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INSTITUTE OF NUCLEAR & PARTICLE PHYSICS

A New Approach to the 3D Faddeev Equation for Three-Body Scattering

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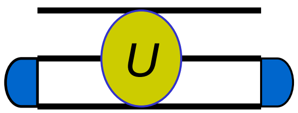
Office of Nuclear Physics

Exploring Nuclear Matter - Quarks to Stars

Three-Body Scattering

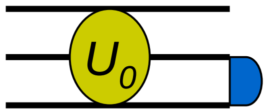
- Transition operator for elastic scattering

$$U = PG_0^{-1} + PT$$



- Transition operator for breakup scattering

$$U_0 = (1 + P)T$$



- Faddeev equation

Free 3N propagator
“nasty” singularities

$$T = tP + tG_0PT$$

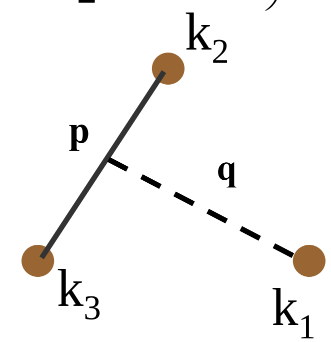
$t = v + vg_0t$ =: NN t-matrix $\quad$ $P = P_{12}P_{23} + P_{13}P_{23} \equiv$ Permutation Operator

3-Body Transition Amplitude

$$T|q_0\varphi_d\rangle = tP|q_0\varphi_d\rangle + tG_0PT|q_0\varphi_d\rangle$$

$$\mathbf{p} = \frac{1}{2}(\mathbf{k}_2 - \mathbf{k}_3)$$

$$\mathbf{q} = \frac{2}{3}\left(\mathbf{k}_1 - \frac{1}{2}(\mathbf{k}_2 + \mathbf{k}_3)\right)$$



The Faddeev Equation in momentum space by using Jacobi Variables

$$G_0^{-1} = (E + i\varepsilon - E'')$$

$$E'' = \frac{1}{m}(p''^2 + \frac{3}{4}q''^2) \ .$$

$$\langle \mathbf{p}'\mathbf{q}|P|\mathbf{p}''\mathbf{q}''\rangle = \delta(\mathbf{p}' + \boldsymbol{\pi}_1)\ \delta(\mathbf{p}'' - \boldsymbol{\pi}_2) + \delta(\mathbf{p}' - \boldsymbol{\pi}_1)\ \delta(\mathbf{p}'' + \boldsymbol{\pi}_2),$$

$$\boldsymbol{\pi}_1 = \frac{1}{2}\mathbf{q} + \mathbf{q}''$$

$$\boldsymbol{\pi}_2 = \mathbf{q} + \frac{1}{2}\mathbf{q}''$$

3-Body Transition Amplitude

$$T(\mathbf{p},\mathbf{q},\mathbf{q}_0) = T_0(\mathbf{p},\mathbf{q},\mathbf{q}_0) + \int d^3q'' \Big[t(\mathbf{p},-\boldsymbol{\pi}_1;\varepsilon)\frac{1}{E+i\epsilon-E''}\, T(\boldsymbol{\pi}_2,\mathbf{q}'',\mathbf{q}_0) + t(\mathbf{p},\boldsymbol{\pi}_1;\varepsilon)\frac{1}{E+i\epsilon-E''}\, T(-\boldsymbol{\pi}_2,\mathbf{q}'',\mathbf{q}_0)\Big].$$

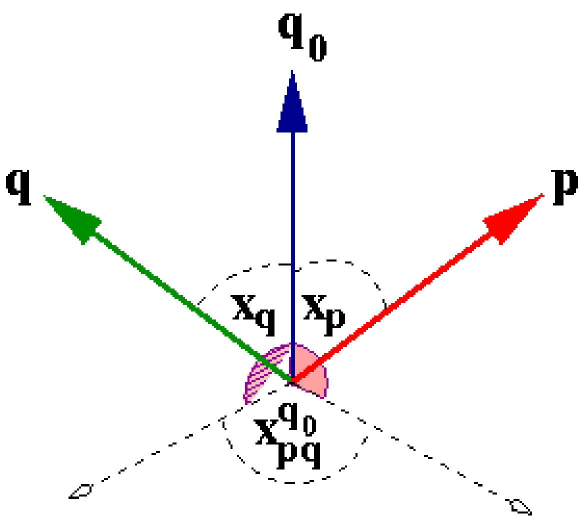
$$\left\langle \mathrm{pq}\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle = \varphi_d\left(\mathrm{q}+\tfrac{1}{2}\mathrm{q}_0\right)\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}_0, E-\tfrac{3}{4m}q^2\right) + \int d^3\mathrm{q}'' \frac{\hat{t}_s\left(\mathrm{p},\tfrac{1}{2}\mathrm{q}+\mathrm{q}'', E-\tfrac{3}{4m}q^2\right)}{E-\tfrac{1}{m}\left(q^2+q''^2+\mathrm{q}\cdot\mathrm{q}''\right)+i\varepsilon}\,\frac{\left\langle \mathrm{q}+\tfrac{1}{2}\mathrm{q}'',\mathrm{q}''\left|\hat{T}\right|\mathrm{q}_0\varphi_d\right\rangle}{E-\tfrac{3}{4m}q''^2-E_d+i\varepsilon}$$

$\hat{t}_s$ ≡ symmetrized 2-body t-matrix

$$t_s(\mathbf{p},\mathbf{p}';z) \equiv \frac{\hat{t}_s(\mathbf{p},\mathbf{p}';z)}{z-E_d}$$

Variables for 3D Calculation

3 distinct vectors in the problem: $\mathbf{q}_0$ $\mathbf{q}$ $\mathbf{p}$



5 independent variables:

$$p = |\mathrm{p}| \ , \ q = |\mathrm{q}|$$

$$x_p = \hat{\mathbf{p}} \cdot \hat{\mathbf{q}}_0, \ x_q = \hat{\mathbf{q}} \cdot \hat{\mathbf{q}}_0$$

$$x_{pq}^{q_0} = (\widehat{\mathbf{q}_0 \times \mathbf{q}}) \cdot (\widehat{\mathbf{q}_0 \times \mathbf{p}})$$

q system : **z || q**

q_0 system : **z || $\mathbf{q}_0$**

Variables invariant under rotation:

freedom to choose coordinate system for numerical calculation

3D Integral Equation in 5 Variables

Solved by Padé Summation

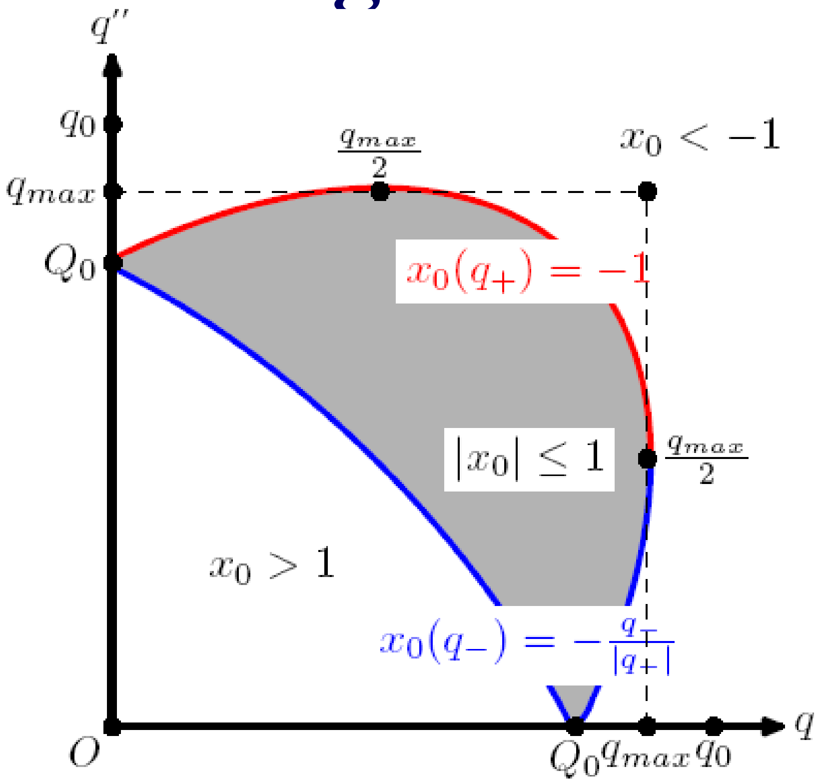
$$
\begin{aligned}
&\langle p, x_p, x_{pq}^{q_0}, x_q, q|\hat{T}|q_0\varphi_d\rangle \\
= \; & \varphi_d\left(\sqrt{q^2+\frac{1}{4}q_0^2+qq_0x_q}\right) \\
\times \; & \hat{t}_s\left(p, \sqrt{\frac{1}{4}q^2+q_0^2+qq_0x_q}, \frac{\frac{1}{2}qy_{pq}+q_0x_p}{\sqrt{\frac{1}{4}q^2+q_0^2+qq_0x_q}}; E-\frac{3}{4m}q^2\right) \\
+ \; & \int_0^\infty dq''q''^2\int_{-1}^{+1}dx''\int_0^{2\pi}d\varphi''\frac{1}{E-\frac{1}{m}(q^2+qq''x''+q''^2)+i\varepsilon} \\
\times \; & \hat{t}_s\left(p, \sqrt{\frac{1}{4}q^2+q''^2+qq''x''}, \frac{\frac{1}{2}qy_{pq}+q''y_{pq''}}{\sqrt{\frac{1}{4}q^2+q''^2+qq''x''}}; E-\frac{3}{4m}q^2\right) \\
\times \; & \frac{\left\langle\sqrt{q^2+\frac{1}{4}q''^2+qq''x''}, \frac{qx_q+\frac{1}{2}q''y_{q_0q''}}{\sqrt{q^2+\frac{1}{4}q''^2+qq''x''}}, \frac{\frac{qx''+\frac{1}{2}q''}{\sqrt{q^2+\frac{1}{4}q''^2+qq''x''}}-x_{\pi_p}x_{\pi_q}}{\sqrt{1-x_{\pi_p}^2}\sqrt{1-x_{\pi_q}^2}}, y_{q_0q''}, q''|\hat{T}|q_0\varphi_d\right\rangle}{E-\frac{3}{4m}q''^2-E_d+i\varepsilon}
\end{aligned}
$$

Singularities:

Position depends on q, q”, x”

Fixed “deuteron” pole

Singularities of 3N Propagator



("moving" singularities)

Discussed e.g. in

- Schmied, Ziegelmann (1974)
- E.F. Redish (1976)

$$\int_0^{\infty} dq'' \int_{-1}^{+1} dx'' \frac{F(q'',x'')}{(x_0 - x'' + i\varepsilon)(q_0^2 - q''^2 + i\varepsilon)}$$

$$= \int_0^{q_{\max}} dq'' \int_{-1}^{+1} dx'' \frac{1}{(q_0^2 - q''^2)} \frac{F(q'',x'')}{(x_0 - x'' + i\varepsilon)} + \int_{q_{\max}}^{\infty} dq'' \int_{-1}^{+1} dx'' \frac{1}{(x_0 - x'')} \frac{F(q'',x'')}{(q_0^2 - q''^2 + i\varepsilon)}$$

Logarithmic Singularity in the q″ Integration

$$\int_0^{q_{max}} dq'' \int_{-1}^{1} dx'' \frac{F(q'',x'')}{x''-x_0-i\varepsilon} = \int_0^{q_{max}} dq'' F(p'',x_0) \int_{-1}^{1} dx \boxed{\frac{1}{x''-x_0-i\varepsilon}} + \int_0^{q_{max}} dq'' \int_{-1}^{1} dx \frac{F(p'',x)-F(p'',x_0)}{x''-x_0}$$

$$\Rightarrow \ln\left|\frac{1+x_0}{1-x_0}\right| + i\pi\Theta(1-|x_0|)$$

$$\ln\left|\frac{1+x_0}{1-x_0}\right| = \left(-\frac{q_-}{|q_-|}\ln\left|q''+|q_-|\right| - \ln\left|q''+|q_+|\right|\right)$$

$$+\boxed{\left(+\frac{q_-}{|q_-|}\ln\left|q''-|q_-|\right| + \ln\left|q''-|q_+|\right|\right)}$$

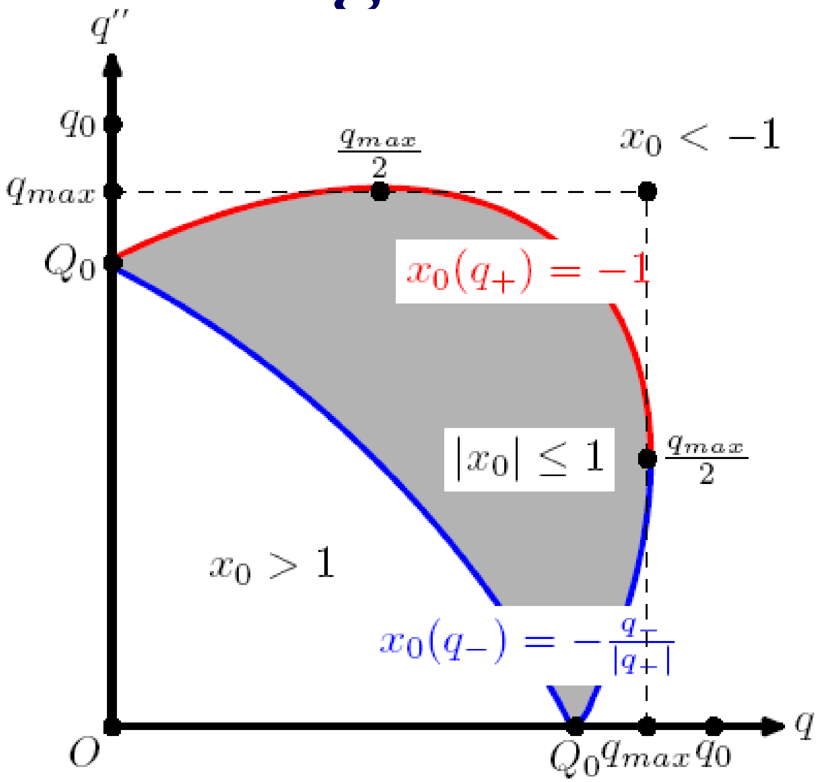
$$\Rightarrow \int_{q_-^{\varepsilon}}^{q_+^{\varepsilon}} dq'' F(q'',x_0) \ln\left|q''-|q_\pm|\right|$$

If $F(q'',x_0)$ is approximated by a cubic spline $\Rightarrow$

analytic integration !

Realized in H.Liu, Ch. Elster, W. Glöckle, PRC 72, 054004 (2005)

Singularities of 3N Propagator



("moving" singularities)

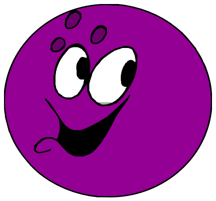
$$\int_0^\infty dq'' \int_{-1}^{+1} dx'' \frac{F(q'',x'')}{(x_0 - x'' + i\varepsilon)(q_0^2 - q''^2 + i\varepsilon)}$$

$$= \int_0^{q_{\max}} dq'' \int_{-1}^{+1} dx'' \frac{1}{(q_0^2 - q''^2)} \frac{F(q'',x'')}{(x_0 - x'' + i\varepsilon)} + \int_{q_{\max}}^{\infty} dq'' \int_{-1}^{+1} dx'' \frac{1}{(x_0 - x'')} \frac{F(q'',x'')}{(q_0^2 - q''^2 + i\varepsilon)}$$

Alternate view of the δ-functions

$$\begin{aligned}\boldsymbol{\pi}_1 &= \frac{1}{2}\mathbf{q} + \mathbf{q}'' \\ \boldsymbol{\pi}_2 &= \mathbf{q} + \frac{1}{2}\mathbf{q}''\end{aligned}$$

$$\begin{aligned}\delta(\mathbf{p}' + \boldsymbol{\pi}_1)\,\delta(\mathbf{p}'' - \boldsymbol{\pi}_2) &= \frac{\delta(p' - \pi_1)}{p'^2}\frac{\delta(p'' - \pi_2)}{p''^2}\delta(\hat{\mathbf{p}}' + \hat{\boldsymbol{\pi}}_1)\delta(\hat{\mathbf{p}}'' - \hat{\boldsymbol{\pi}}_2) \\ \delta(\mathbf{p}' - \boldsymbol{\pi}_1)\,\delta(\mathbf{p}'' + \boldsymbol{\pi}_2) &= \frac{\delta(p' - \pi_1)}{p'^2}\frac{\delta(p'' - \pi_2)}{p''^2}\delta(\hat{\mathbf{p}}' - \hat{\boldsymbol{\pi}}_1)\delta(\hat{\mathbf{p}}'' + \hat{\boldsymbol{\pi}}_2)\end{aligned}$$



→ **Separate momentum and angle variables**

$$\begin{aligned}T(\mathbf{p},\mathbf{q},\mathbf{q}_0) &= T_0(\mathbf{p},\mathbf{q},\mathbf{q}_0) + \int d^3q''\,dp'\,dp''\delta(p' - \pi_1)\,\delta(p'' - \pi_2) \\ &\quad \frac{1}{E + i\epsilon - \frac{1}{m}(p''^2 + \frac{3}{4}q''^2)}(t_s(\mathbf{p}, p'\hat{\boldsymbol{\pi}}_1;\varepsilon)T(p''\hat{\boldsymbol{\pi}}_2, \mathbf{q}'', \mathbf{q}_0)\end{aligned}$$

looks more complicated ! *However* …

Rewrite δ-functions δ (p'-π_i)

Reminder: $|\boldsymbol{\pi}_1| = \sqrt{\frac{1}{4}q^2 + q''^2 + qq''x''}$ $\quad |\boldsymbol{\pi}_2| = \sqrt{q^2 + \frac{1}{4}q''^2 + qq''x''}$

$$\delta(p' - \pi_1) = \frac{2p'}{qq''}\,\delta(x'' - x_0)\,\Theta(1 - |x_0|)$$

$$x_0 = \frac{1}{qq''}\left(p'^2 - \frac{1}{4}q^2 - q''^2\right) = \frac{1}{qq''}\left(p''^2 - \frac{1}{4}q''^2 - q^2\right)$$

$$\delta(p' - \pi_2) = \delta\left(p'' - \sqrt{p'^2 + \frac{3}{4}q^2 - \frac{3}{4}q''^2}\right)\,\Theta\left(p'' - \sqrt{p'^2 + \frac{3}{4}q^2 - \frac{3}{4}q''^2}\right)$$

Faddeev Equation

$$
\begin{aligned}
T(\mathbf{p},\mathbf{q},\mathbf{q}_0) &= T_0(\mathbf{p},\mathbf{q},\mathbf{q}_0) \\
&+ \int d^3q''\, dp'\, dp'' \frac{2p'}{qq''}\, \delta(x''-x_0)\, \Theta(1-|x_0|) \\
&\quad \delta\left(p'' - \sqrt{p'^2 + \frac{3}{4}q^2 - \frac{3}{4}q''^2}\right)\ \Theta\left(p'^2 + \frac{3}{4}q^2 - \frac{3}{4}q''^2\right) \\
&\quad \frac{1}{E + i\epsilon - \frac{1}{m}(p''^2 + \frac{3}{4}q''^2)}\, t_s(\mathbf{p}, p'\hat{\boldsymbol{\pi}}_1; \varepsilon)\ T(p''\hat{\boldsymbol{\pi}}_2, \mathbf{q}'', \mathbf{q}_0).
\end{aligned}
$$

integrate

$$
\begin{aligned}
\hat{T}(\mathbf{p},\mathbf{q},\mathbf{q}_0) &= \hat{T}_0(\mathbf{p},\mathbf{q},\mathbf{q}_0) \\
&+ \frac{2}{q}\int d\hat{\mathbf{q}}''\, dq''\, dp'\ \delta(x''-x_0)\ \Theta(1-|x_0|)\ \Theta\left(p'^2 + \frac{3}{4}q^2 - \frac{3}{4}q''^2\right) \\
&\quad \frac{p'q''}{E + i\epsilon - \frac{1}{m}(p'^2 + \frac{3}{4}q^2)} \hat{t}_s(\mathbf{p}, p'\hat{\boldsymbol{\pi}}_1; \varepsilon)\ \frac{\hat{T}(p''\hat{\boldsymbol{\pi}}_2, \mathbf{q}'', \mathbf{q}_0)}{E + i\epsilon - E_d - \frac{3}{4m}q''^2},
\end{aligned}
$$

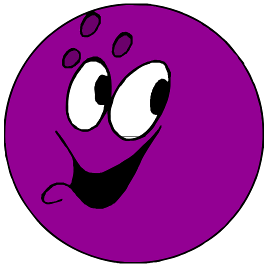
with $\quad t_s(\mathbf{p},\mathbf{p}';z) \equiv \dfrac{\hat{t}_s(\mathbf{p},\mathbf{p}';z)}{z - E_d}$ treats deuteron pole explicitly

Propagator:

$$\frac{1}{E+i\epsilon-\frac{1}{m}(p'^2+\frac{3}{4}q^2)}\frac{1}{E+i\epsilon-E_d-\frac{3}{4m}q''^2}=$$
$$\left[\frac{1}{E+i\epsilon-\frac{1}{m}(p'^2+\frac{3}{4}q^2)}-\frac{1}{E+i\epsilon-E_d-\frac{3}{4m}q''^2}\right]\frac{1}{-E_d-\frac{3}{4m}q''^2+\frac{1}{m}(p'^2+\frac{3}{4}q^2)}$$

Last piece $\bar{G}(q,q'',p')=\frac{1}{-E_d-\frac{3}{4m}q''^2+\frac{1}{m}(p'^2+\frac{3}{4}q^2)}$

Can **not** become singular:

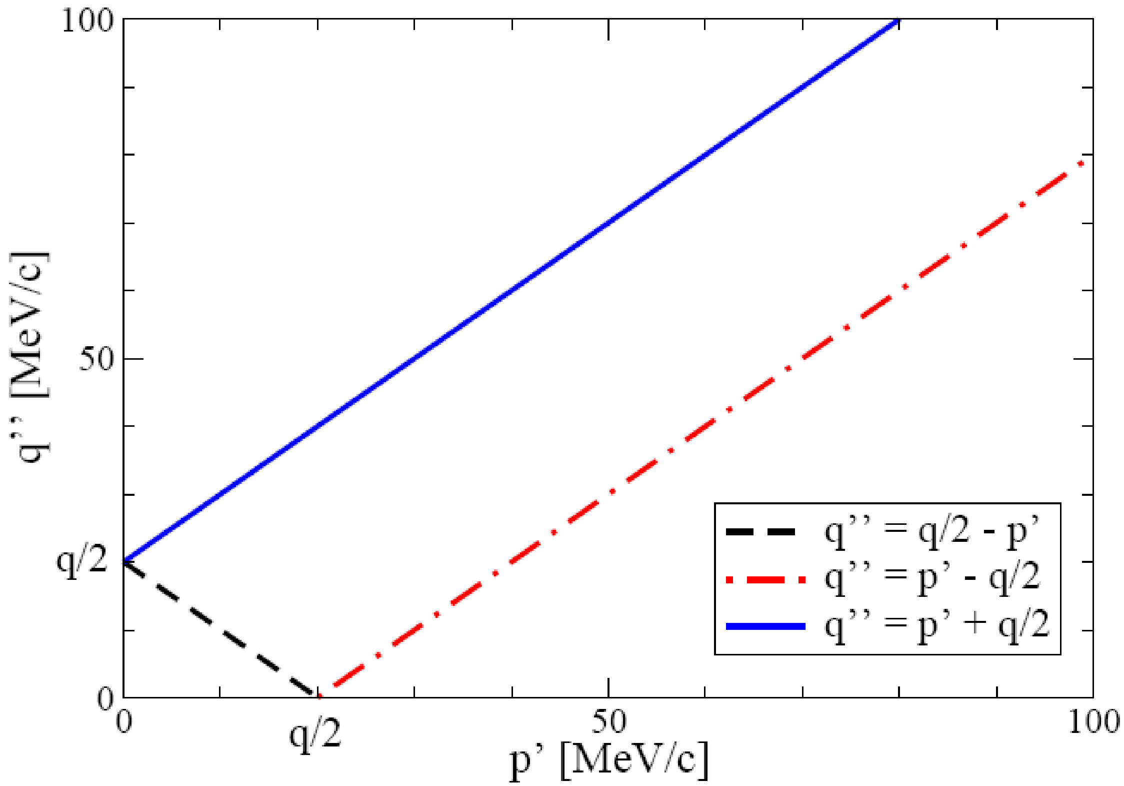


$$\bar{G}(q,q'',p')=\frac{1}{-E_d+\frac{1}{m}p''^2}=\frac{1}{|E_d|+\frac{1}{m}p''^2}>0$$

Faddeev Equation:

$$
\begin{aligned}
\hat{T}(\mathbf{p},\mathbf{q},\mathbf{q_0}) \;=\;& \hat{T}_0(\mathbf{p},\mathbf{q},\mathbf{q_0}) \\
+\;& \frac{2}{q}\int_0^\infty dp'p' \frac{1}{E+i\epsilon-\frac{1}{m}(p'^2+\frac{3}{4}q^2)} \int_{|q/2-p'|}^{q/2+p'} dq''q''\;\bar{G}(q,q'',p') \\
& \int d\hat{\mathbf{q}}''\;\delta(x''-x_0)\hat{t}_s(\mathbf{p},p'\hat{\boldsymbol{\pi}}_1;\varepsilon)\;\hat{T}(p''\hat{\boldsymbol{\pi}}_2,\mathbf{q}'',\mathbf{q_0}) \\
-\;& \frac{2}{q}\int_0^\infty dq''q'' \frac{1}{E+i\epsilon-E_d-\frac{3}{4m}q''^2} \int_{|q/2-q''|}^{q/2+q''} dp'p'\;\bar{G}(q,q'',p') \\
& \int d\hat{\mathbf{q}}''\delta(x''-x_0)\hat{t}_s(\mathbf{p},p'\hat{\boldsymbol{\pi}}_1;\varepsilon)\;\hat{T}(p''\hat{\boldsymbol{\pi}}_2,\mathbf{q}'',\mathbf{q_0})\;.
\end{aligned}
$$

Domain for momentum integration



For q = 40 MeV/c

Angle Integration is straightforward (but work)

$$
\begin{aligned}
\hat{T}(p, x_p, x_{pq}^{q0}, x_q, q; q_0) &= \hat{T}_0(p, x_p, x_{pq}^{q0}, x_q, q; q_0) \\
&+ \frac{2}{q}\int_0^\infty dp'p' \frac{1}{E + i\epsilon - \frac{1}{m}(p'^2 + \frac{3}{4}q^2)} \int_{|q/2-p'|}^{q/2+p'} dq''q''\, \bar{G}(q, q'', p') \\
&\quad \int_0^{2\pi} d\varphi''\, \hat{t}_s\left(p, p', \frac{\frac{1}{2}qy_{pq} + y_{pq''}(x_0)}{\sqrt{\frac{1}{2}q^2 + q''^2 + qq''x_0}}; \varepsilon\right) \\
&\quad \hat{T}\left(p'', \frac{qx_q + \frac{1}{2}q''y_{q_0q''}(x_0)}{\sqrt{q^2 + \frac{1}{4}q''^2 + qq_0x_0}}, x^{q0}_{\pi 2 y_{q_0q''}}(x_0), y_{q_0q''}(x_0), q''; q_0\right) \\
&- \frac{2}{q}\int_0^\infty dq''q'' \frac{1}{E + i\epsilon - E_d - \frac{3}{4m}q''^2} \int_{|q/2-q''|}^{q/2+q''} dp'p'\, \bar{G}(q, q'', p') \\
&\quad \int_0^{2\pi} d\varphi''\, \hat{t}_s\left(p, p', \frac{\frac{1}{2}qy_{pq} + y_{pq''}(x_0)}{\sqrt{\frac{1}{2}q^2 + q''^2 + qq''x_0}}; \varepsilon\right) \\
&\quad \hat{T}\left(p'', \frac{qx_q + \frac{1}{2}q''y_{q_0q''}(x_0)}{\sqrt{q^2 + \frac{1}{4}q''^2 + qq_0x_0}}, x^{q0}_{\pi 2 y_{q_0q''}}(x_0), y_{q_0q''}(x_0), q''; q_0\right),
\end{aligned}
$$

FBS 45, 1 (2009)

Summary

- Two "tricks"
  - Rewriting of δ-function of the permutation operator
  - Rewriting of the product of two singular propagators into a sum
- Turn "usual" integration over the Faddeev kernel with "moving" singularities
- Into two integrations (over the kernel) with simple poles
- Instead of integration over q'', x'', and φ''
  - and `half-moon' with double singularities + log singularities
- $\Rightarrow$ now integration over q'', p', and φ''
  - with 2 integrals, simple poles and `stripe' of momentum integration

Tested in pw for s-waves: H. Witała, W. Glöckle, Eur.Phys.J A37, 871 (2008)