Relativistic Description of
Two- and Three-Body Systems
in Nuclear Physics,
Oct. 19-23, ECT*, Trento

Application of Nonperturbative Renormalization Scheme in Light-Front Dynamics to a Scalar Model

A.V. Smirnov

Lebedev Physical Institute, Moscow

in collaboration with

V.A. Karmanov, *Lebedev Physical Institute, Moscow*

J.-F. Mathiot, *Laboratoire de Physique Corpusculaire, Univ. Blaise Pascal, Clermont-Ferrand*

Dynamical equation for the state vector

We will solve the following eigenstate equation:

$$\boxed{\hat{P}^2 \,|\, p\rangle = M^2 \,|\, p\rangle}$$

where $|\, p\rangle$ is the state vector of a physical system considered, defined on the invariant light-front surface

$$\omega \cdot x = 0, \quad \text{with} \quad \omega^2 = 0$$

and $\hat{P}^\rho$ is the momentum operator:

$$\hat{P}^\rho = \hat{P}^\rho_{free} + \omega^\rho \int d^4x \; \delta(\omega \cdot x) \; H_{\text{int}}(x)$$

$H_{\text{int}}(x)$ is the interaction Hamiltonian. The interaction coupling constant is not assumed to be small.

The standard choice of the LF plane $t + z = 0$ corresponds to $\omega = (1,0,0,-1)$

Physical model and interaction Hamiltonian

We consider a scalar system of interacting scalar bosons described by the fields

$B(x)$ – charged heavy boson with the physical mass m

$b(x)$ – neutral light boson with the physical mass $\mu < m$

Initial (bare) interaction Hamiltonian is

$$H_{\text{int}}(x) = -g_0 B^+(x)B(x)b(x) - \delta m^2 B^+(x)B(x) - \delta\mu^2 b^2(x)$$

where g_0 is the bare coupling constant. The physical coupling constant is g. The dimensionless constant which determines the interaction strength is

$$\alpha \equiv \frac{g^2}{16\pi m^2}$$

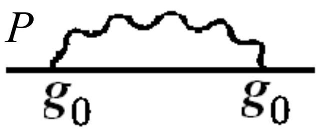
Bare fields satisfy free Klein-Gordon equations with the physical masses:

$$(\partial^\nu \partial_\nu + m^2)B(x) = 0, \qquad (\partial^\nu \partial_\nu + \mu^2)b(x) = 0$$

Divergences and regularization

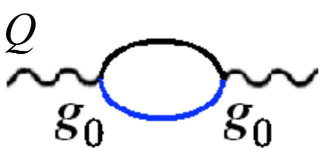
We will encounter two types of diagrams with divergent amplitudes:

Primary boson self-energy (mass operator):



$$= g_0^2 \Sigma(P^2)$$

Secondary boson self-energy (polarization operator):



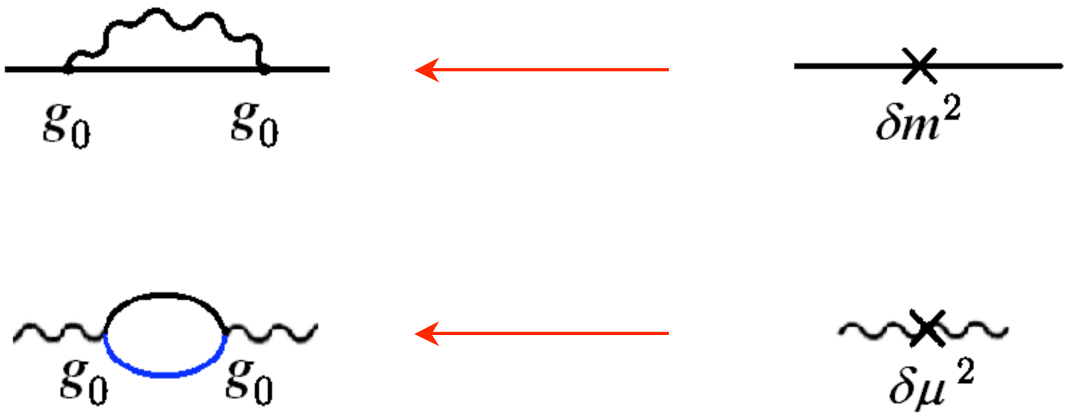
$$= g_0^2 \Pi(Q^2)$$

Both amplitudes diverge logarithmically at high internal momenta

To regularize divergences we use the Pauli-Villars (PV) regularization procedure

Counterterms

To remove dependence of physical results on the PV masses, we need to introduce counterterms:



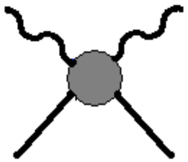
The mass counterterms δm^2 and $\delta \mu^2$ are found as eigenvalues from the dynamical equations for the state vectors of bosons B and b.

From physical point of view each mass counterterm compensates the shift of mass caused by the interaction, so that the “dressed” boson mass coincides with that of the bare boson.

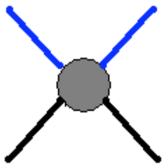
The bare coupling constants are found from additionally imposed renormalization conditions

Renormalization conditions

We formulate the renormalization conditions as follows: the elastic two-body scattering amplitudes $B+b \to B+b$ and $B+\bar{B} \to B+\bar{B}$ near their poles at $s \to m^2$ are, respectively,



$$= \quad T_{Bb\to Bb}(s \to m^2) = \frac{g^2}{s-m^2}$$



$$= \quad T_{B\bar{B}\to B\bar{B}}(s \to \mu^2) = \frac{g^2}{s-\mu^2}$$

Any of these relations can serve as a definition of the physical coupling constant g .

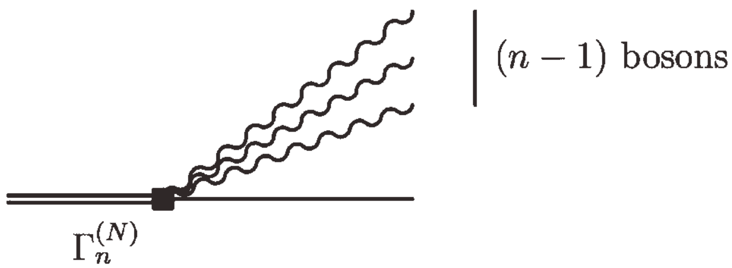
Fock sector decomposition of the state vector

$$|\,p\rangle = \psi_1\,|1\rangle + \psi_2\,|2\rangle + \psi_3\,|3\rangle + \ldots + \psi_n\,|n\rangle + \ldots$$

$\psi_n = \psi_n(p_1, p_2, \ldots p_n, \omega\tau \,|\, p)$ are light-front wave functions. It is convenient to introduce vertex functions Γ_n:

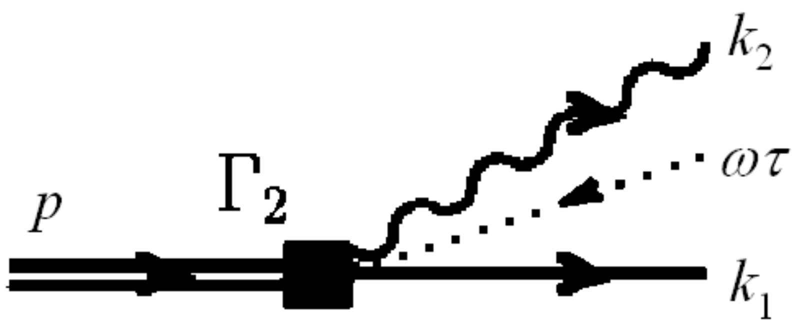
$$\psi_n = \frac{\Gamma_n}{s_n - M^2}$$

where $s_n = (p_1 + p_2 + \ldots + p_n)^2 = (p + \omega\tau)^2$. In diagram technique the vertex function is represented as



N is the order of approximation: we retain Fock sectors with $n \le N$ and neglect those with $n > N$

Parameterization of the two-body vertex function



Dotted line with the four-momentum $\omega\tau$ denotes the departure of the vertex from the energy shell:

$$k_1 + k_2 = p + \omega\tau$$

where $k_1^2 = m^2,\ k_2^2 = \mu^2,\ p^2 = M^2,\ (\omega\tau)^2 = 0.$

The two-body vertex, just like a scattering amplitude $0+0 \to 0+0$, depends on TWO invariant variables.
We will choose the latter as follows:

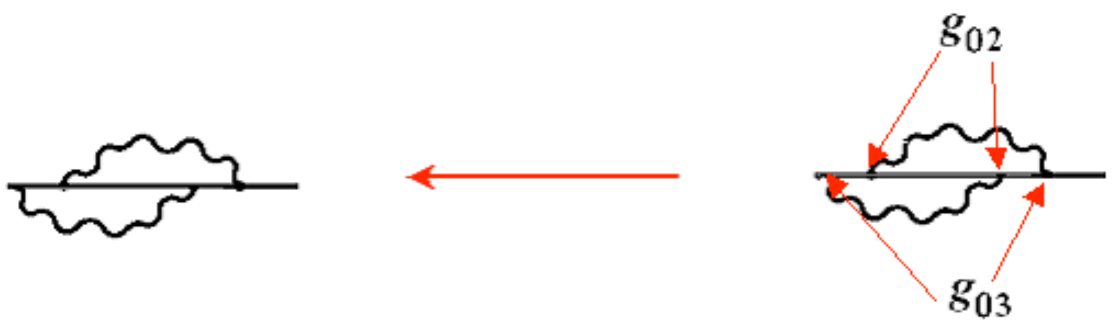
$$s = (k_1 + k_2)^2, \qquad x = \frac{\omega \cdot k_2}{\omega \cdot p}$$

Fock sector dependent renormalization scheme

The counterterm δm^2 being a correction to the self-energy in the TWO-BODY Fock sector formally acts in the ONE-BODY sector. In order to make it of the same order, we supply δm^2 by subscripts indicating the Fock sector it belongs to, e.g.

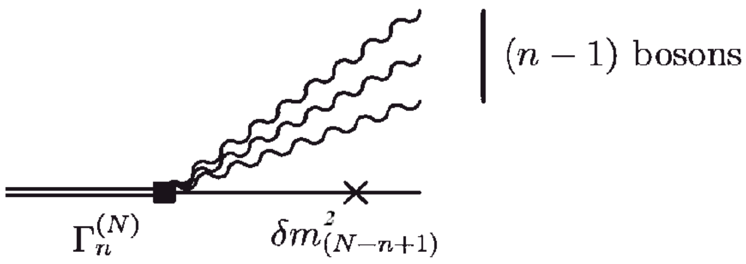
$\delta m^2 \to \delta m_2^2$ (now δm^2 has the same order of magnitude in the number of particles as the self-energy)

Bare coupling constants are treated simultaneously, e.g. in the three-body approximation

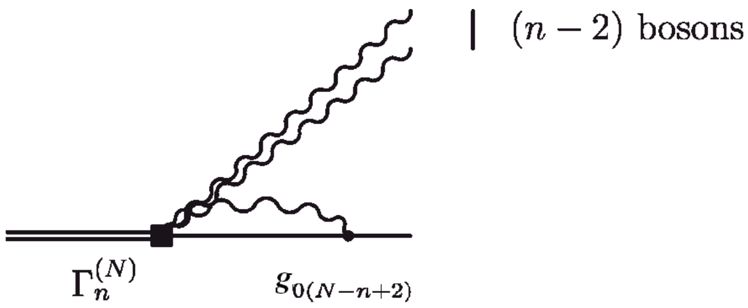


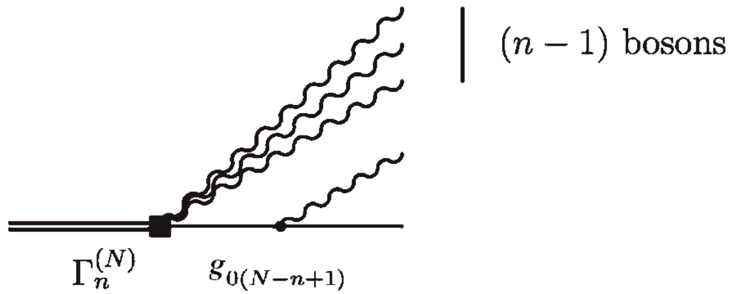
Fock sector dependent renormalization scheme

Suppose we truncate Fock space up to sectors containing not more than N particles (N-body approximation)



General rule to assign the subscript to δm_k^2: k + (number of particles in flight) = N



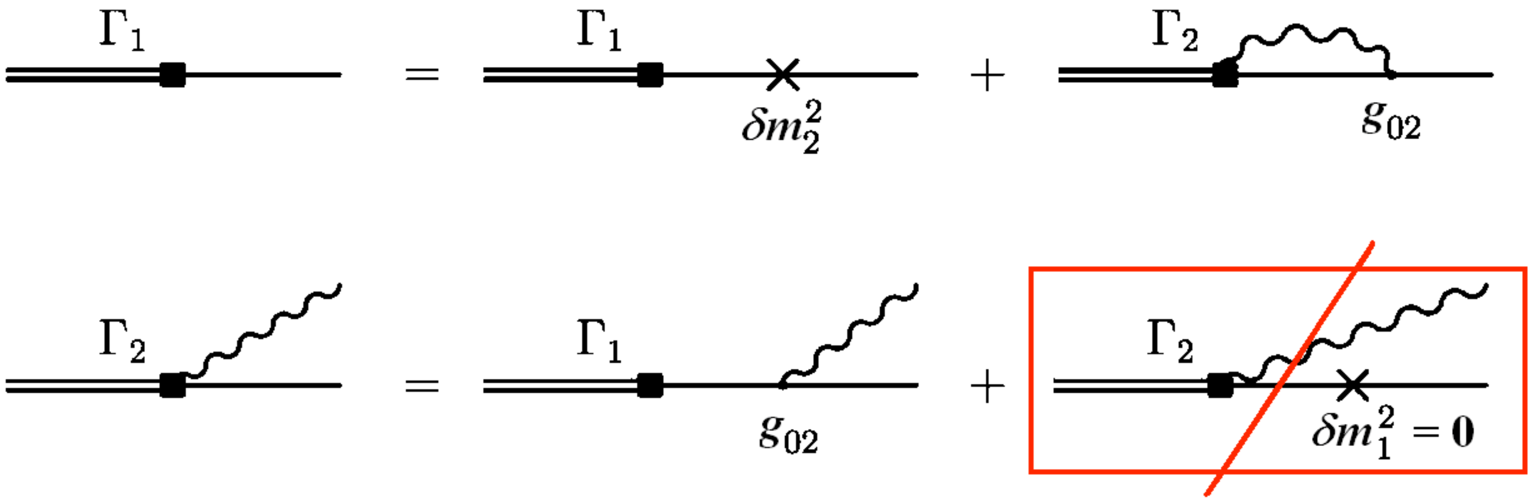


General rule to assign the subscript to g_{0k}: k + (number of particles in flight) = N

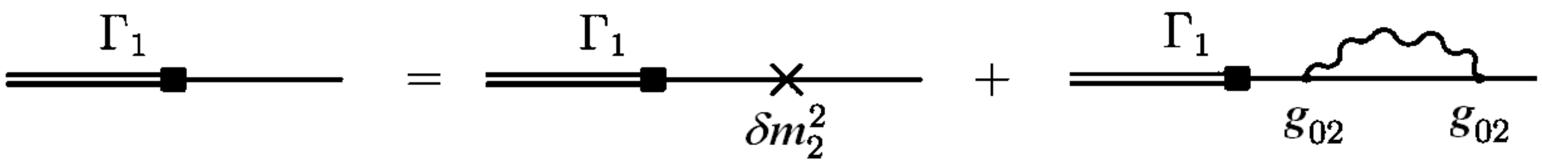
We always have $k \le N$. Taking $N = 1$, we get

$\delta m_1^2 = 0$ (no mass renormalization)
$g_{01} = g$ (no charge renormalization)

System of eigenstate equations for the vertex functions in the TWO-BODY approximation



Substitution of Γ_2 from the second equation into the first one gives



In analytical form

$$\Gamma_1 = \frac{\Gamma_1}{m^2 - M^2}\delta m_2^2 + \frac{\Gamma_1}{m^2 - M^2} g_{02}^2 \Sigma(M^2)$$

This equation should be solved in the limit $M^2 \equiv p^2 \to m^2$

System of eigenstate equations for the vertex functions in the TWO-BODY approximation

It is convenient to introduce the one-body wave function $\psi_1 = \dfrac{\Gamma_1}{m^2 - M^2}$. The equation becomes

$$\psi_1 \left[\delta m_2^2 + g_{02}^2 \Sigma(M^2) + (M^2 - m^2) \right] = 0$$

From here follows, in the limit $M^2 \to m^2$:

$$\delta m_2^2 = -g_{02}^2 \Sigma(m^2)$$

Solution for Γ_2 has the form

$$\Gamma_2 = g_{02} \psi_1$$

ψ_1 is NOT determined from this equation and can be found from an additionally imposed NORMALIZATION CONDITION

$$\langle 1 | 1 \rangle + \langle 2 | 2 \rangle \equiv N_1^{(2)} + N_2^{(2)} = 1$$

The norms are

$$N_1^{(2)} = \psi_1^2, \qquad N_2^{(2)} = \frac{1}{(2\pi)^3} \int \frac{d^2 k_t dx}{2x(1-x)} \frac{\Gamma_2^2}{\left(\dfrac{k_t^2 + \mu^2}{x} + \dfrac{k_t^2 + m^2}{1-x} - m^2 \right)^2}$$

System of eigenstate equations for the vertex functions in the TWO-BODY approximation

The quantity g_{02} is found from the RENORMALIZATIION CONDITION

$$T_{Bb\to Bb}(s=m^2)=\frac{g^2}{s-m^2}=\frac{\Gamma_2^2(s=m^2,x)}{s-m^2}\;\Rightarrow\;\Gamma_2(s=m^2,x)=g$$

Thus we find the final renormalized solution:

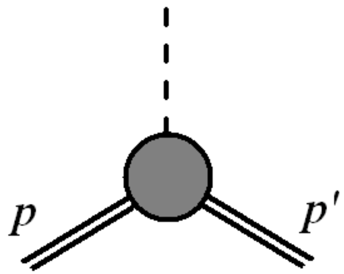
$$\psi_1=\sqrt{1-N_2^{(2)}},\qquad \Gamma_2=g$$

$$N_2^{(2)}=\frac{g^2}{(2\pi)^3}\int\frac{d^2k_t dx}{2x(1-x)}\,\frac{1}{\left(\frac{k_t^2+\mu^2}{x}+\frac{k_t^2+m^2}{1-x}-m^2\right)^2}$$

The counterterms are

$$g_{02}^2=\frac{g^2}{\sqrt{1-N_2^{(2)}}},\qquad \delta m_2^2=-g_{02}^2\Sigma(m^2),\qquad \delta\mu_2^2=-g_{02}^2\Pi(\mu^2)$$

Electromagnetic vertex of a scalar particle in CLFD



$$G^{\rho} = \langle p'|J^{\rho}|p\rangle = (p+p')^{\rho}F(Q^2) + \frac{m^2\omega^{\rho}}{\omega\cdot p}B(Q^2)$$

where $Q^2 = (p'-p)^2 \equiv -q^2$ and an additional condition is imposed on the four-vector ω:

$$\omega\cdot q = \omega\cdot(p'-p) = 0$$

that is equivalent to $q^+ = 0$ in ordinary LFD on the plane $x^+ = 0$. The electromagnetic vertex is determined by TWO form factors: $F(Q^2)$ – physical form factor and $B(Q^2)$ – nonphysical form factor.

In EXACT calculations or in of PERTURBATION THEORY (if ALL contributions of a given order are taken into account)

$$B(Q^2) \equiv 0$$

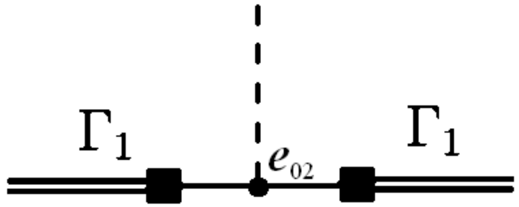
In APPROXIMATE calculations (i.e. if Fock space is truncated), generally speaking, $B(Q^2) \neq 0$.

Since $\omega^2 = 0$, the physical form factor can be extracted as

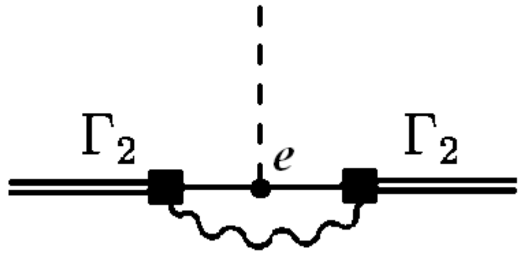
$$F(Q^2) = \frac{\omega_{\rho}G^{\rho}}{2(\omega\cdot p)}$$ which is equivalent to $$F(Q^2) = \frac{\langle p'|J^+|p\rangle}{2p^+}$$ in ordinary LFD

Electromagnetic vertex of a scalar particle in CLFD in the TWO-BODY approximation

Contribution from the ONE-BODY sector



Contribution from the TWO-BODY sector



$$eF(Q^2) = e_{02}\psi_1^2 + eF_2^{(2)}(Q^2)$$

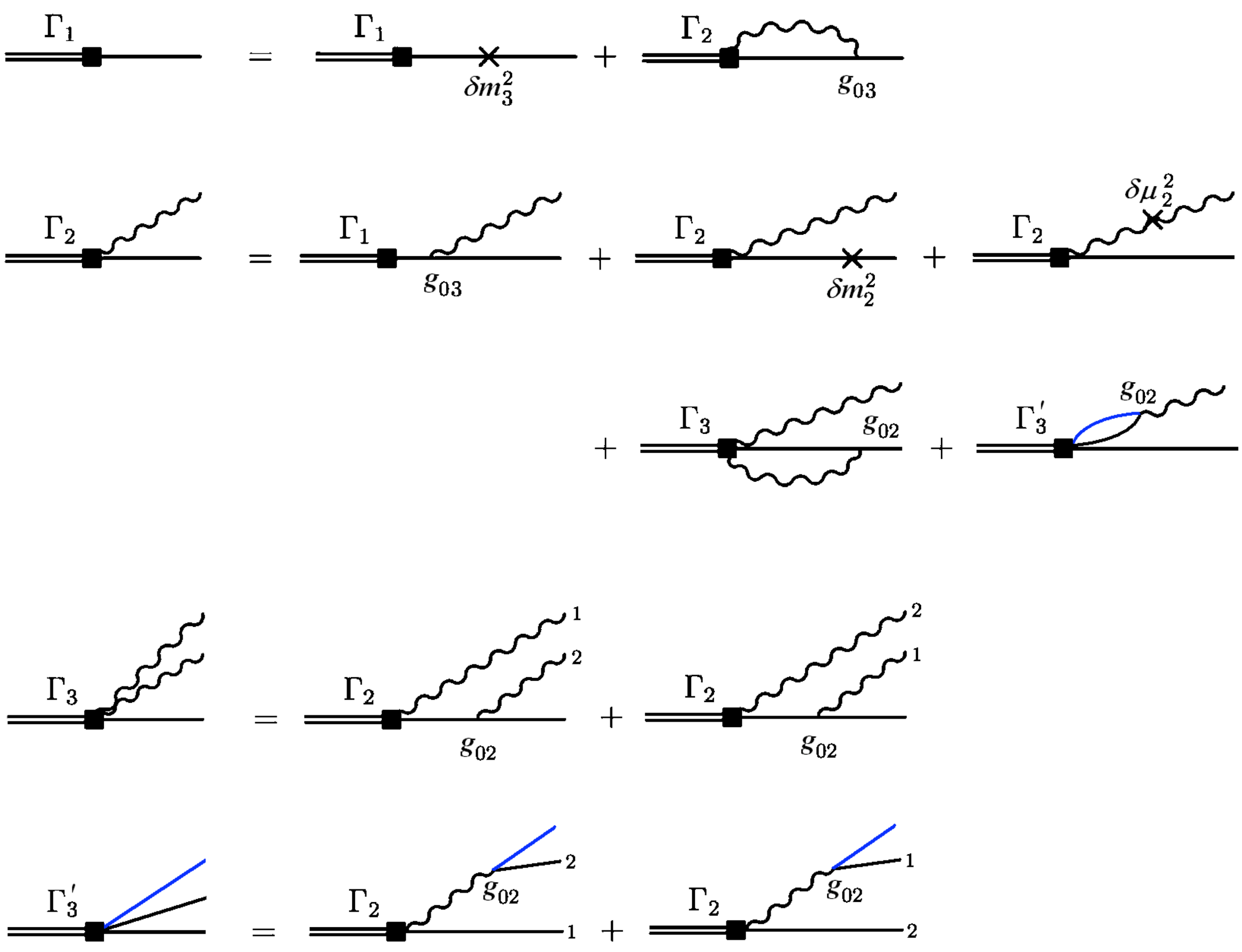
The bare coupling constant e_{02} is found from the condition $F(0)=1$. We have

$$\boxed{e = e_{02}N_1^{(2)} + eN_2^{(2)} \quad \Rightarrow \quad e_{02} = e}$$ The "external" electromagnetic bare coupling constant is not renormalized at all!

Form factor: $$\boxed{F(Q^2) = 1 + F_2^{(2)}(Q^2) - F_2^{(2)}(0)}$$

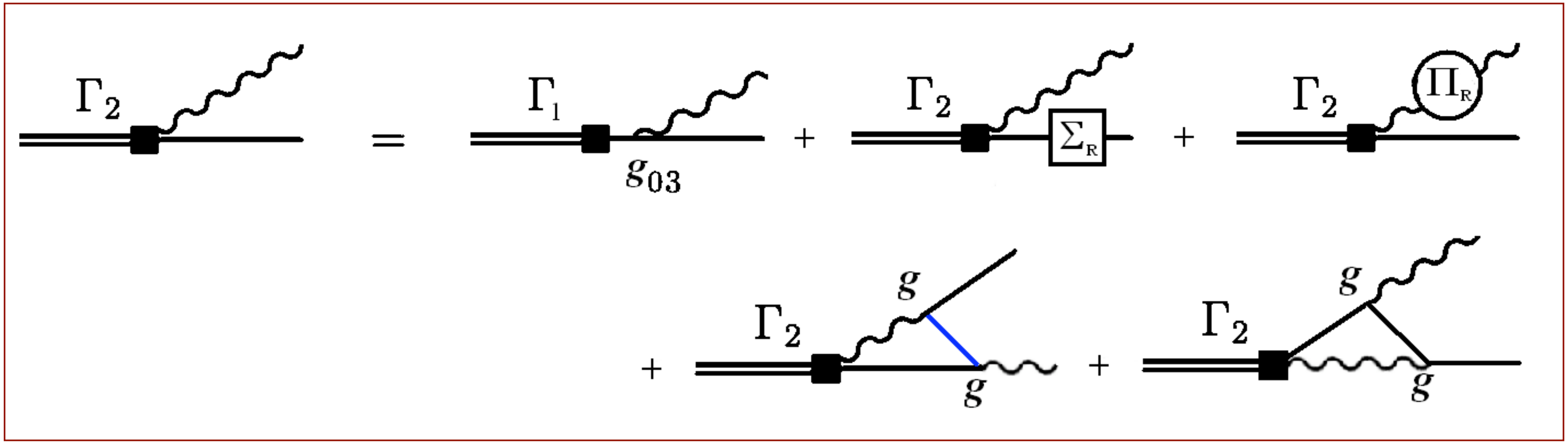
The same result is obtained in the second order of perturbation theory, though we have not done any decompositions in powers of the coupling constant.

System of eigenstate equations for the vertex functions in the THREE-BODY approximation



System of eigenstate equations for the vertex functions in the THREE-BODY approximation

Closed equation for Γ_2:



$$= \quad g^2\left[\Sigma(P^2) - \Sigma(m^2) - \Sigma(m^2)(P^2 - m^2)\right]$$

$$= \quad g^2\left[\Pi(Q^2) - \Pi(\mu^2) - \Pi'(\mu^2)(Q^2 - \mu^2)\right]$$

These equations do NOT contain any divergent contributions and allow direct going over to the limit of INFINITE PV masses

System of eigenstate equations for the vertex functions in the THREE-BODY approximation

Analytical form of the equation:

$$\Gamma_2(s,x) = g_{03}\psi_1 + \left[\frac{\Sigma_R(P^2)}{m^2 - P^2} + \frac{\Pi_R(Q^2)}{\mu^2 - Q^2}\right]\Gamma_2(s,x) + \int d\Omega'\, V(s,x,s',x')\Gamma_2(s',x')$$

Apply the renormalization condition to Γ_2, i.e. set $s = m^2$ and demand

$$\Gamma_2(m^2,x) = g$$

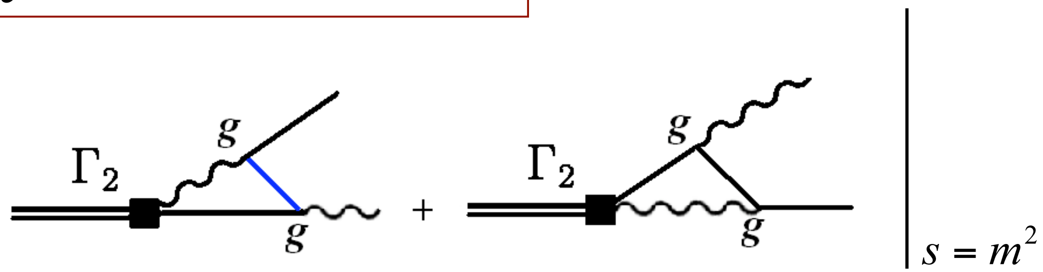
So, exact $\Gamma_2(m^2,x)$ must be independent of x! In approximate solution x-dependence may survive.

If $s = m^2$, we have $P^2 = m^2$ and $Q^2 = \mu^2$. Since $\Sigma_R(P^2) \propto (P^2 - m^2)^2$ and $\Pi_R(Q^2) \propto (Q^2 - \mu^2)^2$,

$$g = g_{03}\psi_1 + \int d\Omega'\, V(m^2,x,s',x')\Gamma_2(s',x')$$

and the function

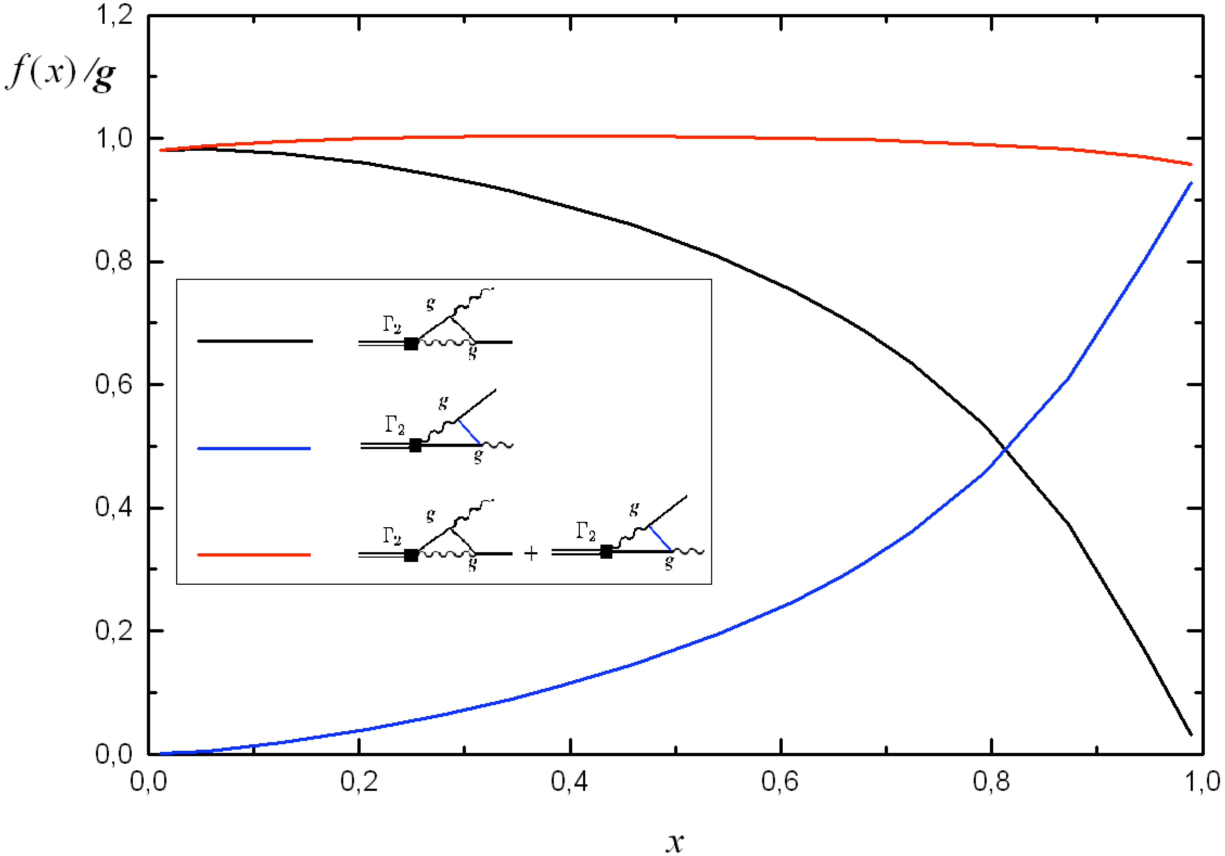
$$f(x) \equiv \int d\Omega'\, V(m^2,x,s',x')\Gamma_2(s',x') =$$



must be x-independent!

System of eigenstate equations for the vertex functions in the THREE-BODY approximation

Numerical calculation: $m = 0.95,\ \mu = 0.15,\ \alpha = g^2/16\pi m^2 = 2$

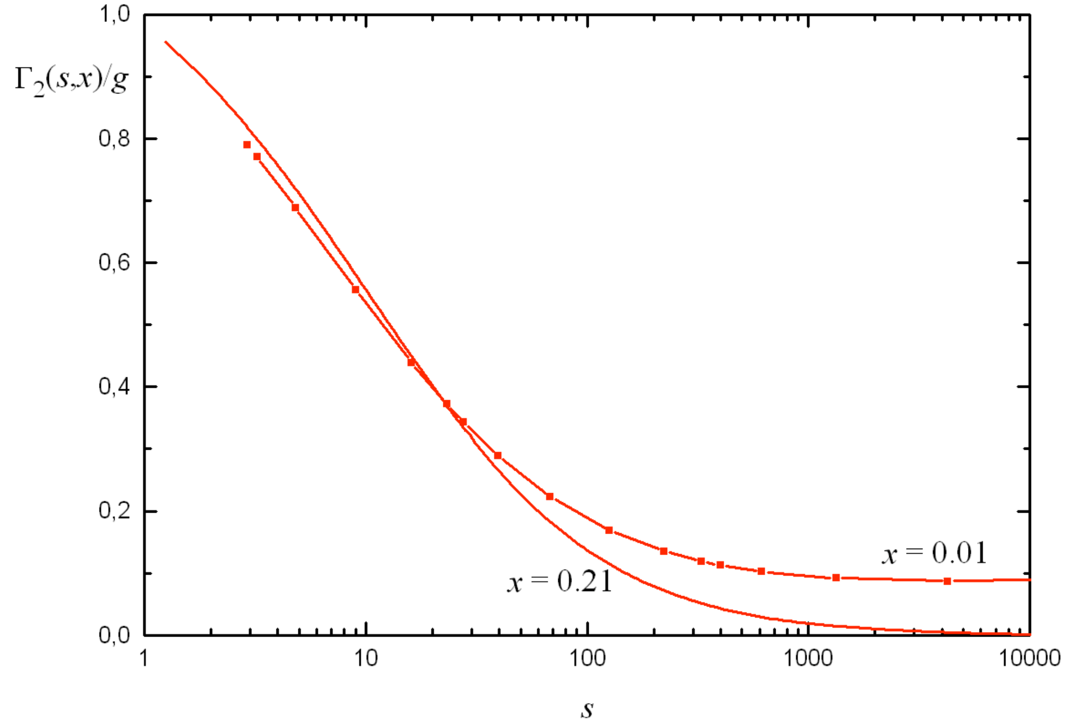


Since $f(x)/g$ is very close to 1, $g_{03}^2 = \dfrac{[g - f(x)]^2}{1 - N_2^{(3)} - N_3^{(3)}} << g^2$

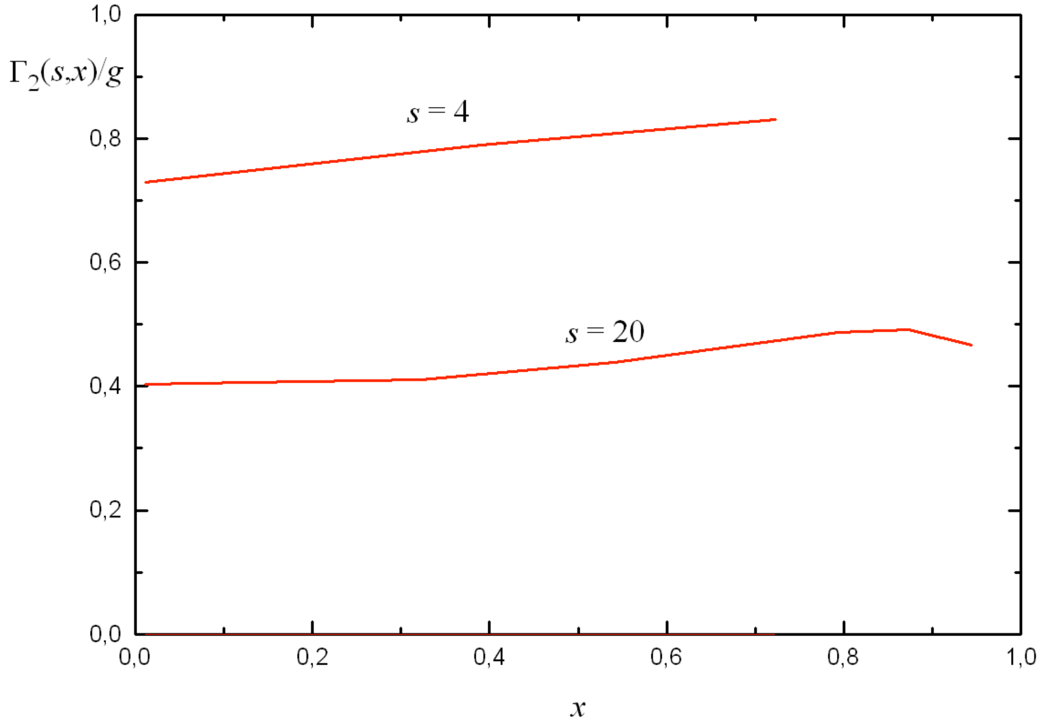
This possibly indicates rather fast convergence of the Fock sector decomposition of the state vector!

System of eigenstate equations for the vertex functions in the THREE-BODY approximation

Dependence of Γ_2 on s at fixed x

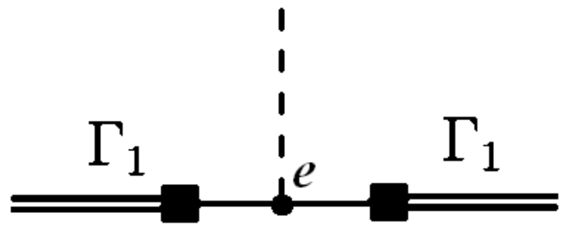


Dependence of Γ_2 on x at fixed s

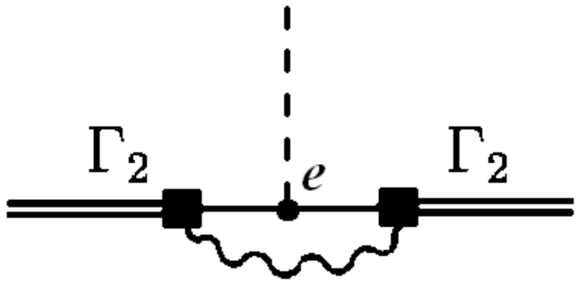


Electromagnetic vertex of a scalar particle in CLFD in the THREE-BODY approximation

Contribution from the ONE-BODY sector

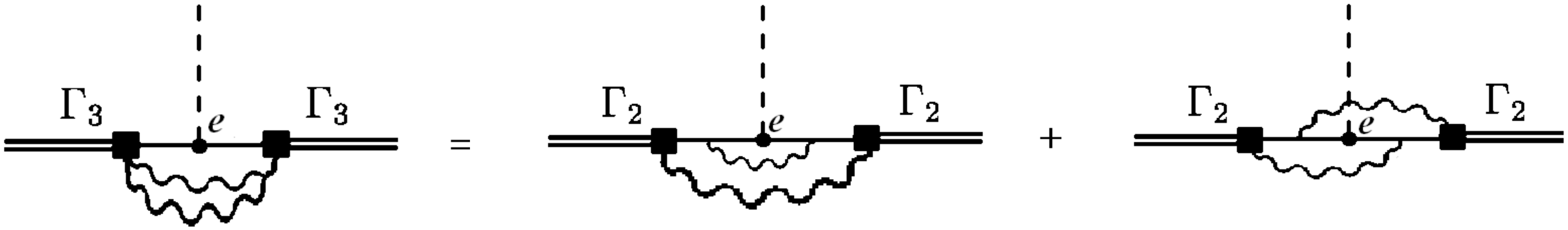


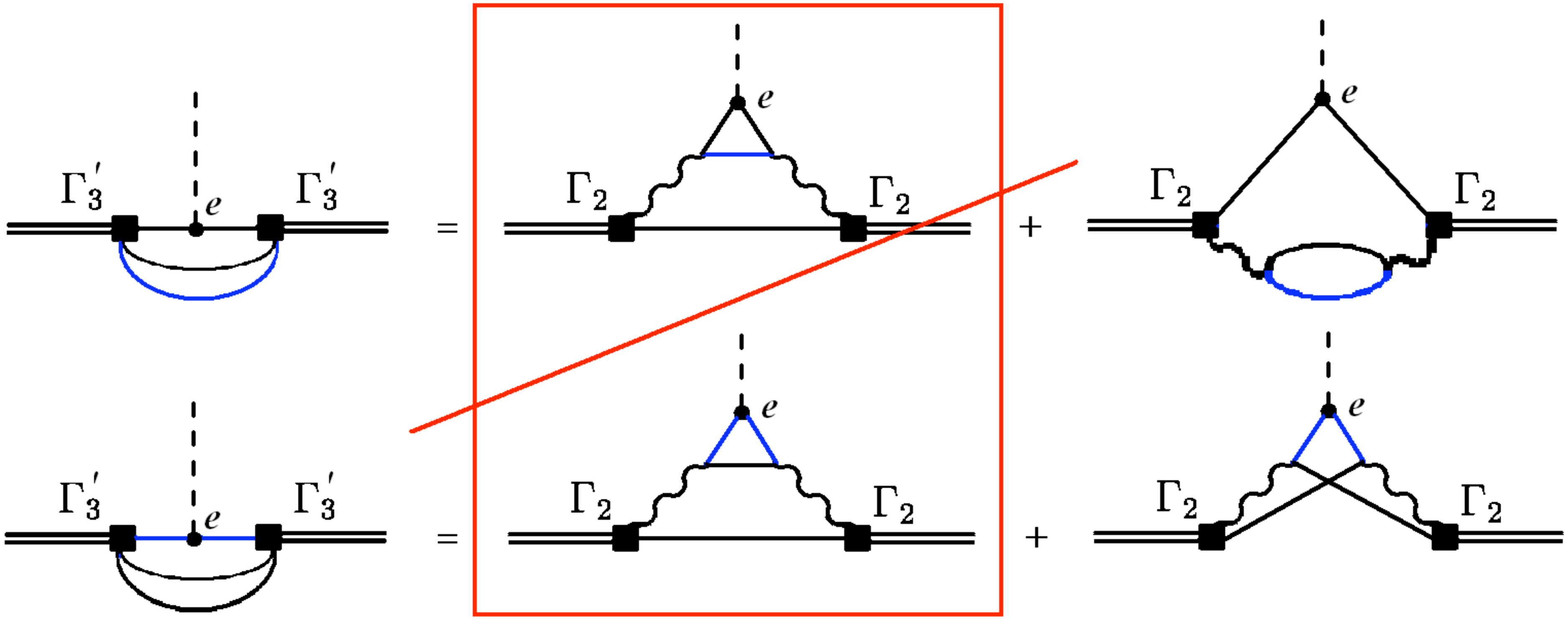
Contribution from the TWO-BODY sector



Electromagnetic vertex of a scalar particle in CLFD in the THREE-BODY approximation

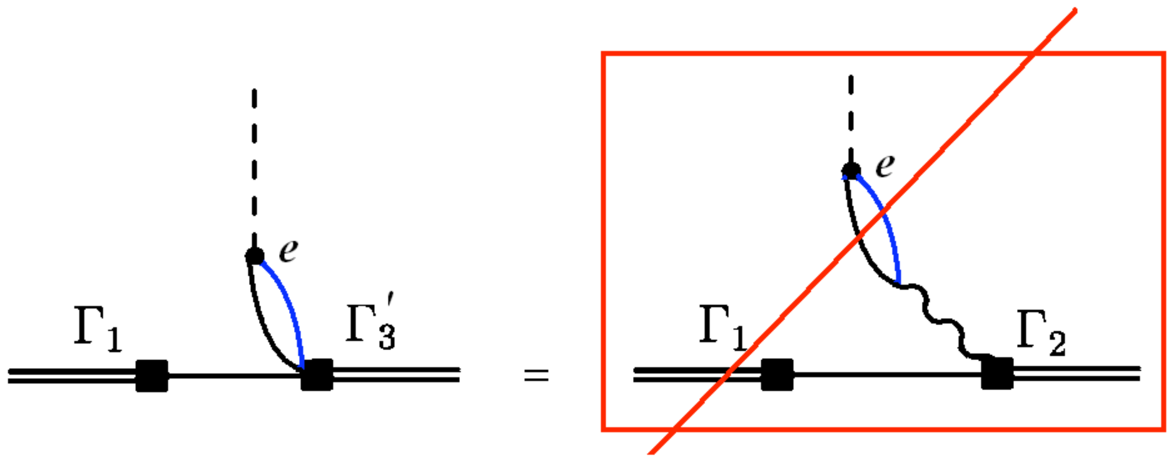
Contributions from the THREE-BODY sector(s)





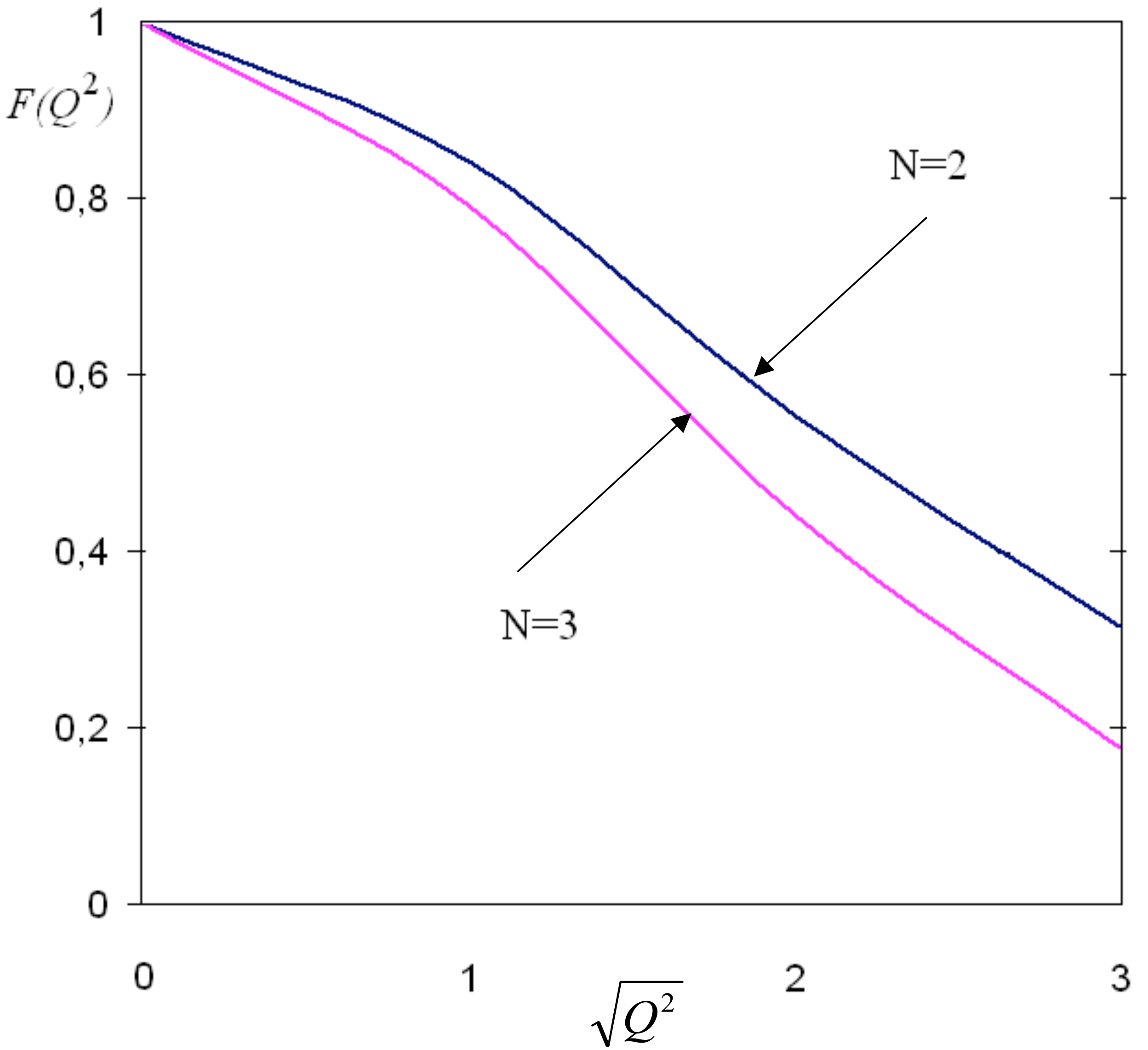
Electromagnetic vertex of a scalar particle in CLFD in the THREE-BODY approximation

Contributions from the THREE-BODY sector(s)



This diagram contributes to the nonphysical form factor $B(Q^2)$ only!

Electromagnetic form factor. Results of calculations



Conclusion

1. Within covariant formulation of light-front dynamics we formulated a self-consistent procedure of nonperturbative calculation of a compound particle state vector based on its Fock sector decomposition with the subsequent Fock space truncation.

2. Fock sector dependent renormalization scheme allows to keep under control cancellation of ultra-violet field-theoretical divergences and ensures step-by-step determination of all renormalization counterterms.

3. The method developed has been successfully applied to a toy model of a scalar system made up from massive scalar bosons, considered in the two- and three-body approximations. Direct numerical computations showed that the inclusion of three-body sectors in the heavy boson state vector almost exhausts the state vector contents, i. e. in this case the Fock sector decomposition converges rapidly.

4. The closest perspectives are connected with the use of the approach for fermionic systems. In case of success, it may become an alternative to lattice calculations.