M^2)$

$$\Gamma(p^2, p \cdot P; P^2) = g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{[(q-p)^2 + \mu^2]} \frac{\Gamma(q^2, q \cdot P; P^2)}{[(P/2 + q)^2 + m^2]\,[(P/2 - q)^2 + m^2]}$$

Lightfront two-body bound state equation

- Two-body bound state equation in ladder truncation

$$\left(\frac{p^2+m^2}{x(1-x)} - M^2\right)\psi(p,x) \quad = \quad \frac{g^2}{8\pi^3}\int_0^\infty k\,dk\int_0^{2\pi}d\alpha\int_0^1\frac{dy}{2\,y\,(1-y)}$$

$$V(p,x;k,y;\cos\alpha;\mu^2,m^2,M^2)\,\psi(k,y)$$

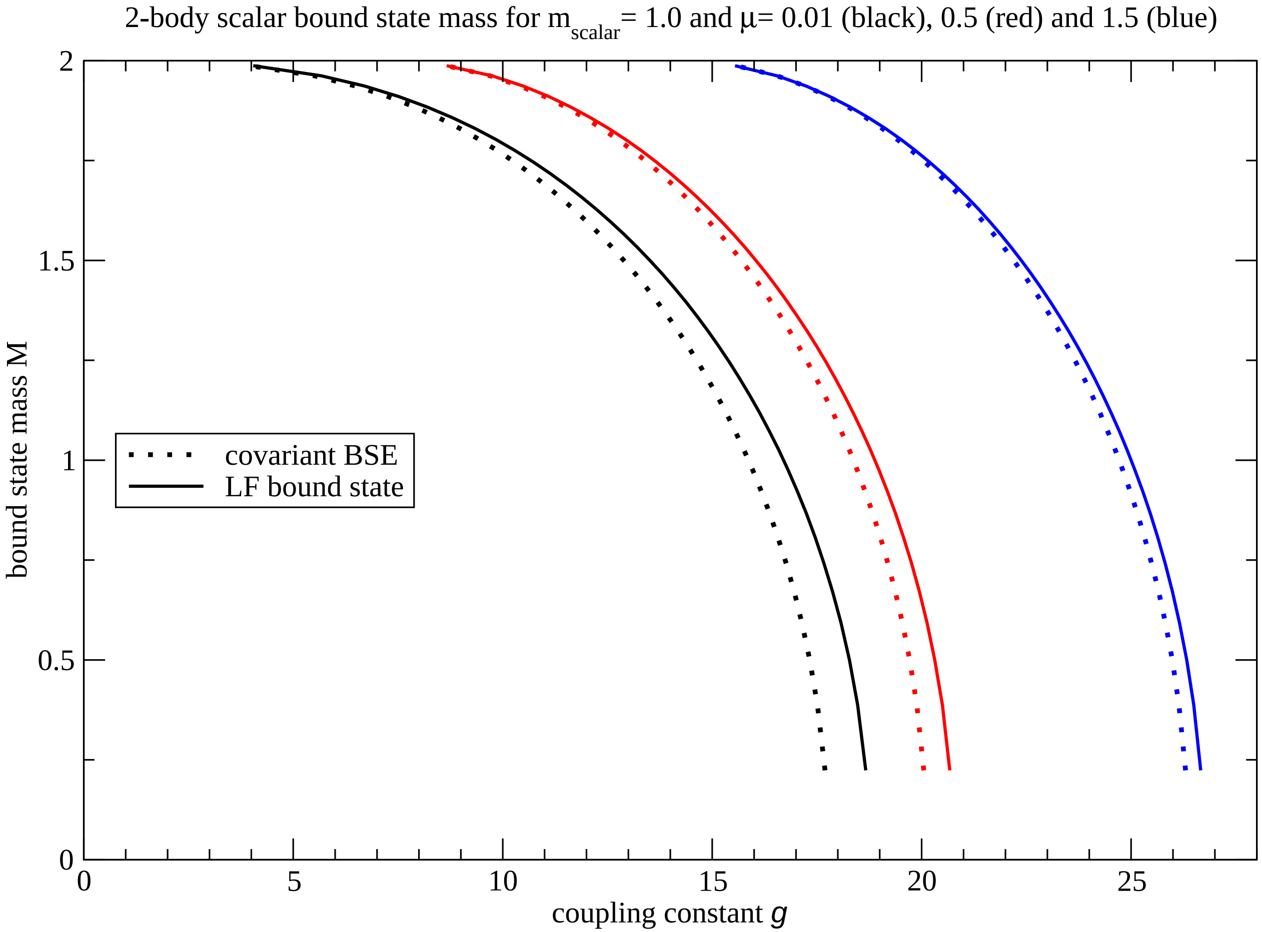
  - x, y fraction of total momentum carried by constituent
  - p, k: 2-dim transverse vectors; $\cos\alpha = \frac{p_\perp\cdot k_\perp}{p\,k}$

- One-particle exchange kernel in LF dynamics

$$V = \begin{cases} \dfrac{1}{\mu^2+m^2\left(\frac{x}{y}-2+\frac{y}{x}\right)+\frac{x}{y}\,k^2-2\,p\,k\,\cos\alpha+\frac{y}{x}\,p^2+(x-y)\left(\frac{m^2+p^2}{x\,(1-x)}-M^2\right)} & \text{for } y<x \\[2ex] \dfrac{1}{\mu^2+m^2\left(\frac{y}{x}-2+\frac{x}{y}\right)+\frac{y}{x}\,p^2-2\,p\,k\,\cos\alpha+\frac{x}{y}\,k^2+(y-x)\left(\frac{m^2+k^2}{y\,(1-y)}-M^2\right)} & \text{for } y>x \end{cases}$$

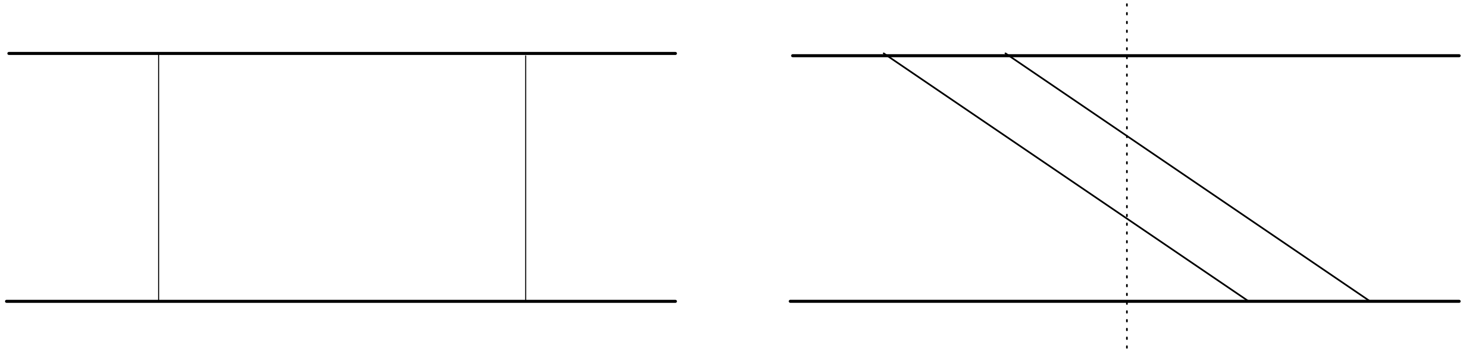
Lightfront vs. Bethe–Salpeter results

- LF two-body bound state equation very similar to explicitly covariant two-body BS eqn



Lightfront vs. Bethe–Salpeter results

- LF two-body bound state equation very similar to explicitly covariant two-body BS eqn
- Difference between LF and BS eqn: not more than one exchange particles "in flight" in truncated LF eqn



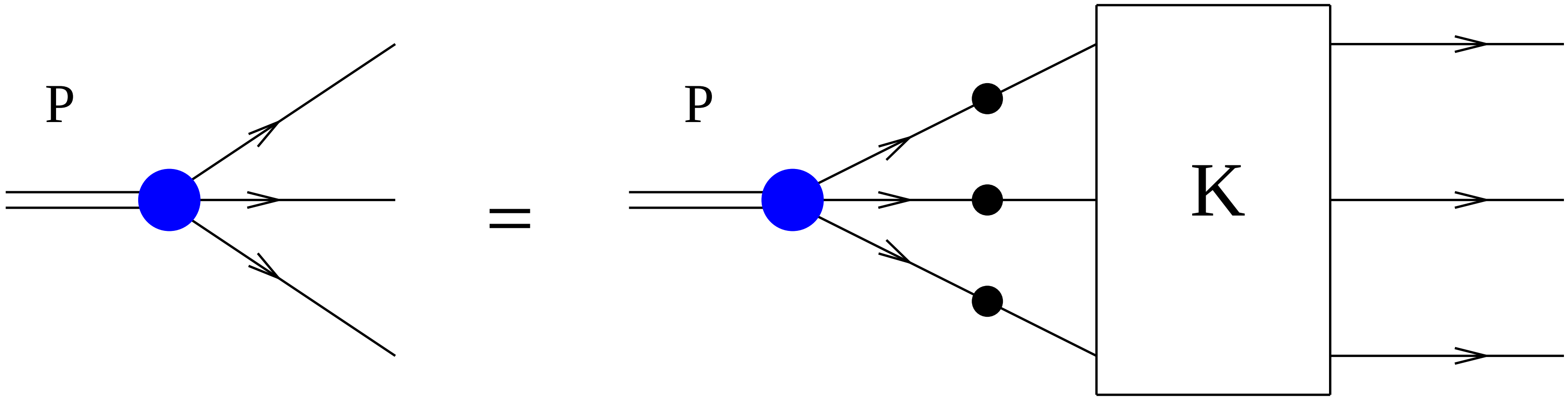
  - Apparently, contributions "stretched boxes" small
    - at least in this scalar theory

Schoonderwoerd, Bakker, Karmanov, Phys. Rev. **C58**, 3093 (1998) [arXiv:nucl-th/9806365];

Carbonell, Karmanov, Eur. Phys. J. **A27**, 11 (2006); [arXiv:hep-th/0505262]

Three-body bound states

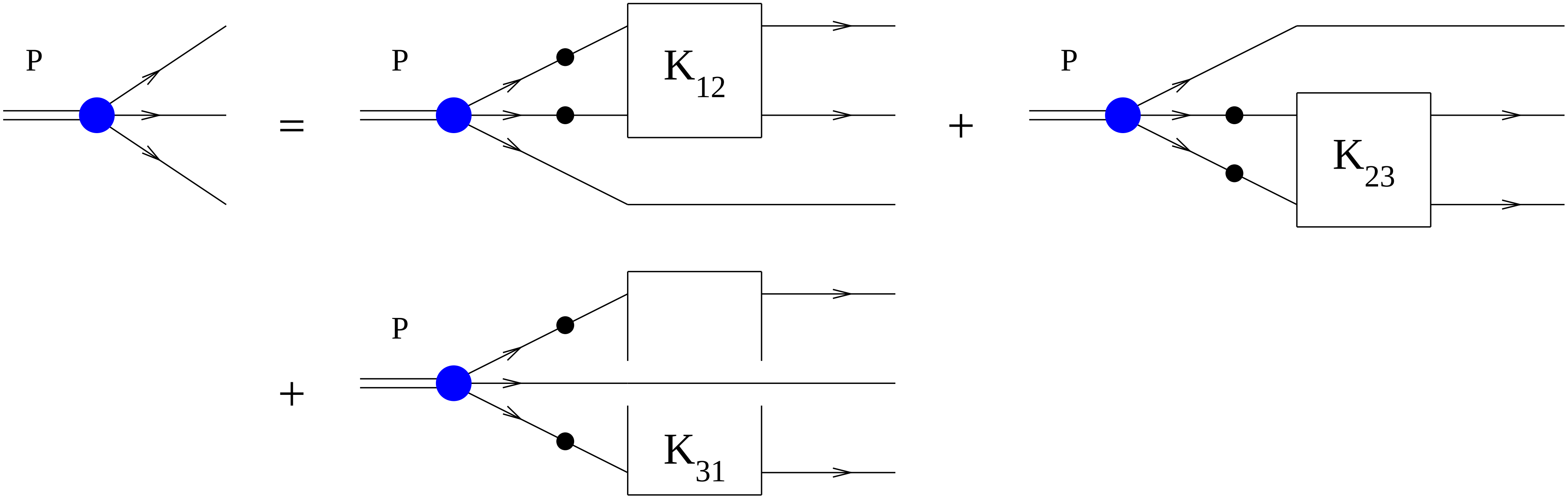
- Bound state equation for BS amplitude $\Gamma(p_1, p_2, p_3; P)$



  - at mass pole
  - momentum conservation: $p_1 + p_2 + p_3 = P$
  - in principle, should use dressed propagators
  - three-body scattering kernel K

Three-body bound states

- Ignoring intrinsic three-body kernel



with K_{ij} the ij two-body scattering kernel

same as in two-body Bethe–Salpeter equation

- Solve numerically for BS amplitude $\Gamma(p_1, p_2, p_3; P)$ at mass pole $P^2 = -M^2$(Euclidean metric)

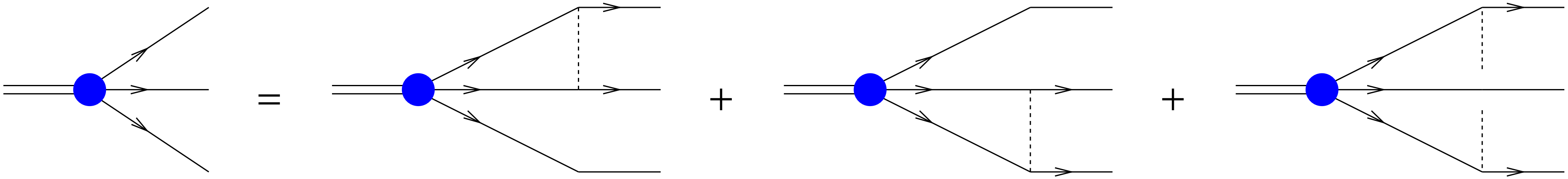
Scalar field theory – three-body bound states

- Constituent scalar particles: mass m
- Interaction: exchange of a scalar particle of mass μ
- Bare propagators

$$\begin{aligned}\Delta(p) &= \frac{1}{p^2+m^2}\\ D(k) &= \frac{1}{k^2+\mu^2}\end{aligned}$$

Scalar field theory – three-body bound states

- Constituent scalar particles: mass m
- Interaction: exchange of a scalar particle of mass μ
- Bound state equation in ladder truncation



$$\Gamma(p_1, p_2, p_3; P) \quad = \quad g^2 \sum \int \frac{d^4k}{(2\pi)^4} \, \Delta(p_1^+) \, \Delta(p_2^-) \, D(k) \, \Gamma(p_1^+, p_2^-, p_3; P)$$

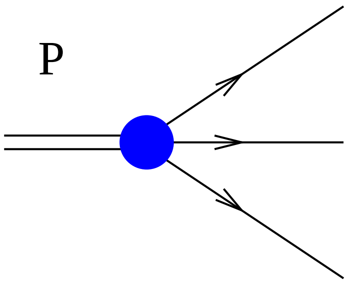
summed over cyclic permutations, and with $p_i^\pm = p_i \pm k$

Momentum choices

- Momentum conservation $p_1 + p_2 + p_3 = P$
- Total bound state momentum P fixed: $P^2 = -M^2$
- BS amplitude of scalar bound state:
  scalar function of two independent 4-vectors

$$\Gamma(p_1, p_2, p_3; P) \quad = \quad \Gamma(p, q; P)$$

- Standard Jacobi coordinates for 3-body system



$$\begin{aligned} p_1 &= \frac{1}{3}P + q + p \\ p_2 &= \frac{1}{3}P + q - p \\ p_3 &= \frac{1}{3}P - 2\,q \end{aligned}$$

Numerical details

- $\Gamma(p,q;P)$ function of five independent scalar variables

$$\Gamma(p,q;P) \quad = \quad \Gamma(p^2, p\cdot P, q^2, q\cdot P, p\cdot q; P)$$

- Notation (rest-frame)

$$\begin{aligned} p_\mu &= p[\cos\theta_p, \sin\theta_p\cos\phi_{pq}, \sin\theta_p\sin\phi_{pq}, 0] \\ q_\mu &= q[\cos\theta_q, \sin\theta_q, 0, 0] \\ k_\mu &= k[\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi\cos\alpha, \sin\theta\sin\phi\sin\alpha] \end{aligned}$$

- In practice: $\Gamma(p,q;P)$ function of p^2, q^2, θ_p, θ_q, and ϕ_{pq}
- Four-dimensional integral

$$\int d^4k \quad = \quad \int_0^\infty k^3\,dk \int_0^\pi \sin^2\theta\,d\theta \int_0^\pi \sin\phi\,d\phi \int_0^{2\pi} d\alpha$$

More numerical details

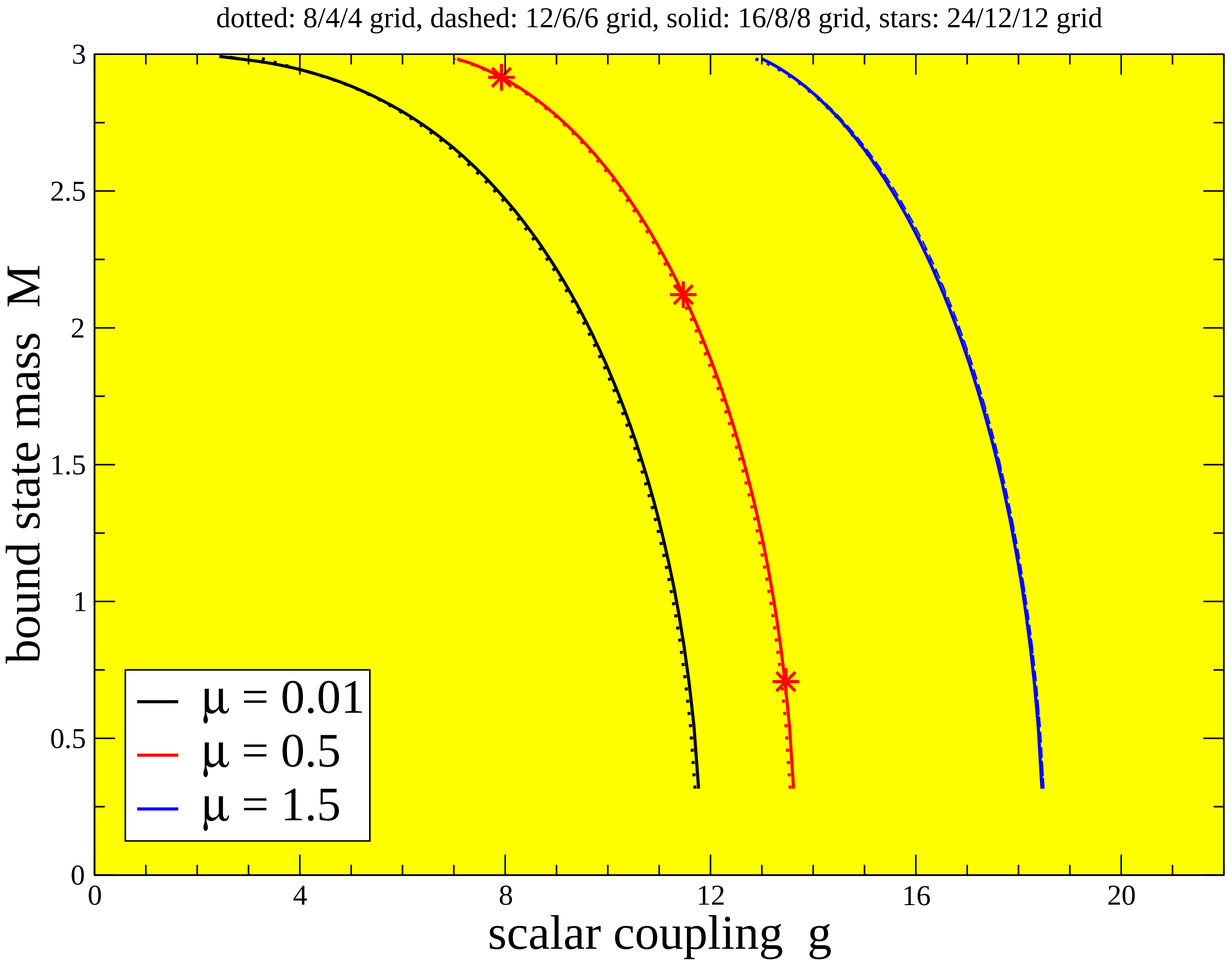
Scalar bound state equation in ladder truncation

$$\Gamma(p^2, q^2, \theta_p, \theta_q, \phi_{pq}; P^2) \;=\; g^2 \int \frac{d^4k}{(2\pi)^4}$$
$$\sum_i \mathcal{K}_1(p^2, q^2, \theta_p, \theta_q, \phi_{pq}; k^2, \theta, \phi, \alpha; P^2)\,\Gamma(\ldots; P^2)$$

with the kernels

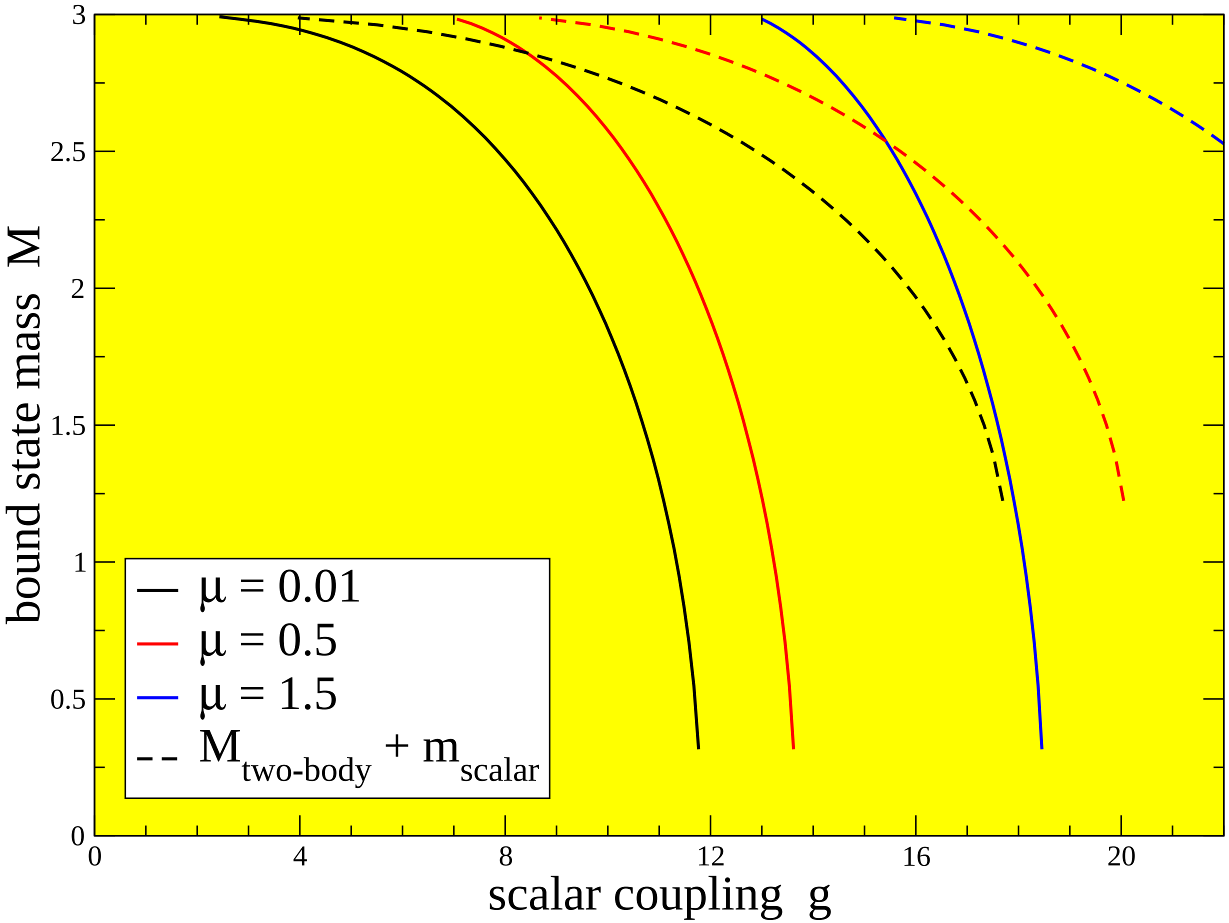
$$\begin{aligned}
\mathcal{K}_1 &= D(k)\,\Delta\left(\tfrac{1}{3}P + q + p + k\right)\,\Delta\left(\tfrac{1}{3}P + q - p - k\right)\\
\mathcal{K}_2 &= D(k)\,\Delta\left(\tfrac{1}{3}P + q - p + k\right)\,\Delta\left(\tfrac{1}{3}P - 2q - k\right)\\
\mathcal{K}_3 &= D(k)\,\Delta\left(\tfrac{1}{3}P - 2q + k\right)\,\Delta\left(\tfrac{1}{3}P + q + p - k\right)
\end{aligned}$$

Numerical results



- Estimated numerical accuracy
  better than 1% to about 2% (weak coupling of $\mu = 0.01$)
- Independent of choice of momenta (within accuracy)

Numerical results – continued



- Three-body state indeed bound state

  $M_3 < M_2 + m$ at fixed coupling constant
- Symmetric under exchange of internal momenta

Lightfront three-body bound state equation

Light-front 3-body wave function

$$\psi(\vec{p}_1, x_1; \vec{p}_2, x_2; \vec{p}_3, x_3)$$

with momentum conservation

$$\sum_i \vec{p}_i \;=\; 0 \qquad \sum_i x_i \;=\; 1$$

function of 5 independent variables (Jacobi coordinates)

$$\psi(p, x_p; q, x_q; \cos\theta)$$

- p and q 2-dim. transverse momenta with $\cos\theta = \frac{p\cdot k}{p\,k}$
- x_p and x_q momentum fractions

Lightfront three-body bound state equation

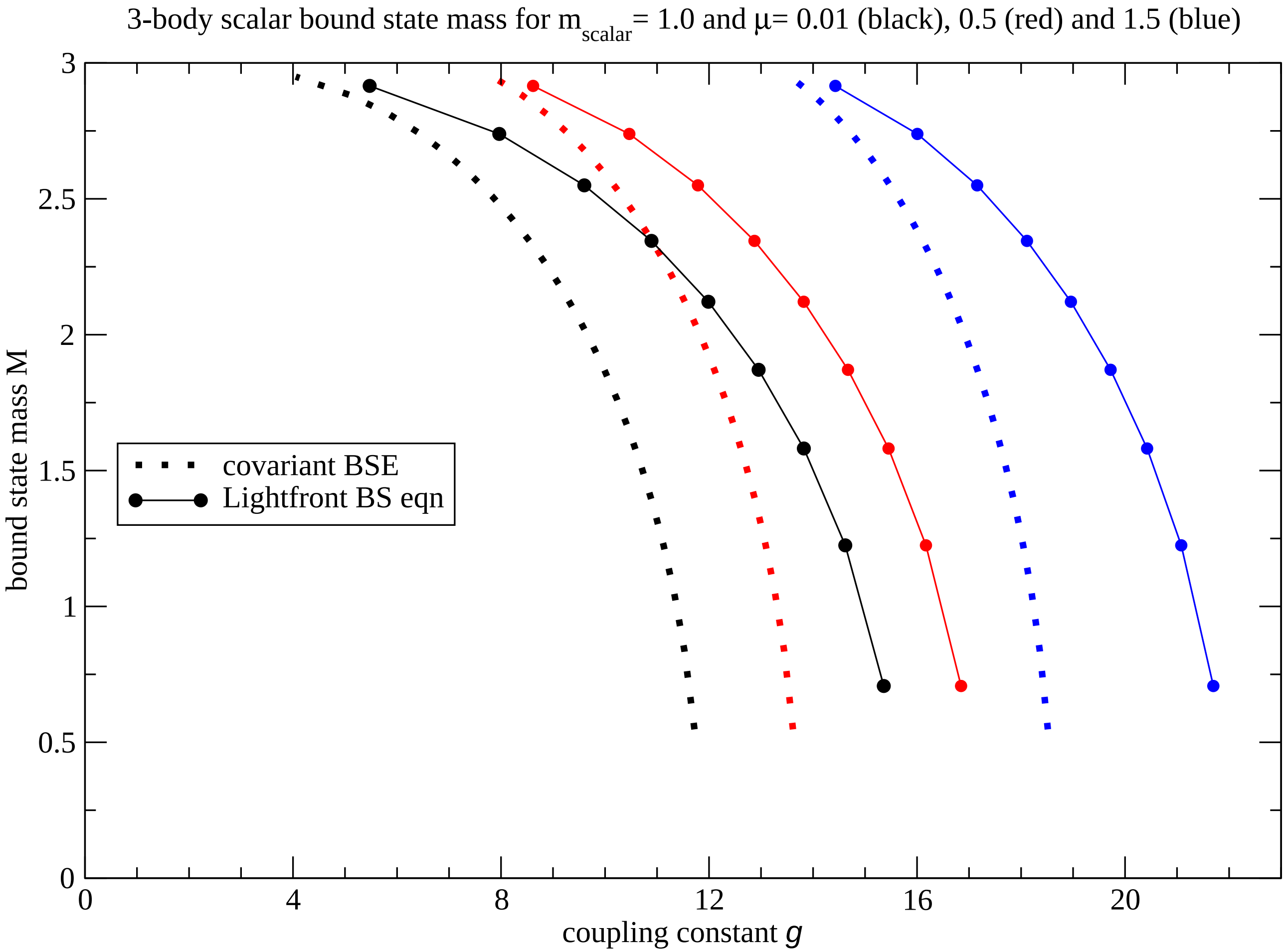
Light-front 3-body wave function

$$
\begin{aligned}
\psi(p,x_p;q,x_q;\cos\theta) \;=\; & \frac{g^2}{8\pi^3}\int_0^\infty k\,dk\int_0^{2\pi} d\alpha\int_0^1 \frac{dy}{2\,y\,(1-y)} \\
& \frac{1}{\frac{k^2+m^2}{y}+\frac{k^2+m^2}{1-y}-M_{pq}^2} V(p,x_p;k,y;\cos\alpha;\mu^2,m^2,M_{pq}^2) \\
& \times\Big[\psi(r_{12},z_{12};r'_{12},z'_{12};\cos\phi_{12})+\psi(r_{23},z_{23};r'_{23},z'_{23};\cos\phi_{23}) \\
& \qquad +\psi(r_{31},z_{31};r'_{31},z'_{31};\cos\phi_{31})\Big]
\end{aligned}
$$

where V as before and

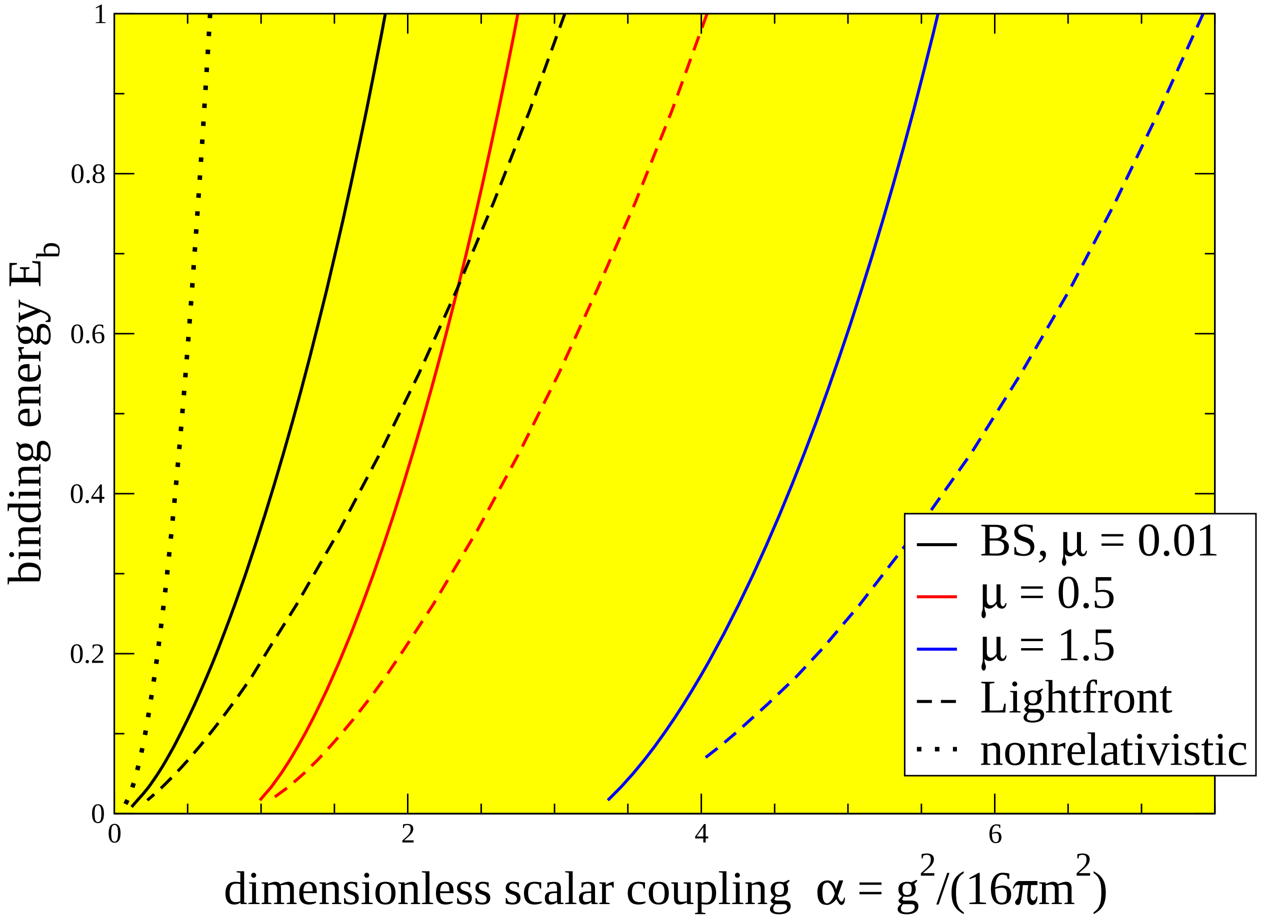
$$
M_{pq}^2 \;=\; (1-x_q)M^2-\frac{q^2+(1-x_q)m^2}{x_q}
$$

Lightfront three-body bound state results



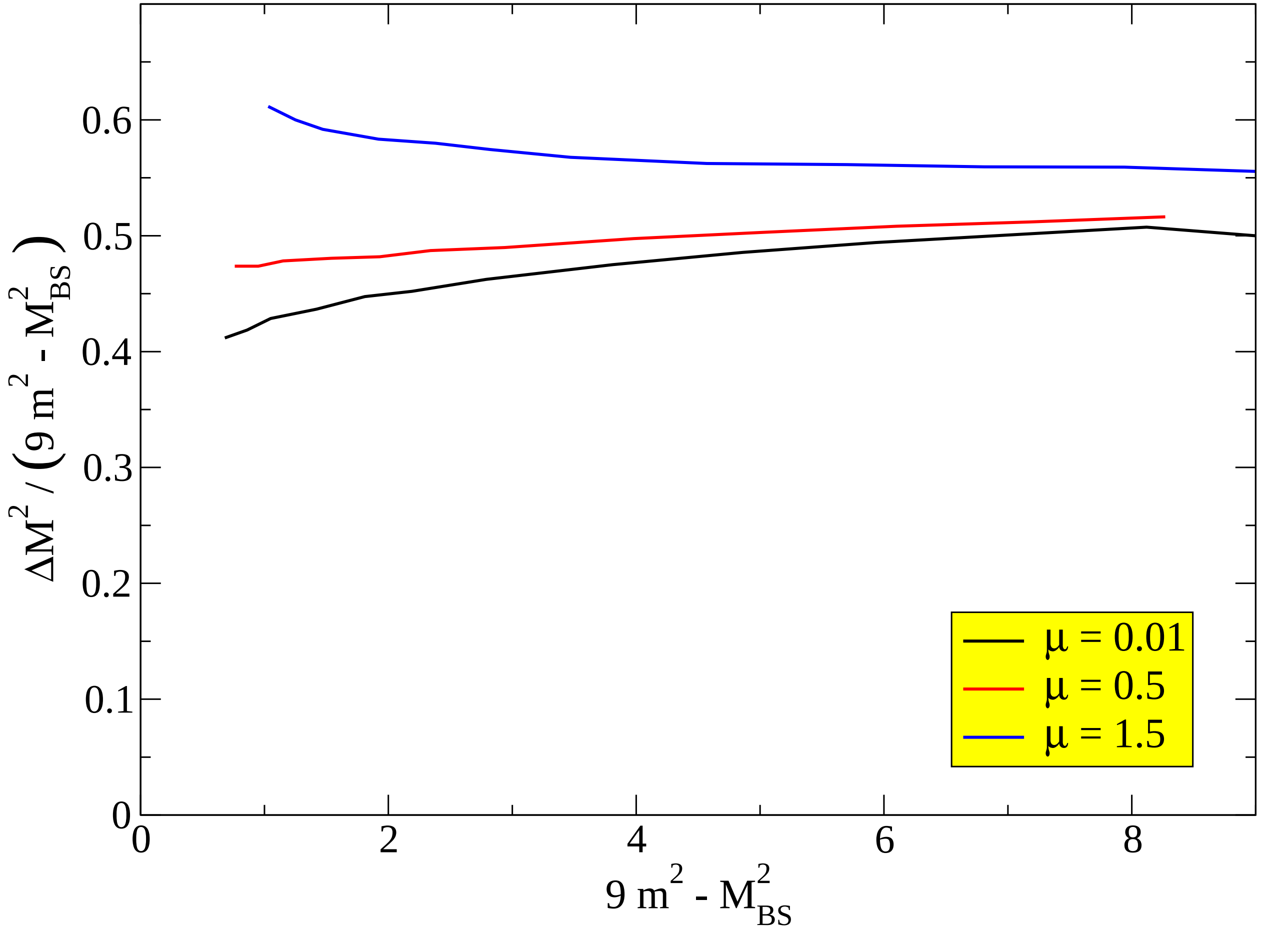
- LF 3-body bound state equation qualitatively similar to covariant 3-body BS eqn, but underbinds significantly

Lightfront three-body bound state results



- LF underbinds compared to covariant 3-body BS eqn even in weak-binding limit

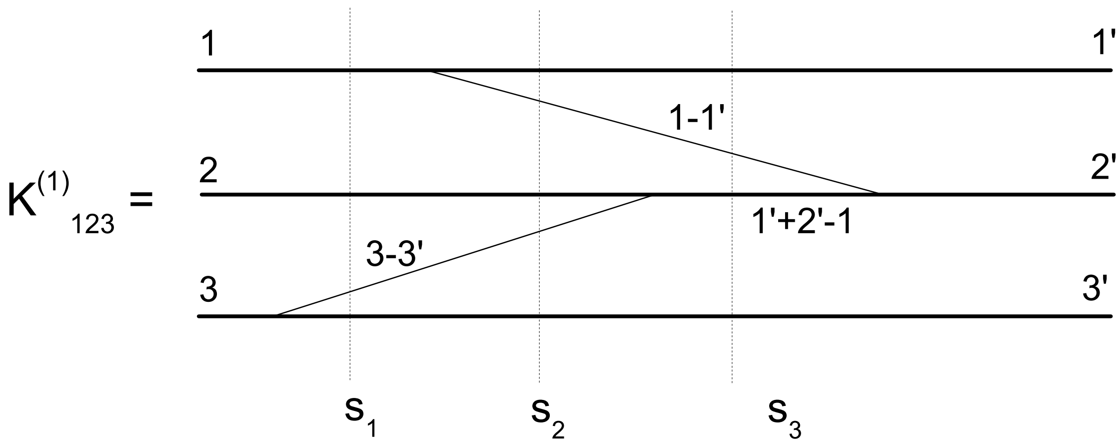
Lightfront three-body bound state results



- Difference between LF and BS binding approximately independent of coupling constant and exchange mass

Lightfront vs. Bethe–Salpeter ladder truncation

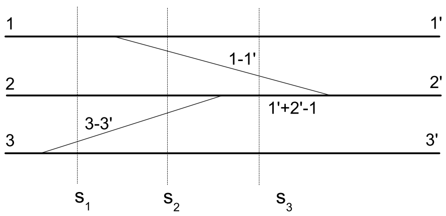
- Main difference between LF 3-body bound state eqn and covariant BS eqn due to “two exchange particles in flight”



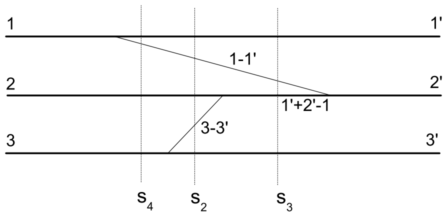
Lightfront vs. Bethe–Salpeter ladder truncation

- Main difference between LF 3-body bound state eqn and covariant BS eqn due to “two exchange particles in flight”

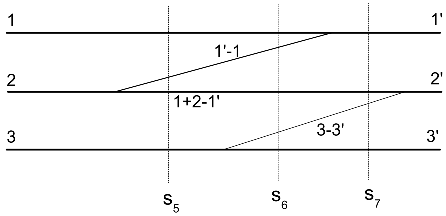
$K^{(1)}_{123} =$



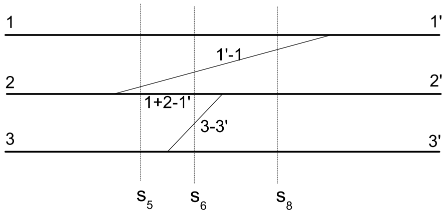
$K^{(2)}_{123} =$



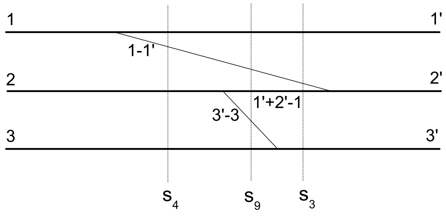
$K^{(3)}_{123} =$



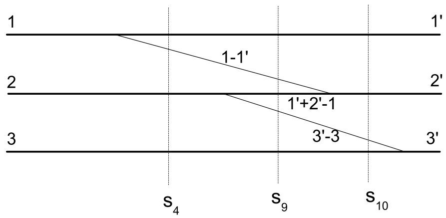
$K^{(4)}_{123} =$



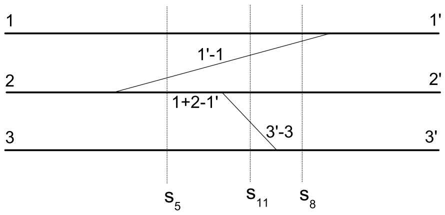
$K^{(5)}_{123} =$



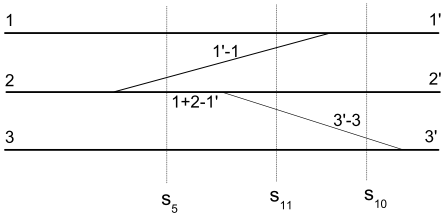
$K^{(6)}_{123} =$



$K^{(7)}_{123} =$

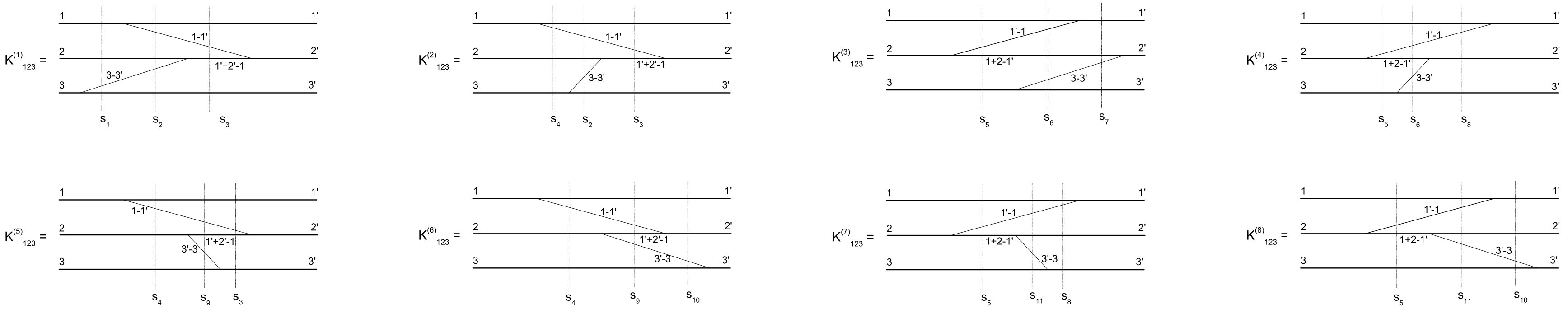


$K^{(8)}_{123} =$

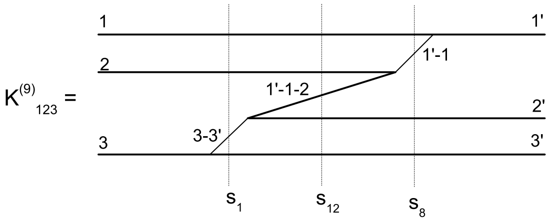


Lightfront vs. Bethe–Salpeter ladder truncation

- Main difference between LF 3-body bound state eqn and covariant BS eqn due to “two exchange particles in flight”



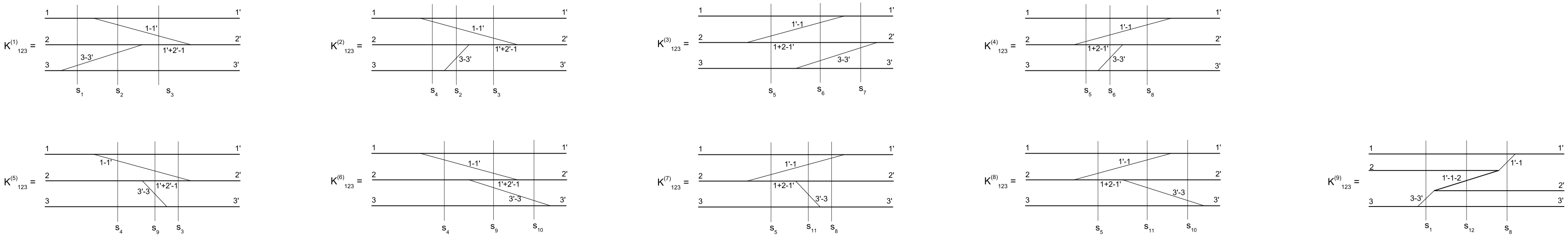
- Also contribution from pair creation



- Relativistic effective 3-body force in LF 3-body bound state eqn

Relativistic 3-body-force correction

- Additional contribution to LF 3-body bound state eqn



- Perturbative estimate of correction

$$\Delta M_3^2 \;=\; -\int \frac{d^2 k_{3\perp}}{(2\pi)^3}\,\frac{d^2 k_{\perp}}{(2\pi)^3}\,\frac{d^2 k'_{3\perp}}{(2\pi)^3}\,\frac{d^2 k'_{\perp}}{(2\pi)^3}$$

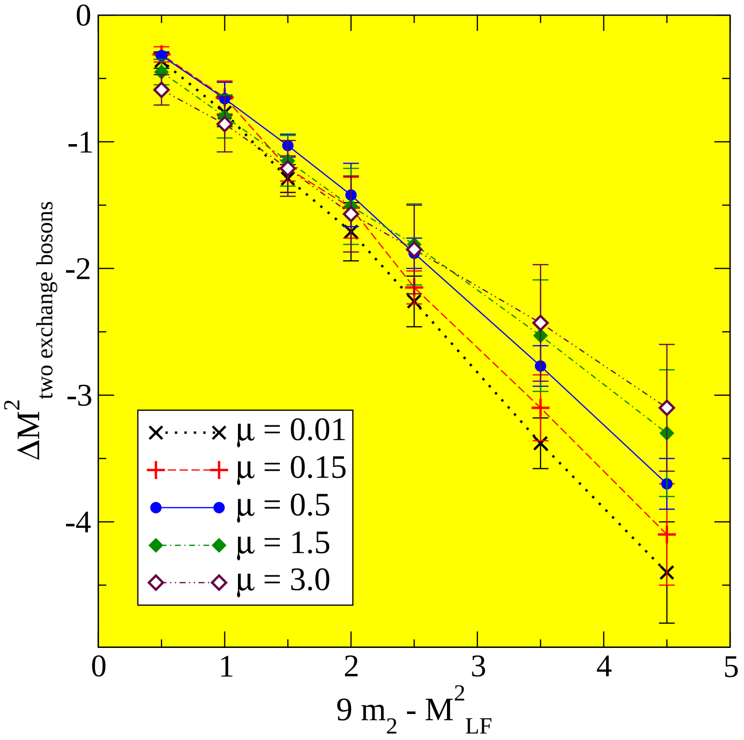
$$\int_0^1 \frac{dx_3}{2x_3(1-x_3)}\,\frac{dx}{2x(1-x)}\,\frac{dx'_3}{2x'_3(1-x'_3)}\,\frac{dx'}{2x'(1-x')}$$

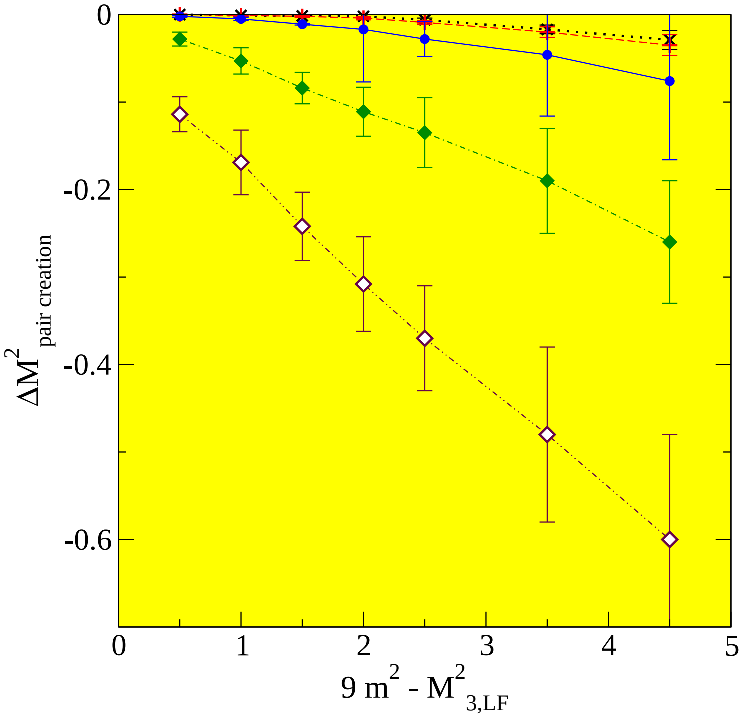
$$\times\,\psi(\vec{k}_\perp, x; \vec{k}_{3\perp}, x_3)\, K_{123}(\vec{k}_\perp, x; \vec{k}_{3\perp}, x_3; \vec{k'}_\perp, x'; \vec{k'}_{3\perp}, x'_3)\, \psi(\vec{k'}_\perp, x'; \vec{k'}_{3\perp}, x'_3)$$

- 11-dimensional integral, to be done numerically . . .

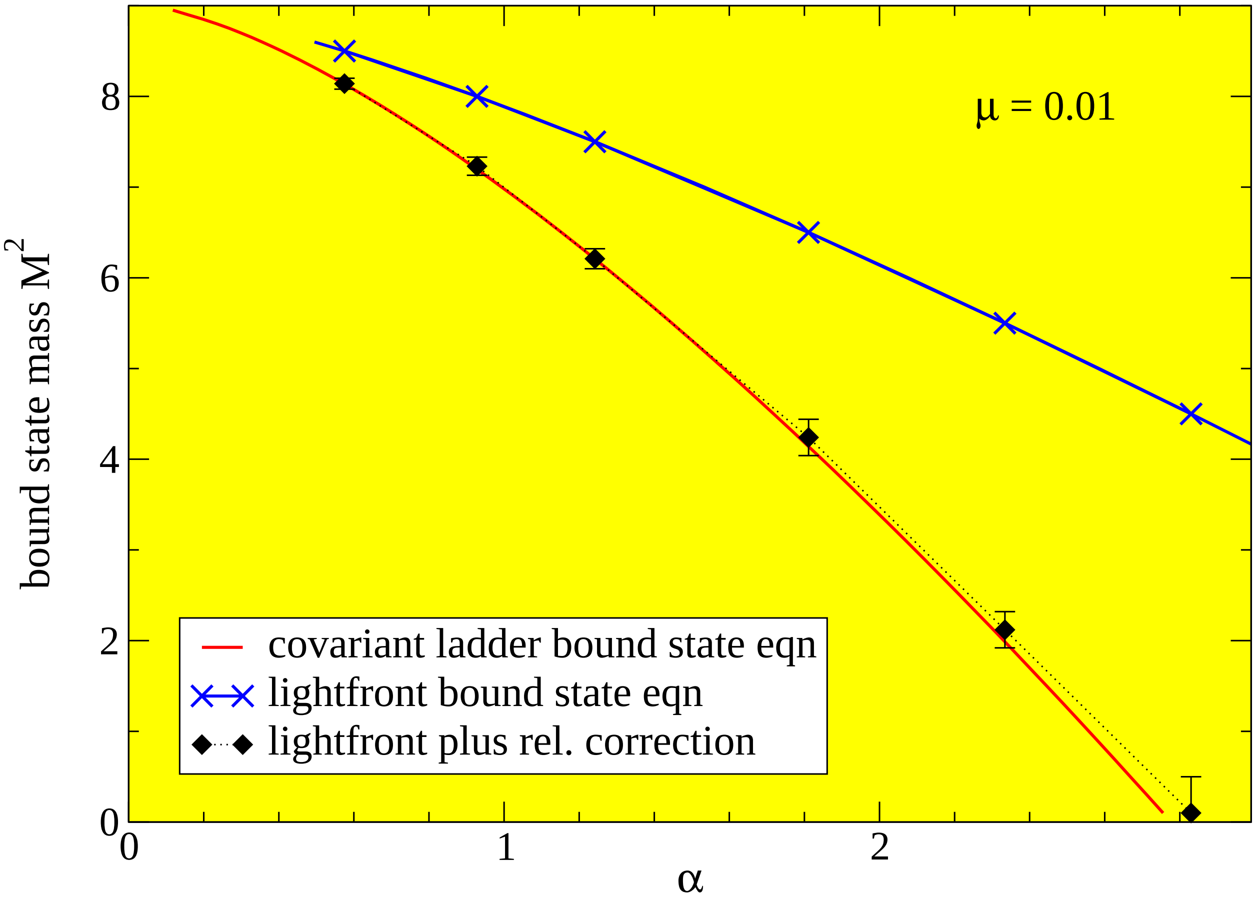
Numerical results LF correction

- Dominated by “two exchange particles in flight”
  - remarkably independent of exchange mass
- Contribution due to pair-creation
  does depend on ratio constituent over exchange mass



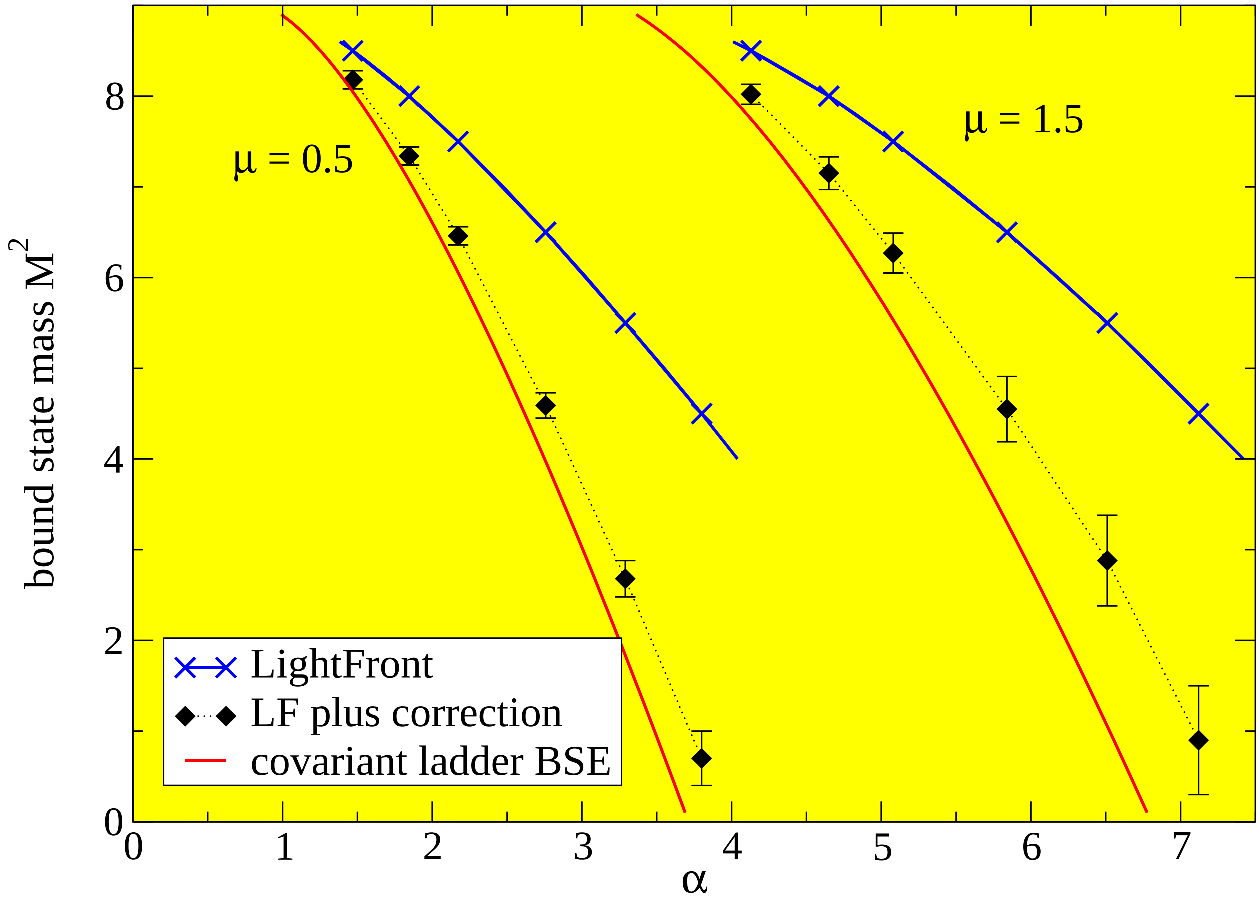


Numerical results lightfront vs. BS eqn



- LF 3-body bound state eqn + relativistic 3-body force correction **agrees** with explicitly covariant 3-body BSE

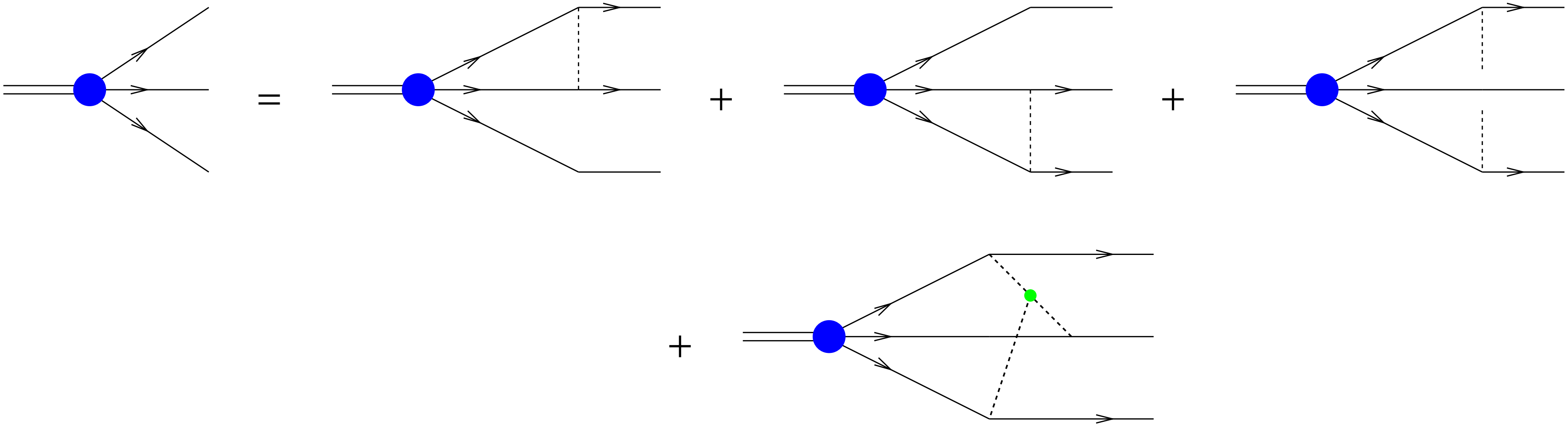
Numerical results lightfront vs. BS eqn



- LF 3-body bound state eqn + relativistic 3-body force correction gives ∼ 90% binding compared to 3-body BSE

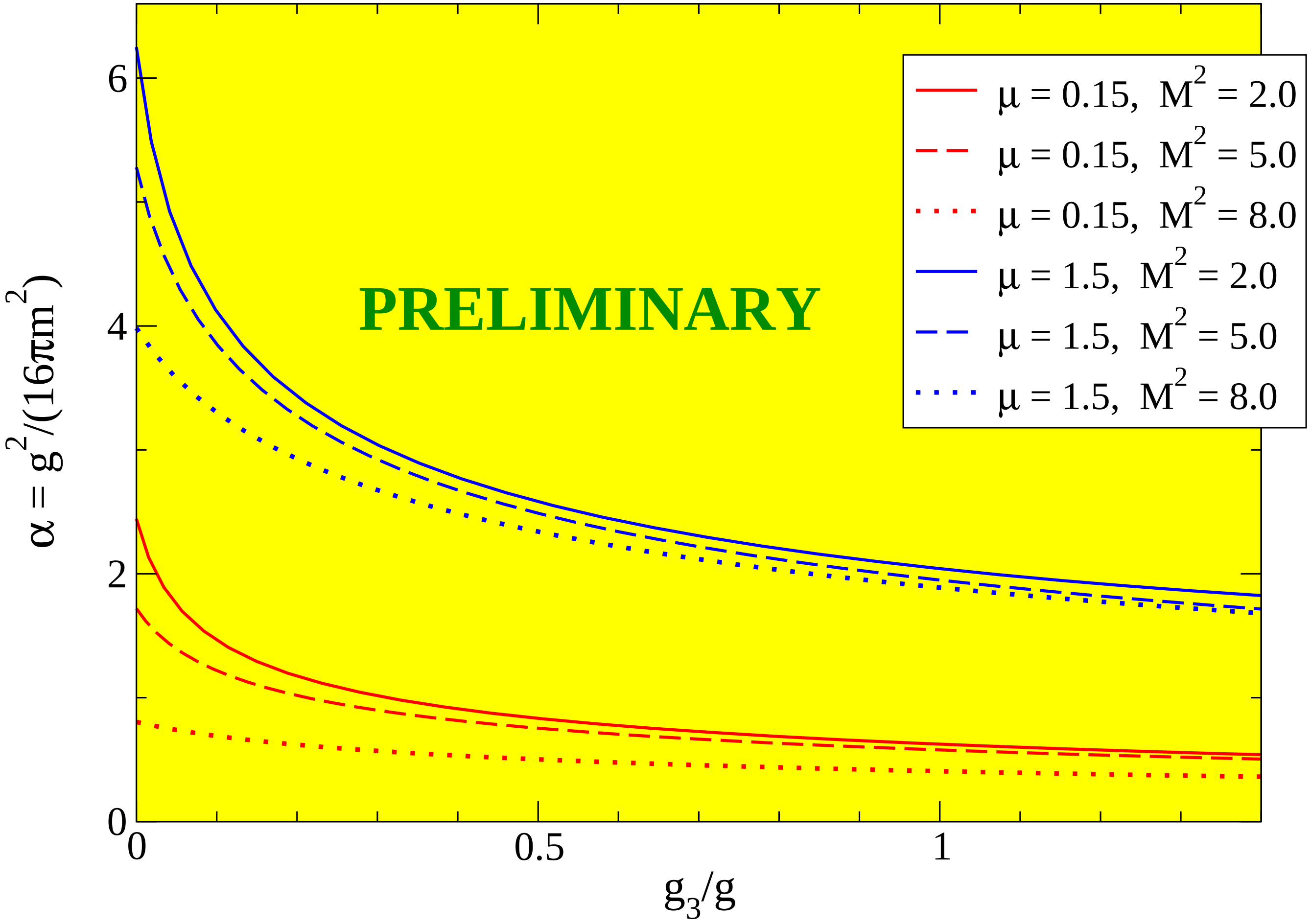
Intrinsic three-body bound force

- Add self-interaction $g_3\,\psi^3$ for exchange particle (motivated by triple gluon vertex)
- Consider intrinsic threebody force contribution in addition to ladder kernel



- Eight-dimensional integral equation for $\Gamma(p,q;P)$
- $\Gamma(p,q;P)$ function of 5 indep. variables at fixed $P^2=-M^2$

Numerical results (work in progress . . .)



estimated numerical accuracy about 5% to 10%

- equal strength for ladder kernels and intrinsic 3-body interaction term reduce α by factor of 2 to 4 to keep E_b the same

Conclusions

- Manifestly covariant 3-body Bethe–Salpeter eqn in ladder truncation can be solved without further approximations
  - Also with intrinsic 3-body forces

- Lightfront three-body bound state eqn in ladder truncation can be solved numerically without further approximations

- Differences between BS and LF results due to contributions from “two exchange particles in flight” missing from LF eqn $\Longrightarrow$ relativistic 3-body force

- Future work
  - Form factors, distribution functions, . . .
  - Baryons as three quark bound states ?