Renormalization of higher twist operators in QCD

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[1] V. Braun, J. Rohrwild,

Outline

Operator Product Expansion

$$J(x)J(0) \sim \sum_N C_N(x^2, \mu^2)\, \mathcal{O}_N(\mu^2) \tag{1}$$

Twist: $t = \text{dimension} - \text{spin}: \quad \mathcal{O}^{t=2}_{\mu_1\ldots\mu_N} = \text{Sym}\, \bar{q}\gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_N} q - \text{Traces}$

Higher twist effects

- exclusive and semi-inclusive reactions
  Belitsky, Mueller, NPB 589, 2000,
  Kivel, Polyakov, Vanderhaeghen, PRD, **63** 2001)
- diffractive electroproduction of vector mesons
  Anikin, Ivanov, Pire, Szymanowski, Wallon, 2009
- single spin asymmetry
  Eguchi, Koike, Tanaka, 2007; Kang, Qiu, 2009, etc
- higher twist hadronic wave functions
  Braun, Filyanov, 90; Ball, Braun, Lenz, 2006, Braun, Fries, Mahnke, Stein, 2000, etc

Higher twist operators= quasipartonic + nonquasipartonic.

- Quasipartonic operators:
  Bukhvostov, Frolov, Lipatov, Kuraev, 1985 (BFLK)
- Nonquasipartonic operators:
  Avoid them if it is possible

To understand the renormalization of the twist-4 baryonic wave functions.

Light-ray operators:

$$\mathcal{O}^{t=2}_{\mu_1\ldots\mu_N} = \mathrm{Sym}\,\bar q\gamma_{\mu_1}D_{\mu_2}\ldots D_{\mu_N}q - \mathrm{Traces}$$

$$\mathcal{O}^{t=2}_{\mu_1\ldots\mu_N} \Leftrightarrow \mathcal{O}_N(n) = n^{\mu_1}\ldots n^{\mu_N}\mathcal{O}^{t=2}_{\mu_1\ldots\mu_N}\,, \qquad n^2 = 0$$

Nonlocal-operator = generating function for local operators

$$\mathcal{O}(nz) = \sum_{N=1}^{\infty}\frac{z^N}{N!}\mathcal{O}_N(n) = \bar q(0)\gamma_+[0,zn]q(zn)\,.$$

Including total derivatives:

$$\mathcal{O}(nz) \to \mathcal{O}(nz_1,nz_2) \to \mathcal{O}(z_1,z_2) = \bar q(z_1 n)\gamma_+[z_1 n, z_2 n]q(z_2 n)$$

RG-equation

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\frac{\alpha_s}{2\pi}\mathbb{H}\right)[\mathcal{O}(z_1,z_2)]_R=0\,, \tag{2}$$

where $\mathbb{H}$ is the integral operator

$$\begin{aligned}[\mathbb{H}\cdot\mathcal{O}](z_1,z_2) \quad &= \quad 2C_F\Big\{\int_0^1\frac{d\alpha}{\alpha}\Big[2\mathcal{O}(z_1,z_2)-\bar\alpha\mathcal{O}(z_{12}^\alpha,z_2)-\bar\alpha\mathcal{O}(z_1,z_{21}^\alpha)\Big] \\ &\qquad -\int_0^1 d\alpha\int_0^{\bar\alpha}d\beta\,\mathcal{O}(z_{12}^\alpha,z_{21}^\beta)-\frac{3}{2}\mathcal{O}(z_1,z_2)\Big\}\,,\end{aligned} \tag{3}$$

where $z_{12}^{\alpha}=z_1(1-\alpha)+z_2\alpha$.

$\mathbb{H}$ is invariant under $SL(2,R)$ transformations of light-ray, $z\to\dfrac{az+b}{cz+d}$.

Equation (3)⇒DGLAP, ERBL, GPD

$$\varphi_{AB}(z_1,z_2)=\langle A|\mathcal{O}(z_1,z_2)|B\rangle\,.$$

RG-equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \frac{\alpha_s}{2\pi}\mathbb{H}\right)[\mathcal{O}(z_1,z_2)]_R = 0\,, \qquad (2)$$

where $\mathbb{H}$ is the integral operator

$$\begin{aligned}[\mathbb{H}\cdot\mathcal{O}](z_1,z_2) &= 2C_F\left\{\int_0^1 \frac{d\alpha}{\alpha}\left[2\mathcal{O}(z_1,z_2) - \bar\alpha\mathcal{O}(z_{12}^\alpha,z_2) - \bar\alpha\mathcal{O}(z_1,z_{21}^\alpha)\right]\right.\\ &\qquad \left. - \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\mathcal{O}(z_{12}^\alpha,z_{21}^\beta) - \frac{3}{2}\mathcal{O}(z_1,z_2)\right\}, \end{aligned} \qquad (3)$$

where $z_{12}^\alpha = z_1(1-\alpha) + z_2\alpha$.

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Bukhvostov, Frolov, Lipatov, Kuraev, 1985 (BFLK)
Renormalization of quasipartonic operators:

$$\mathcal{O}(z_1,\ldots,z_N) = \Phi(z_1)\otimes\Phi(z_2)\ldots\Phi(z_N)$$

t=N (number of fields in the operator)

- Quasipartonic Operators are closed under renormalization.
- Hamiltonian $\mathbb{H}$ has two-particle form

$$\mathbb{H} = \sum_{ik}\mathbb{H}_{ik}$$

- All two-particle kernels are $SL(2,R)$ invariant.
- In this work:

Extend this analysis to nonquasipartonic operators

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Building blocks for operators:

$$q,\ \bar{q},\ F_{\mu\nu},\ D_\mu q,\ D_\mu D_\nu q,\ D_\mu D_\nu D_\rho q, \ldots$$

- Tensor properties
- Equations of motion (EOM)

Baryonic twist-4 operator

$$\boxed{\epsilon^{ijk} u_i^{\uparrow}(z_1) C \not{n} \gamma_\perp \not{p} u_j^{\downarrow}(z_2) \gamma^\perp \not{n} d^{\downarrow}(z_3)}$$

$$\boxed{x_{\alpha\dot\alpha} = x_\mu (\sigma^\mu)_{\alpha\dot\alpha}\,, \qquad \bar{x}^{\dot\alpha\alpha} = x_\mu (\bar\sigma^\mu)^{\dot\alpha\alpha}\,,}$$

where $\sigma^\mu = (\mathbb{1}, \vec\sigma)$, $\bar\sigma^\mu = (\mathbb{1}, -\vec\sigma)$.

$$q = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}, \qquad \bar q = (\chi^\beta, \bar\psi_{\dot\alpha}) \tag{4}$$

where ψ_α, $\bar\chi^{\dot\beta}$ are two-component Weyl spinors, $\bar\psi_{\dot\alpha} = (\psi_\alpha)^\dagger$, $\chi^\alpha = (\bar\chi^{\dot\alpha})^\dagger$. The gluon strength tensor $F_{\mu\nu}$ is decomposed as

$$F_{\alpha\beta,\dot\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}F_{\mu\nu} = 2\left(\epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} - \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}\right) \tag{5}$$

where $f_{\alpha\beta}$ and $\bar f_{\dot\alpha\dot\beta}$ are chiral and antichiral symmetric tensors, $f^* = \bar f$

$$i\widetilde{F}_{\alpha\beta,\dot\alpha\dot\beta} = 2(\epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} + \epsilon_{\alpha\beta}\bar f_{\dot\alpha\dot\beta}) \tag{6}$$

$u^\alpha = \epsilon^{\alpha\beta}u_\beta$, $\bar u^{\dot\alpha} = \bar u_{\dot\beta}\epsilon^{\dot\beta\dot\alpha}$, $(uv) = (u^\alpha v_\alpha)$, $(\bar u\bar v) = (\bar u_{\dot\alpha}\bar v^{\dot\alpha})$.

Fiertz identity: $\boxed{(ab)(cd) = (ac)(bd) - (ad)(bc)}$

$$\boxed{\psi_\alpha, \chi_\alpha, \bar\psi_{\dot\alpha}, \bar\chi_{\dot\alpha}, f_{\alpha\beta}, \bar f_{\dot\alpha\dot\beta}}$$

$$\Phi(x) = \{\psi_\xi = \xi^\alpha \psi_\alpha(x), f_\xi(x) = \xi^\alpha \xi^\beta f_{\alpha\beta}(x)\ \ldots\}$$

$$\begin{aligned}
i[\mathbf{P}_{\alpha\dot\alpha}, \Phi(x)] &= \partial_{\alpha\dot\alpha}\Phi(x) \\
i[\mathbf{D}, \Phi(x)] &= \frac{1}{2}\left(x_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha} + 2t + \xi^\alpha \frac{\partial}{\partial \xi^\alpha} + \bar\xi_{\dot\alpha}\frac{\partial}{\partial \bar\xi_{\dot\alpha}}\right)\Phi(x) \\
i[\mathbf{M}_{\alpha\beta}, \Phi(x)] &= \frac{1}{4}\left(x_{\alpha\dot\gamma}\partial_\beta{}^{\dot\gamma} + x_{\beta\dot\gamma}\partial_\alpha{}^{\dot\gamma} - 2\xi_\alpha \frac{\partial}{\partial \xi^\beta} - 2\xi_\beta \frac{\partial}{\partial \xi^\alpha}\right)\Phi(x) \qquad (7) \\
i[\bar{\mathbf{M}}_{\dot\alpha\dot\beta}, \Phi(x)] &= \frac{1}{4}\left(x_{\gamma\dot\alpha}\partial^\gamma{}_{\dot\beta} + x_{\gamma\dot\beta}\partial^\gamma{}_{\dot\alpha} - 2\bar\xi_{\dot\alpha}\frac{\partial}{\partial \bar\xi^{\dot\beta}} - 2\bar\xi_{\dot\beta}\frac{\partial}{\partial \bar\xi^{\dot\alpha}}\right)\Phi(x) \\
i[\mathbf{K}_{\alpha\dot\alpha}, \Phi(x)] &= \left(x_{\alpha\dot\gamma}x_{\gamma\dot\alpha}\,\partial^{\gamma\dot\gamma} + 2t x_{\alpha\dot\alpha} + 2\xi_\alpha \bar x_{\dot\alpha}{}^{\beta}\frac{\partial}{\partial \xi^\beta} + 2\bar\xi_{\dot\alpha} x_{\alpha\dot\beta}\frac{\partial}{\partial \bar\xi_{\dot\beta}}\right)\Phi(x)\,.
\end{aligned}$$

$M_{\alpha\beta}, \bar M_{\dot\alpha\dot\beta}$ are the generators of Lorentz group.

$$T^{s,\bar s}: \qquad T_{\underbrace{\alpha\beta\ldots\gamma}_{2s},\underbrace{\dot\alpha\dot\beta\ldots\dot\gamma}_{2\bar s}}$$

$$\mathcal{O}_{\alpha\beta\gamma,\dot\alpha\dot\beta\dot\gamma} = D_{\alpha\dot\alpha}\bar\psi_{\dot\gamma} D_{\beta\dot\beta}\psi_\gamma$$

- Total symmetrization: $\mathcal{O}_{\alpha\beta\gamma,\dot\alpha\dot\beta\dot\gamma} \to \lambda^\alpha\lambda^\beta\lambda^\gamma\bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta}\bar\lambda^{\dot\gamma}\mathcal{O}_{\alpha\beta\gamma,\dot\alpha\dot\beta\dot\gamma} \equiv \mathcal{O}_{+++,+++}$
- Partial symmetrization $\epsilon^{\alpha\beta}\mathcal{O}_{\alpha\beta\gamma,\dot\alpha\dot\beta\dot\gamma} \to \lambda^\gamma\bar\lambda^{\dot\alpha}\bar\lambda^{\dot\beta}\bar\lambda^{\dot\gamma}\epsilon^{\alpha\beta}\mathcal{O}_{\alpha\beta\gamma,\dot\alpha\dot\beta\dot\gamma} \equiv \epsilon^{\alpha\beta}\mathcal{O}_{\alpha\beta+,+++}$

$$(\mu\lambda)\epsilon^{\alpha\beta}\mathcal{O}_{\alpha\beta+,+++} = (\mu^\alpha\lambda^\beta - \lambda^\alpha\mu^\beta)\mathcal{O}_{\alpha\beta+,+++} = \mathcal{O}_{-++,+++} - \mathcal{O}_{+-+,+++} =$$
$$= D_{-+}\bar\psi_+ D_{++}\psi_+ - D_{++}\bar\psi_+ D_{-+}\psi_+$$

$''+''$ - contraction with λ, $''-''$ - contraction with μ $(\psi_+ = \lambda^\alpha\psi_\alpha,\ \psi_- = \mu^\alpha\psi_\alpha)$

λ and μ define two light-like vectors:

$$n_{\alpha\dot\alpha} = \lambda_\alpha\bar\lambda_{\dot\alpha},\ \tilde n_{\alpha\dot\alpha} = \mu_\alpha\bar\mu_{\dot\alpha},\quad n^2 = \tilde n^2 = 0$$

$$\boxed{x_{\alpha\dot\alpha} = z\,\lambda_\alpha\bar\lambda_{\dot\alpha} + \tilde z\,\mu_\alpha\bar\mu_{\dot\alpha} + w\,\lambda_\alpha\bar\mu_{\dot\alpha} + \bar w\,\mu_\alpha\bar\lambda_{\dot\alpha}}$$

$$\begin{aligned}(\mu\lambda)\psi_\alpha &= \lambda_\alpha\psi_- - \mu_\alpha\psi_+\,,\\ (\mu\lambda)^2 f_{\alpha\beta} &= \lambda_\alpha\lambda_\beta f_{--} - (\lambda_\alpha\mu_\beta + \mu_\alpha\lambda_\beta)f_{+-} + \mu_\alpha\mu_\beta f_{++}\end{aligned}$$

$$\boxed{\psi_+, \psi_-, f_{++}, f_{+-}\,\ldots\ldots, D_{++}, D_{--}, D_{-+}, D_{+-}}$$

$$D_{++} = 2\partial_z,\ D_{--} = 2\partial_{\tilde z},\ D_{+-} = 2\partial_w,\ D_{-+} = 2\partial_{\bar w}$$

$SL(2,R)$ subgroup which maps line $z\lambda_\alpha\lambda_{\dot\alpha}$ to itself.

$$S_+ = \frac{i}{2}K_{--} \qquad S_- = -\frac{i}{2}P_{++}, \qquad S_0 = \frac{i}{2}(D - M_{-+} - \bar{M}_{-+}) \tag{8}$$

$$[S_+, S_-] = 2S_0 \qquad [S_0, S_\pm] = \pm S_\pm \tag{9}$$

$$z \longrightarrow z' = \frac{az+b}{cz+d}$$

$\Phi(z) \equiv \Phi(z,0,0,0)$

$$\Phi(z) = \frac{1}{(cz+d)^{2j}}\Phi(z')$$

$$S_+ = z^2\partial_z + 2jz \qquad S_0 = z\partial_z + j \qquad S_- = -\partial_z$$

$$\begin{aligned} j = 1/2: &\quad \psi_-,\ \chi_-,\ \bar\psi_-,\ \bar\chi_-,\ f_{--},\ \bar f_{--} \\ j = 1: &\quad \psi_+,\ \chi_+,\ \bar\psi_+,\ \bar\chi_+,\ f_{+-},\ \bar f_{+-} \\ j = 3/2: &\quad f_{++},\ \bar f_{++} \end{aligned} \tag{10}$$

Operators with derivatives ??

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Operators with derivatives ??

V. Braun, A. M., J. Rohrwild (2008)

- Basis with all derivatives is overcompleted
- In general fields with derivatives have “bad” $SL(2,R)$ transformation properties.

$[D_{\bar w}\psi_+](z)$–does not transform in proper way under $SL(2,R)$

$$\psi_+(x) \to \frac{1}{(1+z\epsilon)^2}\left\{\psi_+\left(\frac{z}{1+\epsilon z},\tilde{z},\frac{w}{1+\epsilon z},\frac{\bar{w}}{1+\epsilon z}\right)+\epsilon z\bar{w}\psi_-\left(\dots\right)\right\}$$

$$[D_w D_{\tilde{z}}\psi_+](z)=\frac{1}{(1+z\epsilon)^3}[D_w D_{\tilde{z}}\psi_+]\left(\frac{z}{1+\epsilon z}\right)$$

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$$[D_w D_{\tilde{z}}\psi_+](z) = \frac{1}{(1+z\epsilon)^3}[D_w D_{\tilde{z}}\psi_+]\left(\frac{z}{1+\epsilon z}\right)$$

$$\psi_+(z,\tilde z,w,0)=\sum_{n,k}\frac{\tilde z^k}{k!}\frac{w^n}{n!}[D_w^n D_{\tilde z}^k\psi_+](z)$$

$$\psi_-(z,\tilde z,0,\bar w)=\sum_{n,k}\frac{\tilde z^k}{k!}\frac{\bar w^n}{n!}[D_{\bar w}^n D_{\tilde z}^k\psi_-](z)$$

$$\boxed{\bar\psi_+(z,\tilde z,0,\bar w),\ \bar\psi_-(z,\tilde z,w,0)\ \ f_{++}(z,\tilde z,w,0),\ f_{+-}(z,\tilde z,0,0),\dots}$$

Operators which appear in the expansion of these fields form the complete basis

$$[D_{\bar w}\psi_+](z)\equiv[D_{-+}\psi_+](z)=[D_{++}\psi_-](z)+EOM=2\partial_z\psi_-(z)+EOM$$

Representation of full conformal group

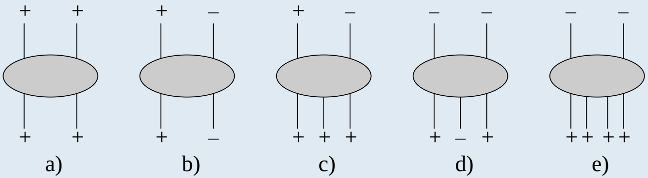
	$j=1/2$	$j=1$	$j=3/2$	$j=2$	$j=5/2$
$E=1$		ψ_+			
$E=2$	ψ_-		$D_w\psi_+$		
$E=3$		$D_{\bar w}\psi_-,\ D_{\tilde z}\psi_+$		$D_w^2\psi_+$	
$E=4$	$D_{\tilde z}\psi_-$		$D_{\bar w}^2\psi_-,\ D_wD_{\tilde z}\psi_+$		$D_w^3\psi_+$

A generic operator has the form

$$S\ \bar{\psi}_{+}^{j_1,k_1}(z_1) \otimes f_{++}^{j_2,k_2}(z_2) \otimes \psi_{-}^{j_3,k_3}(z_3)$$

- Quasipartonic operators=only "+" fields BFLK,1985
- This work="+" fields+(one "−" field or one transverse derivative D_{+-}, D_{-+}).
  M. Okawa (1980-1981),
  E. Shuryak, A. Vainshtein, 1982

$\Phi(z_1)\otimes\Phi(z_2)-$ operator with open color indices



- Light-cone gauge: $A_{++}=0$ - $SL(2,R)$ invariant condition.
- $$[\Phi\otimes\Phi]_R(z_1,z_2)=[\Phi\otimes\Phi]_B(z_1,z_2)+\frac{\alpha_s}{4\pi\epsilon}[\mathbb{H}^{(2\to2)}\Phi\otimes\Phi](z_1,z_2)$$
$$+\frac{\alpha_s}{4\pi\epsilon}[\mathbb{H}^{(2\to3)}\Phi\otimes\Phi\otimes\Phi](z_1,z_2)+\dots \qquad (11)$$
- Hamiltonians $\mathbb{H}^{(2\to2)}$, $\mathbb{H}^{(2\to3)}$ are $SL(2,R)$ invariant operators.

$$[\mathcal{H}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_0^1 d\beta\,\bar\alpha^{2j_1-2}\bar\beta^{2j_2-2}\,\omega\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right)\varphi(z_{12}^{\alpha},z_{21}^{\beta})\,,$$

$$[\widehat{\mathcal{H}}\ \varphi](z_1,z_2)=\int_0^1\frac{d\alpha}{\alpha}\Big[2\varphi(z_1,z_2)-\bar\alpha^{2j_1-1}\varphi(z_{12}^{\alpha},z_2)-\bar\alpha^{2j_2-1}\varphi(z_1,z_{21}^{\alpha})\Big]\,, \tag{12}$$

$$[\mathcal{H}^{+}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\bar\alpha^{2j_1-2}\bar\beta^{2j_2-2}\,\varphi(z_{12}^{\alpha},z_{21}^{\beta})\,, \tag{13}$$

$$[\widetilde{\mathcal{H}}^{+}\varphi](z_1,z_2)=\int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\bar\alpha^{2j_1-2}\bar\beta^{2j_2-2}\left(\frac{\alpha\beta}{\bar\alpha\bar\beta}\right)\varphi(z_{12}^{\alpha},z_{21}^{\beta})\,, \tag{14}$$

- Eigenfunctions: $\psi_n(z_1,z_2)=(z_1-z_2)^n$:
- Eigenvalues: $\mathcal{H}\psi_n(z_1,z_2)=E_n\psi_n(z_1,z_2) \qquad \mathcal{H}\leftrightarrow E_n$

$$\begin{aligned}\widehat{E}_n&=\psi(n+2j_1)+\psi(n+2j_2)-2\psi(1)\\&=\psi(J+j_1-j_2)+\psi(J+j_2-j_1)-2\psi(1)\end{aligned}$$

where $J=n+j_1+j_2$, $SL(2,R)-$Casimir operator $\mathbb{C}_2=J(J-1)$.

$\Phi_+^A(z_1)\otimes\Phi_+^B(z_2)$, where A, B-color indices

BFLK, 1985

$$[\mathbb{H}\Phi_+^A\otimes\Phi_+^B](z_1,z_2) = -2t^c_{AA'}\otimes t^c_{BB'}[\mathcal{H}_1\Phi_+^{A'}\otimes\Phi_+^{B'}](z_1,z_2) - 2(t^{B'}t^B)_{AA'}[\mathcal{H}_2\Phi_+^{A'}\otimes\Phi_+^{B'}](z_1,z_2)+\dots \tag{15}$$

$t^a = T^a$ for $\psi,\bar\chi$, $t^a=(-T^a)^t$ for $\bar\psi,\chi$, $t^a_{bc}=if^{bac}$ for $f,\bar f$

- Universal term

$$\mathcal{H}_1(J) = \psi(J+\Delta)+\psi(J-\Delta)-2\psi(1) \tag{16}$$

$\Delta=|h_1-h_2|$, h_1,h_2 are the helicities of the fields,

$$\psi_+^{h=1/2},\ \bar\psi_+^{h=-1/2},\ f_{++}^{h=1},\ \bar f_{++}^{h=-1}$$

- One of fields is f or $\bar f$

$$\mathcal{H}_2(J) = (-1)^{J-\Delta}\frac{\Gamma(2\Delta)\Gamma(J-\Delta)}{\Gamma(J+\Delta)} \tag{17}$$

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BFLK, 1985

$$\begin{aligned}[\mathbb{H}\Phi_+^A\otimes\Phi_+^B](z_1,z_2) = &-2t^c_{AA'}\otimes t^c_{BB'}[\mathcal{H}_1\Phi_+^{A'}\otimes\Phi_+^{B'}](z_1,z_2)\\ &-2(t^{B'}t^B)_{AA'}[\mathcal{H}_2\Phi_+^{A'}\otimes\Phi_+^{B'}](z_1,z_2)+\dots\end{aligned}\tag{15}$$

$t^a=T^a$ for $\psi,\bar\chi$, $t^a=(-T^a)^t$ for $\bar\psi,\chi$, $t^a_{bc}=if^{bac}$ for $f,\bar f$

- Universal term

$$\mathcal{H}_1(J)=\psi(J+\Delta)+\psi(J-\Delta)-2\psi(1)\tag{16}$$

$\Delta=|h_1-h_2|$, h_1,h_2 are the helicities of the fields,

$$\psi_+^{h=1/2},\ \bar\psi_+^{h=-1/2},\ f_{++}^{h=1},\ \bar f_{++}^{h=-1}$$

- One of fields is f or $\bar f$

$$\mathcal{H}_2(J)=(-1)^{J-\Delta}\frac{\Gamma(2\Delta)\Gamma(J-\Delta)}{\Gamma(J+\Delta)}\tag{17}$$

$$\Phi_- \otimes \Phi_+ = \{\psi_- \otimes \psi_+,\ D_{-+}\bar{\psi}_+ \otimes \psi_+,\ f_{+-} \otimes f_{++}\ \ldots\}.$$

$\mathbb{V}_\Phi$-the special representation of full conformal group, $\mathbb{V}_\psi = \{\psi_+, \psi_-, D_w\psi_+, \ldots\}$
Tensor product decomposition:

$$\mathbb{V}_\Phi \otimes \mathbb{V}_\Phi = \sum_n \mathcal{V}_n \qquad \mathcal{H}\mathcal{V}_n = E_n \mathcal{V}_n$$

$$\mathcal{V}_n = (\Phi_+ \otimes \Phi_+) \oplus (\Phi_- \otimes \Phi_+) \oplus \ldots$$

Beisert, 2004, Beisert et al, 2005: $\boxed{\mathcal{H}(J) \to \mathcal{H}(\mathbf{J}), \quad \text{e.g} \ \ \psi(J) \to \psi(\mathbf{J}).}$

- $SL(2,R)$: $\mathbb{C}_2^{SL(2)} = J(J-1)$
- $SO(4,2)$: $\mathbb{C}_2^{SO(4,2)} = \mathbf{J}(\mathbf{J}-1)$

$$\begin{aligned}[\mathbb{H}\Phi^A \otimes \Phi^B](z_1,z_2) = &- 2t^c_{AA'} \otimes t^c_{BB'}[\mathcal{H}_1(\mathbf{J})\Phi^{A'} \otimes \Phi^{B'}](z_1,z_2) \\ &- 2(t^{B'}t^B)_{AA'}[\mathcal{H}_2(\mathbf{J})\Phi^{A'} \otimes \Phi^{B'}](z_1,z_2) + \ldots\end{aligned}$$

$$\psi_+(z_1)\otimes\psi_+(z_2) = \sum_{N,q}\psi_{N,q}(z_1,z_2)\mathcal{O}_{N,q}$$

$$[\mathbf{C}_2, \psi_+(z_1)\otimes\psi_+(z_2)] = \sum_{N,q}\psi_{N,q}(z_1,z_2)[\mathbf{C}_2\mathcal{O}_{N,q}] = \sum_{N,q}[\mathbb{C}_2\psi_{N,q}(z_1,z_2)]\mathcal{O}_{N,q}$$

$$\mathbb{C}_2^{SO(4,2)}\psi_+\otimes\psi_+ = \mathbb{C}_2^{SL(2)}\psi_+\otimes\psi_+ = -\partial_1\partial_2 z_{12}^2\psi_+\otimes\psi_+$$

$$\mathbb{C}_2^{SO(4,2)}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix} = \begin{pmatrix}\mathbb{C}_2^{SL(2)}+1/4 & \partial_2 z_{21}\\ \partial_1 z_{12} & \mathbb{C}_2^{SL(2)}+1/4\end{pmatrix}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix}$$

$$\mathbb{C}_2^{SO(4,2)} = \widehat{\mathbf{J}}(\widehat{\mathbf{J}}-1), \qquad \widehat{\mathbf{J}} = -\begin{pmatrix}0 & \partial_2 z_{21}\\ \partial_1 z_{12} & 0\end{pmatrix}$$

Eigenfunctions

$$\varphi_n^{\pm}(z_1,z_2) = \begin{pmatrix}1\\ \pm 1\end{pmatrix} z_{12}^n :$$

$$\mathbb{C}_2^{SO(4,2)}\varphi_n^+ = (n+2)(n+1)\varphi_n^+ \qquad J = n+2$$

$$\mathbb{C}_2^{SO(4,2)}\varphi_n^- = (n+1)n\varphi_n^- \qquad J = n+1$$

Hamiltonian

$$\mathbb{H}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix}=\begin{pmatrix}\mathbb{H}_{11} & \mathbb{H}_{12}\\ \mathbb{H}_{21} & \mathbb{H}_{22}\end{pmatrix}\begin{pmatrix}\psi_-\otimes\psi_+\\ \psi_+\otimes\psi_-\end{pmatrix}$$

$$\mathbb{H}\begin{pmatrix}a\\ b\end{pmatrix}z_{12}^n=\mathbb{H}\left[\frac{a+b}{2}\begin{pmatrix}1\\ 1\end{pmatrix}z_{12}^n+\frac{a-b}{2}\begin{pmatrix}1\\ -1\end{pmatrix}z_{12}^n\right]=\frac{a+b}{2}\mathbb{H}\varphi_n^++\frac{a-b}{2}\mathbb{H}\varphi_n^-$$
$$=\frac{a+b}{2}E(n)\varphi_n^++\frac{a-b}{2}E(n-1)\varphi_n^-=\begin{pmatrix}h_{11}(n) & h_{12}(n)\\ h_{21}(n) & h_{22}(n)\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}z_{12}^n$$

$$h_{11}(n)=\psi(n+2)+\psi(n+1)-2\psi(1)\qquad h_{12}(n)=\frac{1}{n+1}$$

$\mathcal{O}_1^{ij}(z_1,z_2)=\psi_-^i(z_1)\psi_+^j(z_2)$, $\mathcal{O}_2^{ij}(z_1,z_2)=\psi_+^i(z_1)\psi_-^i(z_2)$

$$[\mathbb{H}\mathcal{O}_1^{ij}](z_1,z_2)=-2t^a_{ii'}t^a_{jj'}\left\{\int_0^1\frac{d\alpha}{\alpha}[2\mathcal{O}_1^{i'j'}(z_1,z_2)-\mathcal{O}_1^{i'j'}(z_{12}^\alpha,z_2)-\bar{\alpha}\mathcal{O}_1^{i'j'}(z_1,z_{21}^\alpha)]\right.$$
$$\left.+\int_0^1 d\alpha\,\mathcal{O}_2^{i'j'}(z_{12}^\alpha,z_2)\right\}$$

What to do with $\mathbb{H}^{(2\to3)}$??

Does the conformal symmetry fix off-diagonal kernels ?

Yes, It does!

One can derive the set of equations which completely determine $\mathbb{H}^{(2\to3)}$ kernels.

Let us consider the operators

$$\mathcal{O}_1^{ij}(z_1,z_2)=\psi_-^i(z_1)\otimes\psi_+^j(z_2) \qquad \mathcal{O}_2^{ij}(z_1,z_2)=\psi_+^i(z_1)\otimes\psi_-^j(z_2)$$

$$\mathcal{O}_+^{ij}(z_1,z_2)=\psi_+^i(z_1)\otimes\psi_+^j(z_2) \qquad \mathcal{O}_f^{ija}(z_1,z_2,z_3)=\psi_+^i(z_1)\otimes\psi_+^j(z_2)\otimes\bar{f}_{++}^a(z_3)$$

What we are looking for is

$$[\mathcal{O}_k^{ij}(z_1,z_2)]_R\sim\frac{1}{\epsilon}[\mathbb{H}^{(k\to f)}\mathcal{O}_f]^{ij}(z_1,z_2) \tag{18}$$

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$[\mathcal{O}]'_R = [\mathcal{O}]_R - \mathcal{O}$

First step: Transverse derivative

$$\partial_{\mu\bar\lambda}\psi_+ = [D_{\mu\bar\lambda}\psi_+]^i + igt^b_{ii'}A^b_{\mu\bar\lambda}\psi^{i'}_+ = 2\partial_+\psi_- + igA_{\mu\bar\lambda}\psi_+ + \text{ EOM}\,,$$

$$\begin{aligned}\partial_{\mu\bar\lambda}[\psi_+(z_1)\otimes\psi_+(z_2)]'_R &= [\partial_{\mu\bar\lambda}\psi_+(z_1)\otimes\psi_+(z_2)]'_R + [\psi_+(z_1)\otimes\partial_{\mu\bar\lambda}\psi_+(z_2)]'_R \\ &= 2\partial_{z_1}[\psi_-(z_1)\otimes\psi_+(z_2)]'_R + 2\partial_{z_2}[\psi_+(z_1)\otimes\psi_-(z_2)]'_R \\ &+ \underline{ig[A_{\mu\bar\lambda}(z_1)\psi_+(z_1)\otimes\psi_+(z_2)]'_R} + ig[\psi_+(z_1)\otimes A_{\mu\bar\lambda}(z_2)\psi_+(z_2)]'_R + \text{EOM}. \qquad (19)\end{aligned}$$

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Second step: Get rid of $A-$fields

$$[\psi_+(z_1)\otimes\psi_+(z_2)]'_R\,A^b_{\mu\bar\lambda}(z_1) + \psi_+(z_1)\otimes[\psi_+(z_2)A^b_{\mu\bar\lambda}(z_1)]'_R + [\psi_+(z_1)A^b_{\mu\bar\lambda}(z_1)]'_R\otimes\psi_+(z_2)$$

$$\begin{aligned}[\psi_+(z_2)A^b_{\mu\bar\lambda}(z_1)]'_R &= [\psi_+(z_2)\left(A^b_{\mu\bar\lambda}(z_1) - A^b_{\mu\bar\lambda}(z_2)\right)]'_R + [\psi_+(z_2)A^b_{\mu\bar\lambda}(z_2)]'_R \\ &= -\,z_{12}(\mu\lambda)\int_0^1 d\tau\,[\psi_+(z_2)\bar f^b_{++}(z^\tau_{12})]'_R\end{aligned}$$

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$$[\mathbb{S}\varphi](z_1,z_2) = \int_{z_2}^{z_1} ds\,\varphi(z_1,z_2,s) = z_{12}\int_0^1 d\tau\,\varphi(z_1,z_2,z_{12}^{\tau})\,. \tag{20}$$

In this notation

$$\psi_+(z_1)^i \otimes [\psi_+(z_2)^j A^b_{\mu\bar\lambda}(z_1)]'_R = -(\mu\lambda)\frac{\alpha_s}{4\pi\epsilon}[\mathbb{S}\,\mathbb{H}_{23}\,\mathcal{O}_f^{ijb}](z_1,z_2) \tag{21}$$

$$\mathcal{O}_f^{ija}(z_1,z_2,z_3) = \psi_+^i(z_1)\otimes\psi_+^j(z_2)\otimes\bar f_{++}(z_3)$$

$$\begin{aligned}
&2\partial_1[\mathcal{O}_1(z_1,z_2)]'_R + 2\partial_2[\mathcal{O}_2(z_1,z_2)]'_R = \\
&\quad = \partial_{\mu\bar\lambda}[\mathcal{O}_+(z_1,z_2)]'_R - ig\Big(A^b_{\mu\bar\lambda}(z_1)(t^b\otimes I) + A^b_{\mu\bar\lambda}(z_2)(I\otimes t^b)\Big)[\mathcal{O}_+(z_1,z_2)]'_R \\
&\qquad + (\mu\lambda)\frac{\alpha_s}{4\pi\epsilon}\Big[\mathbb{S}\,((t^b\otimes I)\mathbb{H}_{23} - (I\otimes t^b)\mathbb{H}_{13})\mathcal{O}_f^b\Big](z_1,z_2)
\end{aligned} \tag{22}$$

$$[\Delta\varphi]^{ij}(z_1,z_2) = \Big\{(t^b_{ii'}\otimes I_{jj'})A^b_{\mu\bar\lambda}(z_1) + (I_{ii'}\otimes t^b_{jj'})A^b_{\mu\bar\lambda}(z_2)\Big\}\,\varphi^{i'j'}(z_1,z_2)\,. \tag{23}$$

$$\sim [\mathbb{H}_{12},\Delta]\mathcal{O}_+(z_1,z_2), \qquad [\mathcal{O}_+(z_1,z_2)]'_R \sim \frac{1}{\epsilon}\mathbb{H}_{12}\mathcal{O}_+(z_1,z_2)$$

$$\sum_{k=1}^{2} \partial_k[\mathbb{H}^{k\to f}\mathcal{O}_f](z_1,z_2) = \frac{1}{2} ig\Big\{[(\mathbb{H}_{12}\Delta - \Delta\mathbb{H}_{12})\mathcal{O}_+](z_1,z_2)$$
$$+ (\mu\lambda)\Big[\mathbb{S}\Big((t^b\otimes I)\mathbb{H}_{23} - (I\otimes t^b)\mathbb{H}_{13}\Big)\mathcal{O}^b_f\Big](z_1,z_2) \qquad (24)$$

$\mathbb{H}_{12} = (t^c\otimes t^c)\times\mathcal{H}_{12}$, $[t^b\otimes I + I\otimes t^b, t^c\otimes t^c] = 0.$ $[\mathbb{H}_{12}, \Delta(A = const)] = 0$

Third step: Insert the expressions for $\mathbb{H}_{12}$, $\mathbb{H}_{13}$, $\mathbb{H}_{23}$ and separate color structures

$$\sum_{k=1}^{2}[\partial_k\mathbb{H}^{k\to f}\mathcal{O}_f](z_1,z_2) = g(\mu\lambda)\sum_{i=1}^{3} C_i[(\mathbb{S}\mathcal{W}_i - \mathcal{T}_i)\mathcal{O}_f](z_1,z_2)\,, \qquad (25)$$

where C_i are the color structures:

$$C_1 = f^{bcd}(t^b\otimes t^c)\,, \qquad C_2 = i(t^b\otimes t^d t^b)\,, \qquad C_3 = -i(t^d t^b\otimes t^b)\,, \qquad (26)$$

$$\mathcal{W}_1 = \widehat{\mathcal{H}}_{23} + \widehat{\mathcal{H}}_{23} - \widehat{\mathcal{H}}_{12} - 2(\mathcal{H}^{+}_{23} + \mathcal{H}^{+}_{13})\,, \qquad \mathcal{W}_2 = 2\mathcal{H}^{-}_{23}\,, \qquad \mathcal{W}_3 = 2\mathcal{H}^{-}_{13} \tag{27}$$

and

$$\mathcal{T}_1 = \mathcal{V}_{13} + \mathcal{V}_{23}\,, \qquad \mathcal{T}_2 = \mathcal{V}_{23}\,, \qquad \mathcal{T}_2 = \mathcal{V}_{13}\,, \tag{28}$$

with

$$\begin{aligned} [\mathcal{V}_{13}\varphi](z_1, z_2) =& z_{12} \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta\, \frac{\bar\alpha}{\alpha}\, \varphi(z_{12}^{\alpha}, z_2, z_{21}^{\beta})\,, \\ [\mathcal{V}_{23}\varphi](z_1, z_2) =& z_{12} \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta\, \frac{\bar\alpha}{\alpha}\, \varphi(z_1, z_{21}^{\alpha}, z_{12}^{\beta})\,. \end{aligned} \tag{29}$$

$z_{12}^{\alpha} = z_1(1-\alpha) + z_2\alpha.$

$\mathbb{LHS} = \mathbb{RHS}$

This equation is not $SL(2,R)$ invariant!

$$S_{123}^{+,(j_1 j_2 j_3)} = S_1^{+,j_1} + S_2^{+,j_2} + S_3^{+,j_3} = \sum_{k=1}^{3} z_k^2 \partial_k + 2 j_k z_k \tag{30}$$

$$(\mathbb{LHS} - \mathbb{RHS}) S_{123}^{+,(1,1,3/2)} = S_{12}^{+,(1,1)} (\mathbb{LHS} - \mathbb{RHS}) + (\widetilde{\mathbb{LHS}} - \widetilde{\mathbb{RHS}})$$

$$\mathbb{LHS} = \sum_{k=1}^{2} \partial_k \mathbb{H}^{k \to f} = g(\mu\lambda) \sum_{i=1}^{3} C_i (\mathbb{S}\mathcal{W}_i - \mathcal{T}_i) = \mathbb{RHS}\,, \tag{31}$$

$$\widetilde{\mathbb{LHS}} = \sum_{k=1}^{2} \partial_k \, z_k \, \mathbb{H}^{k \to f} = g(\mu\lambda) \sum_{i=1}^{3} C_i (\mathbb{S} z_3 \mathcal{W}_i - \mathcal{T}_i z_3) = \widetilde{\mathbb{RHS}}\,, \tag{32}$$

$$\begin{aligned} \partial_1 z_{12}\mathbb{H}^{1\to f} + \mathbb{H}^{2\to f} &= g(\mu\lambda)\sum_{i=1}^{3} C_i(\mathbb{S}\,(z_3 - z_2)\mathbb{W}_i - \mathcal{T}_i z_3 + z_2\mathcal{T}_i) \equiv \sum_{i=1}^{3} C_i\mathcal{A}_i\,, \\ \partial_2 z_{21}\mathbb{H}^{2\to f} + \mathbb{H}^{1\to f} &= g(\mu\lambda)\sum_{i=1}^{3} C_i(\mathbb{S}\,(z_3 - z_1)\mathbb{W}_i - \mathcal{T}_i z_3 + z_1\mathcal{T}_i) \equiv \sum_{i=1}^{3} C_i\mathcal{B}_i\,. \end{aligned} \tag{33}$$

$$\begin{aligned} [\mathcal{A}_1\varphi](z_1,z_2) &= z_{12}^2\left(\int_0^1 d\beta\bar\beta\,\varphi(z_1,z_2,z_{12}^{\beta}) - \int_0^1 d\alpha\int_0^{\bar\alpha} d\beta\,\beta\,\varphi(z_1,z_{21}^{\alpha},z_{12}^{\beta})\right)\,, \\ [\mathcal{A}_2\varphi](z_1,z_2) &= z_{12}^2\int_0^1 d\alpha\int_{\bar\alpha}^1 d\beta\frac{\bar\alpha\bar\beta}{\alpha}\,\varphi(z_1,z_{21}^{\alpha},z_{12}^{\beta})\,, \\ [\mathcal{A}_3\varphi](z_1,z_2) &= z_{12}^2\int_0^1 d\alpha\int_{\bar\alpha}^1 d\beta\frac{\bar\alpha}{\alpha^2}(\bar\alpha-\beta)\,\varphi(z_{12}^{\alpha},z_2,z_{21}^{\beta})\,. \end{aligned} \tag{34}$$

$$[\mathbb{H}^{(1\to f)}\mathcal{O}_f](z_1,z_2) = z_{12}^2\Big\{ f^{abc} t^b \otimes t^c \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\beta\, \mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta) + i(t^a t^b)\otimes t^b \int_0^1 d\alpha \int_{\bar\alpha}^1 d\beta \frac{\bar\alpha\bar\beta}{\alpha}\,\mathcal{O}_f(z_{12}^\alpha, z_2, z_{21}^\beta)\Big\} \quad (35)$$

$\boxed{72/(P\times C - \text{symmetry}) \mapsto 16 \text{ kernels}}$

- The operator basis with good transformation properties is constructed.
- It is shown that the conformal symmetry fix the off-diagonal blocks in mixing matrix.
- The effective method for restoring the evolution kernels for higher twist operators is developed.