Solving the BSE with an integral representation in Minkowski space

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Outline

- Motivation and introduction
- PTIR and what does it means Minkowski BSE solution
- Application: BSE in Wick-Cutkosky models
- A: physically meaningful scalar field models, Higgsonia
- A: fermion-antifermion BSE
- A: toy model BSE with regge behaviour

Motivation and Introduction

Appearance and historical propaganda of BSE - relativistic effect in fine and hyperfine splitting in a hydrogen-like atom QFT application,

Salpeter (1952), Salpeter and Newcomb (1952), Salpeter (1953)

Historical review: Bethe-Salpeter Equation – The Origins, Edwin E. Salpeter, arXiv:0811.1050

Inhomogeneous BSE - quantum field theory based description of relativistic bound states. Suitable for description of (quasi)stable 2body states. The BSE couples to infinite system of SDEs, BSE is a part of them. In strong coupling theories it cannot be studied alone, but must be considered in the framework with the all other GFs, eg. changes from dressing of all GFs appearing in the kernel should be considered.

Recent applications:

- Mesons= $q\bar{q}$ states A. Krassnigg, arXiv:0909.4016 Lei Chang, C.D. Roberts, Phys.Rev.Lett., 103:081601,(2009). A. Hoell, A. Krassnigg, P. Maris, C.D. Roberts, S.V. Wright, Phys. Rev. C71 (2005) 065204.

- beyond SM- technihadrons, composite Higgses, exotic bound states (dark matter Higgsonium), etc...

  A. Doff, A. A. Natale, P. S. Rodrigues da Silva, arXiv:0905.2981 V. Sauli,arXiv:0806.3454 M. Kurachi, R. Shrock, JHEP0612:034 ,2006 M. Harada, M. Kurachi, K. Yamawaki, Prog.Theor.Phys. 115 (2006).

- effective applications (mostly in 3 dim reduction)- Deuteron Kaptari et all, Phys.Rev.C54:986,1996, Bondarenko et all, Phys.Atom.Nucl.70:2054-2065,2007 J. Adam eta all. Phys.Rev. C66 (2002)044003.

All particles have their own times-> BSE has complicated singular kernel stmming from 1(small problem):- BSE is composed from GFs, they have singularities of GFs- poles, branch points and 2(big problem):- singularities in SDEs kernels stemming from integral measure because of Minkowski metric–>

No direct assumption-less Minkowski space solution of BSE is known until the date. Two ways out-

A) Wick rotation , Euclidean space numerical solution,

B) method of IR

- at the end we stay in Minkowski space again. Advantages of B) -Hadrons, atoms and nuclei have $P^2 > 0$, no extrapolation of numerical data is needed, chance for excitations

-One can get frame independent solution! It can be useful when form factors are constructed.

PTIR, and what does it means Minkowski BSE solution

PTIR (Perturbation Theory Integral representation) of Greens functions: Nakanishi- Graph Theory and Feynman Integrals, 1971

Review of applications to SDEs, BSE: V. Sauli, FBS.39:45,(2006)

General PTIR derivation by Nakanishi is based on the analytical structure of Feynman diagrams, i.e. on Feynman parametrization and loop momentum integration. The Wick rotation is performed since the formal continuation to Minkowski space is as clear as in the perturbation theory.

Minkowski solution of BSE by PTIR method = Wick rotation is allowed and performed analytically (but it does not mean that WR is not assumed)

Minkowski PTIR based solution must agree with the Euclidean solutions!

PTIR is unique in a given scheme. PTIR is proved for any Feyman diagram an trivially to any order, however proof is perturbative (not-selfconsistent).

Examples of PTIR (or generalized PTIR):

- 2p GF -Assuming perturbative analyticity of Green's function,e.g. Propagators satisfy KLR:

$$
\begin{aligned}
G^{KLR}(k) &= \int_0^\infty dM^2 \rho(M^2) D(k^2, M^2) \\
D(k, M^2) &= \frac{1}{k^2 - M^2 + i\varepsilon}
\end{aligned}
$$

The inverse propagator (selfenergy, polarization, etc..) DR

$$
G^{-1} = p^2 - \Sigma(p) = polyn. + \int_T^\infty dM^2 \sigma(M^2) D(k^2, M^2)
$$

- 3legs: For the three-leg vertex function $\Gamma(p_1, p_2, p_3)$ analysis of contributing Feynman diagrams leads to the PTIR

$$\Gamma(p_1, p_2, p_3) = \int_0^\infty d\alpha \prod_{i=1}^{3} \int_0^1 dz_i \, \delta(1 - \sum_{i=1}^{3} z_i) \, \frac{\rho_3(\alpha, \vec{z})}{\alpha - (z_1 p_1^2 + z_2 p_2^2 + z_3 p_3^2) - i\epsilon},$$

where the momenta are conserved $p_1 + p_2 + p_3 = 0$, the invariant squares are independent and the single weight function $\rho_3(\alpha, \vec{z})$ is sufficient to describe the sum of all relevant Feynman diagrams.

- 4legs: The scattering matrix $M(p,q;P)$ describes the process $\Phi_1\Phi_2 \to \Phi_3\Phi_4$, where $p=(q_1-q_2)/2$ and $q=(q_3-q_4)/2$ are the initial and final relative momenta, respectively, and P is the total four-momentum.

$$
\begin{aligned}
M(p,q;P) &= \int_0^\infty d\gamma \int_\Omega d\vec{\xi} \left\{ \frac{\rho_{st}(\gamma,\vec{\xi})}{\gamma - \left[\sum_{i=1}^4 \xi_i q_i^2 + \xi_5 s + \xi_6 t\right] - i\epsilon} \right. \\
&+ \frac{\rho_{tu}(\gamma,\vec{\xi})}{\gamma - \left[\sum_{i=1}^4 \xi_i q_i^2 + \xi_5 t + \xi_6 u\right] - i\epsilon} \\
&+ \left. \frac{\rho_{us}(\gamma,\vec{\xi})}{\gamma - \left[\sum_{i=1}^4 \xi_i q_i^2 + \xi_5 u + \xi_6 s\right] - i\epsilon} \right\}
\end{aligned}
$$

where q_i^2 is the square of the four-momentum carried by Φ_i and $s, t,$ and u are the Mandelstam variables.

M for ladder BSE is generated via.:

$$M_{ir}(p,q;P) = \frac{g^2}{\mu^2 - (p-q)^2 - i\epsilon}$$

- Two Body BSE PTIR

  Definitions: Inhomogeneous BSE (SDE for G_4) for the full four-point Green's function G satisfies

$$\begin{aligned} G &= G_{BS} - G_{BS} M G_{BS}\,, \\ M &= V - V G_{BS} M \end{aligned} \tag{1}$$

$G_{BS} = -iG_{(1)}G_{(2)}$ is the free two-particle propagator. The BS wave function $\Phi(x_1, x_2) =< 0|T[\phi(x_1)\bar{\phi}(x_2)]|P >$, is related to the BS vertex function by

$$\Phi = G_{BS}\Gamma$$

The vertex function satisfies the BSE

$$\Gamma = -\int V G_{BS} \Gamma$$

PTIR BSE- like 3leg diagram, but the total momenta is fixed $P^2 = M_B^2$, PTIR for BSE reduces in the region of $0 < P^2 < 4m^2$ to the following two-variable spectral representation:

$$\begin{aligned} \Gamma(p, P) &= \int_{\beta_{min}}^{\infty} d\beta \int_{-1}^{1} dz \frac{\tilde{\rho}(\beta, z; P^2)}{\beta - (p + z\frac{P}{2})^2 - i\epsilon} \\ &= \int_{\alpha_{min}}^{\infty} d\alpha \int_{-1}^{1} dz \frac{\rho(\alpha, z; P^2)}{\alpha - (p^2 + zp \cdot P + \frac{P^2}{4}) - i\epsilon}, \end{aligned}$$

where $P = p_1 + p_2$, $p = (p_1 - p_2)/2$.

The Bethe-Salpeter equation for (pathological) WCMs

Classical Lagrangean of (gauged) WCM

$$
\begin{aligned}
L &= (D^\mu \phi_1)^+ D_\mu \phi_1 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 + \frac{1}{2}\partial_\mu \phi_3 \partial^\mu \phi_3 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi_i)\,, \\
V(\phi_i) &= \left(m_1^2 + g\phi_3 \right) \phi_1^+ \phi_1 + \left(\frac{m_2^2}{2} + \frac{g}{2}\,\phi_3 \right) \phi_2^2 + \frac{1}{2} m_3^2 \phi_3^2\,,
\end{aligned}
$$

BSE for (1,2)

$$
\Gamma(p,P) = i \int \frac{d^4k}{(2\pi)^4} V(p,k;P) G_{(1)}(k+P/2) G_{(2)}(-k+P/2) \Gamma(k,P)
$$

$$\Gamma(p,P) = i\int \frac{d^4k}{(2\pi)^4} V(p,k;P) G_{(1)}(k+P/2) G_{(2)}(-k+P/2) \Gamma(k,P)$$

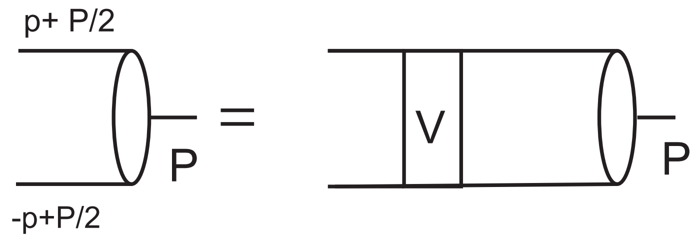

Figure 1: Diagrammatical representation of the BSE for the bound state vertex function.

The both the BSE vertex function and BSE wave functions which satisfy two dim PTIR!, while V, G_i satisfy PTIR for M and for G_i.

ρ is smooth function for massive case (all internal legs carries massive particles), but it can includes delta functions (some of the particles are massless-original WCR model), taking

$$\Gamma(p,P) = \int_{-1}^{1} dz \, \frac{\rho(z)}{m^2 - (p^2 + zp\cdot P + P^2/4) - i\epsilon}$$

one can solve ladder BSE of WCM with massless exchanged scalar boson

Generalized massive WCM by PTIR: Kusaka, Simpson , Williams, PRD 1997, V.Sauli, J.Adam, 2000 PRD (symplifying numerics, removing the integrations in the kernel)

$$\Gamma(p,P) = \int_{-1}^{1} dz \int_{0}^{\infty} d\alpha \, \frac{\rho^{[n]}(z,\alpha)}{[\alpha - (p^2 + zp\cdot P + P^2/4) - i\epsilon]^{[n]}}$$

$$\rho^{[n]}(\alpha', z') = \lambda \int_{-1}^{1} dz \int_{\alpha_{min}(z)}^{\infty} d\alpha \; V^{[n]}(\alpha', z'; \alpha, z)\, \rho^{[n]}(\alpha, z)$$

where we denoted $\lambda = g^2/(4\pi)^2$.

The ladder BSE N=1, $M = m_1 = m_2$ (mistake in def. of D in PRD version)

$$\tilde{V}^{[1]}(\alpha', z'; \alpha, z) \equiv \sum_{i=\pm} \sum_{t=0,T_\pm} \frac{\Theta(x_i(t))\Theta(1 - x_i(t))\Theta(D)}{2J(\alpha', z')|E(x_i(t), S, \alpha')|}$$

$$x_\pm(t) = \frac{-B(t) \pm \sqrt{D(t)}}{2A} \; ; \; D(t) = B(t)^2 - 4Am_3^2 \; ; \; A = \tilde{\alpha} - S \; ; \; B(t) = R(t) - \tilde{\alpha} - m_3^2 \; ;$$

$$E(x) = \tilde{\alpha} - \frac{\mu^2}{x^2} - S \; ; \; R(t) = Jt + M^2 \; ; \; J = \alpha - M^2 \; ; \; S = (1 - \tilde{z}^2)\frac{P^2}{4} \; ; \; T_\pm = \frac{1 \pm \tilde{z}}{1 \pm z} \, .$$

BSE solved in various approximations:

1. ladder BSE 2. ladder BSE with dresses V 3. including corrections to all guys in the ladder BSE

Comparison of spectra for number of parameters $m_{1,2,3}$, calculation of EMG form factors (without numerical boosting) V. Sauli, J. Adam Phys.Rev. D67 (2003)

4. crossed ladder- this conference

Sample results WCM:

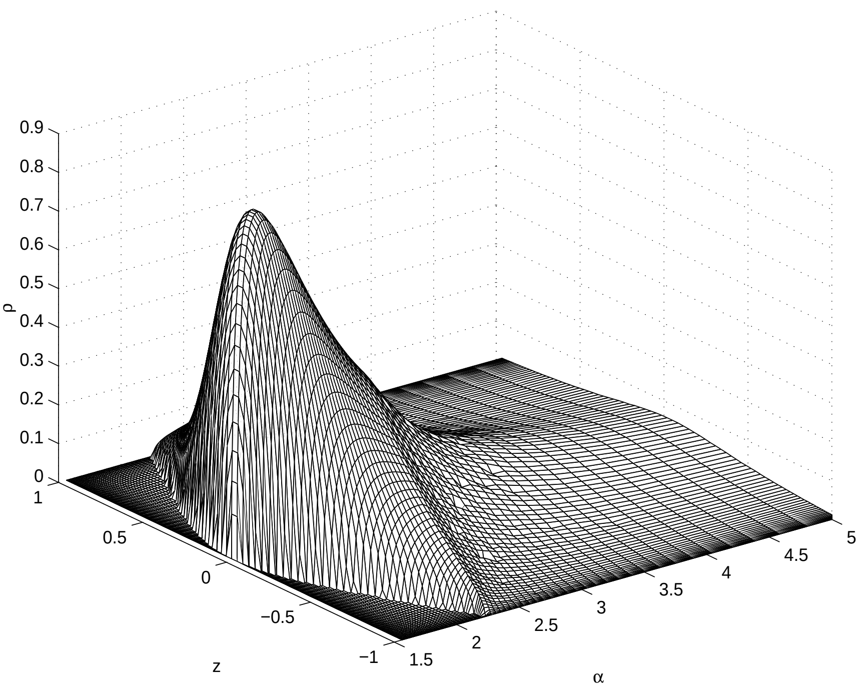

Figure 2: The rescaled weight function $\tilde{\rho}(\alpha, z)$ of the bound-state vertex for $\eta = 0.95$ calculated in bare ladder approximation with $m_3 = 0.5m$.

Sample results for dressed ladder BSE WCM

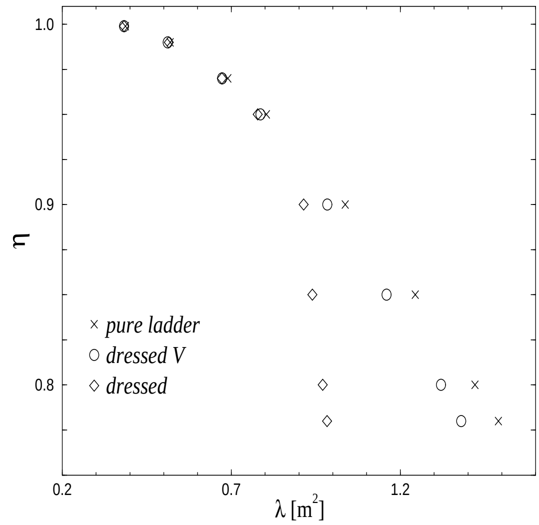

Figure 3: The eigenvalues $\tilde{\lambda}$ calculated for the bare BSE, with dressed kernel V and for dressed ladder BSE as described in the text. Beyond the critical value of coupling $\tilde{\lambda}_{crit} = g^2/(4\pi m)1.5$ only the bare solution is available. $\eta = \sqrt{P^2}/2m$

Since WCM has no true vacuum (potential is unbounded from bellow), it is very likely that $\tilde{\lambda}_{crit}$ is reduced when higher skeleton are considered and perhaps shrink into zero when we would ask for full solution.

WCM conclusion

Main message: considering dressed propagators, only solutions consistent with PTIR can be considered. There is critical coupling, beyond this coupling the results for WCM BSE are meaningless.

Alkofer,Ahlig ,Annals Phys. 275 (1999) 113 -Inconstencies of BSE...

Physically meaningful theories

- Scalar, gauge singlet of any gauge group:

  $V = S^2 m^2/2 + gS^3/3 + \lambda S^4/4$

  Ground state at $S = 0$ for $\lambda > g^2/(4m^2)$

  In BSE S^4 term is contact repelling force between two scalars. What are the relativistic bound states? Do they exist?

- SM and beyond-Can Higgs fields bind itself into the bound states?

  xSM: H Higgs doublet, S-scalar

$$\mathcal{L} = (D_\mu H)^\dagger D^\mu H + \frac{1}{2}\partial_\mu S \partial^\mu S - V(H, S)$$

$$V(H,S) = \lambda(H^\dagger H - \frac{v^2}{2})^2 + \frac{\delta_1}{2} H^\dagger H\ S$$
$$+\frac{\delta_2}{2} H^\dagger H\ S^2 + \delta_1 v^2 S + \frac{\kappa_2}{2} S^2 + \frac{\kappa_3}{3} S^3 + \frac{\kappa_4}{4} S^4 .$$

Higgsonium BSE:

$$\begin{aligned} \Gamma &= \int_k V G^{[2]} \Gamma \\ G^{[2]}(k,P) &= D(k+P/2, M_1^2) D(-k+P/2, M_1^2)\,; \\ D(k, M^2) &= \frac{1}{k^2 - M^2 - \varepsilon} \end{aligned} \tag{2}$$

Trick by G. Rupp, Phys. Lett. B 288, 99 (1992):

$$\Gamma_p(p,P) = \Gamma_I(P)\int_k V_p(k,p,P)G^{[2]}(k,P) + \int_k V_p(k,p,P)G^{[2]}(k,P)\Gamma_p(k,P)$$

where

$$V_I = V_c + V_s\,,\; V_p \quad = \quad V_t + V_u\,,$$

p- relative momenta dependent, I- Inhomogeneous- (only P) dependent term So the full solution of BSE is the sum, $\Gamma_I(P)$ is purely algebraic

$$\Gamma(p,P) = \Gamma_I(P) + \Gamma_p(p,P) \quad , \quad \Gamma_I(P) = \frac{V_I \int_k \Gamma_p(k,P)G^{[2]}(k,P)}{1 - V_I \int_k G^{[2]}(k,P)}$$

Integral Representation:

$$\Gamma(P,p) = \Gamma_I(P) + \int\limits_{-1}^{1} d\eta \int\limits_{-\infty}^{\infty} d\alpha \, \frac{\rho_p(\alpha,\eta)}{[F(\alpha,\eta;P,p)]} \, ,$$

where ρ_p is assumed to be continuous smooth function. On can derived that BSE is equivalent to the

$$\rho_p(\alpha,\eta) = \frac{1}{\alpha - M_1^2} \left[\Gamma_I(P)\rho_I(\alpha,\eta) + \int\limits_{-1}^{1} dz \int\limits_{-\infty}^{\infty} da \, \rho_p(a,z)\mathcal{V}(\alpha,\eta,a,z) \right]$$

where $\rho_I, \mathcal{V}$ are known. The BSE is reduced to two integral equations.

Solutions: V. S.,arXiv:0808.1894, AIPConf.Proc.1030:274-279,2008 arXiv:0806.3454 Dummy parameter $n = 1$ was used to reduce human effort, however it decrease numerical convergence.

1) SM model- to get bound states Higgs must be heavier then 1TeV (according to G.R.)

2) xSM - For 200 MeV Higgs mass eigenstates few bound states have been found not so far from the threshold $\eta = 0.8, 0.95$ for reasonable large values of trilinear coupling of xSM.

3) Study is continuing

Fermion - antifermion pseudoscalar bound states

OGE ladder BSE- V.S. J.Phys.G35:2008 The general structure of denominators in perturbative Feynman diagram does not depend on the spin! One can apply PTIR.

Spin–> more degrees of freedom –> more independent scalars. BSE for pseudoscalar-

$$\Gamma(p,P) = -i\int \frac{d^4k}{(2\pi)^4} V(p,k,P) S_1(k_+,P)\Gamma(p,P)S_2(k_-,P)$$

$$S_i^0(p) = \frac{\not p + m_i}{p^2 - m_i^2 + \varepsilon}$$

Γ_5 approximation with constituent "quarks",

$$\Gamma(q,P) = \gamma_5[\Gamma_A + \Gamma_B q.P \not{q} + \Gamma_C \not{P} + \Gamma_D \not{q} \not{P} + \Gamma_E \not{P} \not{q}]$$

Pinch technique massive gluon propagator Cornwall, Binosi, Papavasilliou,..

$$V(p,q,P) = g^2(\kappa) D_{\mu\nu}(p-q,\kappa)\gamma^\nu \otimes \gamma^\mu$$

$$\Gamma_A(q,P) = \int_0^\infty d\omega \int_{-1}^1 dz \frac{\rho_A^{[N]}(\omega,z)}{[F(\omega,z;P,q)]^N}$$

in general IR has Dirac, tensor structure...

$$\rho(\alpha,x_i)_{\text{scalar theory}} \longrightarrow \sum_j \rho_j(\alpha,x_i)\mathcal{P}_j$$

where α, x_i represent the set of spectral variables, and j runs over all possible independent combinations of Lorentz tensors and Dirac matrices P_j Sample results:

η :	0.8	0.9	0.95	0.99
α	1.20	1.12	1.03	0.816

Coupling $\alpha_s = g^2/(4\pi)$ as a function of binding fraction $\eta = \sqrt{P^2}/(2M)$, for exchanged massive gluon with $m_g = 0.5 M_q$.

The rescaled weight function for almost massless gluon

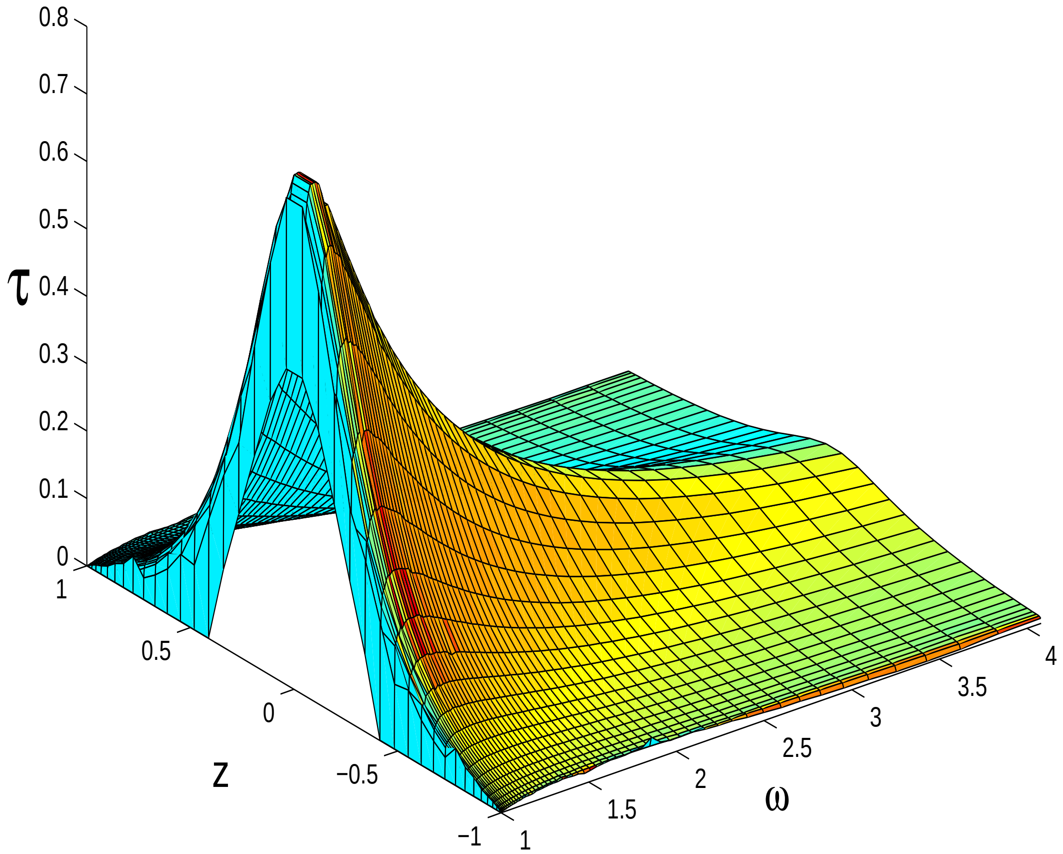

Figure 4: The rescaled weight function $\tau = \frac{\rho^{[2]}(\omega,z)}{\omega^2}$ for the following model parameters: $\eta = 0.95$, $m_g = 0.001M$, $\alpha_s = 0.666$

Toy model with regge behaviour

V.Sauli and P. Bicudo, preliminary results

QCD Mesons- The experimental data finds linear Regge trajectories both for angular and radial excitations,

$$
\begin{aligned}
j &\simeq \alpha_0 + \alpha M^2 \ , \\
n &\simeq \beta_0 + \beta M^2 \ .
\end{aligned}
$$

Lorentz covariant study and solve the BSE for ground state with the "confining" kernel of the form

$$V(q) = \frac{const.}{(q^2 - \mu^2)^2},$$

$$\Gamma(p,P) = i\int \frac{d^4k}{(2\pi)^4} V(p-k) G(k_+,m) G(k_-,m) \Gamma(k,P)\,,$$

where $k_\pm = k \pm P/2$ and $G(k,m)$ is the propagator of scalar field with the mass m and where the Bethe-Salpeter vertex function $\Gamma(p,P)$ for bound state of orbital momentum l is the product of hyperspherical harmonics and the scalar function $\Gamma_S(p,P)$.

$$\Gamma^{[\ell,\ell_z]}(p,P) = -i\mathcal{Y}_\ell^{\ell_z}(\vec{p'})\Gamma_S(p,P)$$

kernel spectral function is functional: $\frac{d}{da}\delta(x-a)$ Is PTIR aplicable?

$$\Gamma_S^{[\ell,\ell_z]}(p,P) = \int_{\alpha_{\rm th}}^{\infty} d\alpha \int_{-1}^{1} dz \frac{\rho_n^{[\ell]}(\alpha,z)}{[\alpha - (p^2 + zp\cdot P + P^2/4) - i\epsilon]^n}\,,$$

Then homogeneous BSE turns to be for particular xhoice $n = 2$

$$\rho^{[2]}(\alpha', z') = \frac{\lambda}{2} \int_{-1}^{1} dz \int_{-\infty}^{\infty} d\alpha\, K(\alpha', z'; \alpha, z) \rho^{[2]}(\alpha, z) + \frac{\lambda}{2} \int_{-1}^{1} dz V(\alpha', z'; \alpha_D, z) \rho^{[2]}(\alpha_D, z)$$

Results $\lambda = \frac{C}{(4\pi)^2} = 0.003 m^4$, $mu/m = 0.2$; $m = 1$ is scale

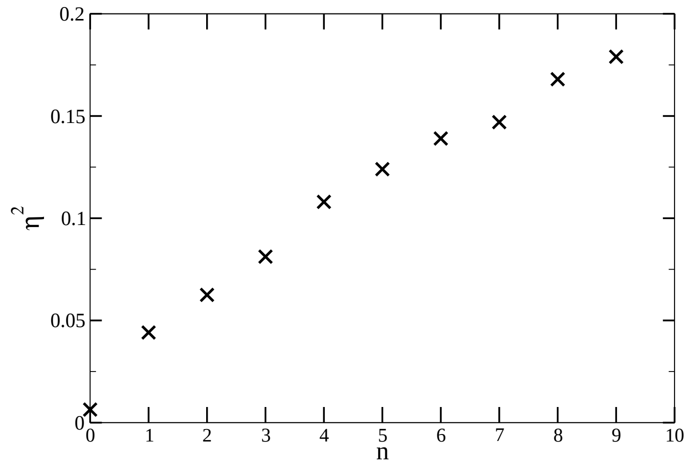

Figure 5: R
egge trajectory of radial excitation as they are seen in our relativistic toy model. $\eta = P^2/4m^2$.

Model is not QCD, end gives very large $\beta = 50GeV^{-1}$ unless $m \simeq 500MeV$, for $\mathbf{7} - 8GeV$ quarks can get correct β.

1. Minkowski PTIR method is applicable (with more or less stable numeric) Limitations -constituent masses, the total mass should be less then sum of the masses

2. We get Regge trajectory (note in our model we get $\beta = 50/4m^2$=1/4 number of equidistantly spread bound states bellow threshold), paper will appear.