

***Symmetries of scattering amplitudes
in $\mathcal{N} = 4$ super-Yang-Mills theory***

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Based on work in collaboration with

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On-shell scattering amplitudes in $\mathcal{N} = 4$ SYM

✓ Scattering amplitudes in $\mathcal{N} = 4$ SYM

✗ Quantum numbers of on-shell states $|i\rangle = |p_i, h_i, a_i\rangle$:
momentum ($p_i^2 = 0$), helicity (h_i), color (a_i)

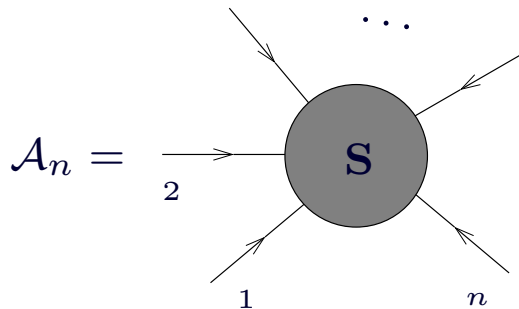
✗ IR divergences \mapsto dimensional regularization

$\mathcal{A}_n = \text{Div}(p_i, 1/\epsilon, \mu) \times \text{Fin}(p_i) \rightarrow$ **subject of this talk**

✗ Perturbative expansion in 't Hooft coupling

$a = g^2 N/8\pi^2$:

$$\mathcal{A}_n(p_i, h_i) = \mathcal{A}_{n;0} + a \sum_H \mathcal{A}_{n;1}^H M_{n;1}^H(p_i) + O(a^2)$$



✓ Simplest example: Maximally Helicity Violating (MHV) amplitudes, e.g. for gluons:

$(- - + \dots +)$, $(- + - + \dots +)$, etc.

Unique helicity structure (tree):

$$\mathcal{A}_n^{\text{MHV}}(p_1^-, p_2^-, p_3^+, \dots, p_n^+) = \mathcal{A}_{n;0}^{\text{MHV}} M_n^{\text{MHV}}(p_i), \quad M_n^{\text{MHV}} = 1 + a M_n^{(1)} + O(a^2)$$

✓ $\mathcal{N} = 4$ SYM is a (super)conformal theory \Rightarrow conformal symmetry of $\mathcal{A}_n(p_i)$? Two problems:

(i) Conformal boosts act non-locally (2nd-order differential operators)

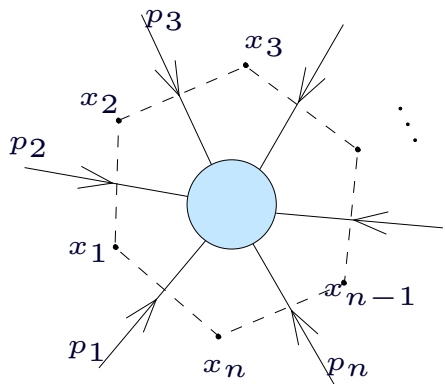
[Witten'03]

(ii) IR divergences break conformal symmetry - how?

Dual conformal symmetry I

- ✓ Hidden symmetry of \mathcal{A}_n of dynamical origin
- ✓ Linear action on the particle momenta in dual space

[Broadhurst'95],[Drummond,Henn,Smirnov,ES'06]



- ✗ $p_i = x_i - x_{i+1} \equiv x_i - x_{i+1} \Leftrightarrow \sum_i p_i = 0$ if $x_{n+1} \equiv x_1$
- ✗ $p_i^2 = 0 \Leftrightarrow x_i^2 - x_{i+1}^2 = 0$
- ✗ Conformal group $SO(4, 2)$ acting on the dual coordinates $x_i \Rightarrow$ dual conformal symmetry.

- ✓ Example: MHV amplitudes

$$\ln M_n^{\text{MHV}} = \ln Z_n(x_i, \epsilon, \mu) + \ln F_n(x_i) + O(\epsilon), \quad \ln Z_n = \sum_{l \geq 1} a^l \sum_{i=1}^n (-x_{i,i+2}^2 \mu^2)^{l\epsilon} \left(\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon} \right)$$

- ✓ Duality MHV amplitude/Wilson loop

[Alday,Maldacena'07], [Drummond,Korchemsky,ES'07], [Brandhuber,Heslop,Travaglini'07]

$$\ln F_n = \ln \langle 0 | \text{Tr P exp} \left(ig \oint_{C_n} dx^\mu A_\mu(x) \right) | 0 \rangle + \text{const} + O(\epsilon)$$

WL has conformal invariance in dual space \Rightarrow Anomalous CWI :

[Drummond,Henn,Korchemsky, ES'07]

$$K^\mu \ln F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^n \ln \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} x_{i,i+1}^\nu \Rightarrow \text{Fixes } \ln F_n \text{ for } n = 4, 5 \text{ but not for } n \geq 6$$

Dual conformal symmetry II

- ✓ Can we generalize dual conformal symmetry to non-MHV amplitudes?
Need to study the helicity structures and the loop corrections

- ✓ Spinor helicity formalism: commuting spinors λ^α (helicity -1/2), $\tilde{\lambda}^{\dot{\alpha}}$ (helicity 1/2) [Xu,Zhang,Chang'87]

$$p_i^2 = 0 \Leftrightarrow p_i^{\alpha\dot{\alpha}} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$$

- ✓ Simplest case: MHV tree level

[Parke, Taylor'86]

$$\mathcal{A}_{n;0}^{\text{MHV}}(\dots i^- \dots j^- \dots) = \delta^{(4)}\left(\sum_{k=1}^n p_k\right) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle}, \quad \langle ij \rangle = -\langle ji \rangle = \epsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta}$$

Is it dual conformal?

- ✓ Dual conformal transformations of spinors

[Drummond,Henn,Korchemsky, ES'08]

- ✗ Conformal group = Poincaré + inversion:

$$I[x^\mu] = \frac{x^\mu}{x^2} \equiv x^{-1}, \quad I[x_i - x_j] = x_i^{-1} (x_i - x_j) x_j^{-1}$$

- ✗ Transforming spinors: $p_i^{\alpha\dot{\alpha}} = (x_i - x_{i+1})^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}} \Rightarrow$

$$I[\lambda_i^\alpha] = \frac{\lambda_i^\alpha (x_i)_{\alpha\dot{\alpha}}}{x_i^2} = \lambda_i^\alpha \frac{(x_{i+1})_{\alpha\dot{\alpha}}}{x_i^2} \Rightarrow I[\langle i i+1 \rangle] = \frac{\langle i i+1 \rangle}{x_i^2}$$

Dual conformal symmetry III

✓ What about the momentum conservation delta function $\delta^{(4)}(\sum_{i=1}^n p_i)$?

✗ Cyclic symmetry : $\sum_{i=1}^n p_i = 0 \Leftrightarrow \sum_{i=1}^n (x_i - x_{i+1}) = 0$ iff $x_{n+1} \equiv x_1$

✗ Relax cyclicity, $x_1 \neq x_{n+1}$, and then impose it by

$$\delta^{(4)}(x_1 - x_{n+1}) \rightarrow \text{manifestly dual conformal}$$

✓ Split-helicity tree amplitudes: all negative-helicity gluons appear contiguously

✗ Known explicitly from recursion relations

[Britto, Cachazo, Feng, Roiban, Spradlin, Volovich, Witten]

✗ Example: split-helicity MHV tree amplitude – **manifestly dual conformal!**

$$\mathcal{A}_n^{\text{MHV}}(- - + \dots +) = \delta^{(4)}(x_1 - x_{n+1}) \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle}$$

✗ Non-split-helicity amplitudes are dual conformal not on their own, but as parts of **superamplitudes**

✗ Superamplitudes in **dual superspace** exhibit **dual superconformal symmetry**.

Superamplitudes in on-shell superspace I

- ✓ $\mathcal{N} = 4$ gluon supermultiplet \rightarrow PCT self-conjugate \rightarrow holomorphic (chiral) description

$$\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \eta^A \eta^B S_{AB}(p) + \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p)$$

η^A ($SU(4)$ index $A = 1 \dots 4$, helicity $1/2$) are Grassmann variables of **on-shell superspace**

- ✓ Superamplitudes $\mathcal{A}_n(\Phi(1) \dots \Phi(n)) =$ expansion in powers of η_i^A

- ✓ Example: **Nair's** description of tree MHV amplitudes

[Nair'88]

$$\mathcal{A}_n^{\text{MHV}} = \frac{\delta^{(4)}(\sum_{i=1}^n p_i) \delta^{(8)}(\sum_{j=1}^n \lambda_{j\alpha} \eta_j^A)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} = \frac{\delta^{(4)}(\sum_{i=1}^n p_i)}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} \left(\langle 12 \rangle^4 \eta_1^4 \eta_2^4 \eta_3^0 \dots \eta_n^0 + \dots \right)$$

- ✓ On-shell $\mathcal{N} = 4$ supersymmetry

✗ super-Poincaré

$$q_\alpha^A = \lambda_\alpha \eta^A, \quad \bar{q}_{A\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \eta^A}, \quad p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} : \quad \{q_\alpha^A, \bar{q}_{B\dot{\alpha}}\} = \delta_B^A p_{\alpha\dot{\alpha}}$$

✗ super-conformal \rightarrow **non-local** (2nd-order)

$$s_A^\alpha = \frac{\partial^2}{\partial \lambda_\alpha \partial \eta^A}, \quad \bar{s}_{\dot{\alpha}}^A = \eta^A \frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}}, \quad k_{\alpha\dot{\alpha}} = \frac{\partial^2}{\partial \lambda^\alpha \partial \tilde{\lambda}_{\dot{\alpha}}} : \quad \{s_A^\alpha, \bar{s}_{\dot{\alpha}}^B\} = \delta_A^B k_{\alpha\dot{\alpha}}$$

Superamplitudes in on-shell superspace II

- ✓ Invariance of the superamplitude: $p_{\alpha\dot{\alpha}} \mathcal{A}_n = q_{\alpha}^A \mathcal{A}_n = 0 \Rightarrow$

$$\mathcal{A}_n(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \delta^{(8)}\left(\sum_{j=1}^n \lambda_j \eta_j\right) \left[\mathcal{A}_n^{(0)} + \mathcal{A}_n^{(4)} + \dots + \mathcal{A}_n^{(4n-16)} \right]$$

- ✓ $\mathcal{A}_n^{(4k)}(\eta)$ – homogeneous polynomials in η of degree $4k$:
 $k = 0 \rightarrow$ MHV, $k = 1 \rightarrow$ Next-to-MHV, \dots , $k = n - 4 \rightarrow \overline{\text{MHV}}$
- ✓ Simplest case – All-order MHV superamplitude:

$$\mathcal{A}_n^{\text{MHV}}(\lambda, \tilde{\lambda}, \eta) = \delta^{(4)}\left(\sum_{i=1}^n p_i\right) \delta^{(8)}\left(\sum_{j=1}^n \lambda_j \alpha \eta_j^A\right) \left[\frac{M_n(p)}{\langle 1 2 \rangle \langle 2 3 \rangle \dots \langle n 1 \rangle} \right]$$

- ✓ Define ‘ratio’ R = general/MHV superamplitude:

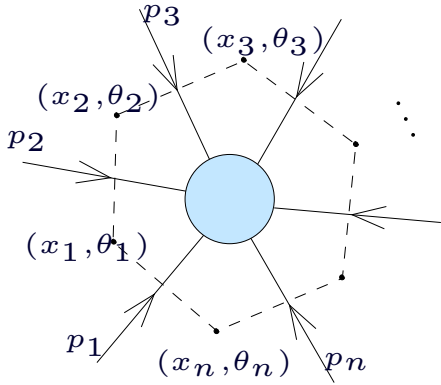
$$\mathcal{A}_n = \mathcal{A}_n^{\text{MHV}} \times \left[R_n(\lambda, \tilde{\lambda}, \eta) + O(\epsilon) \right] = \mathcal{A}_n^{\text{MHV}} \left[1 + R_n^{(4)} + \dots + R_n^{(4n-16)} + O(\epsilon) \right]$$

$R_n^{(4k)}$: **finite** homogeneous polynomials in $\eta \rightarrow$ helicity structures and loop corrections

- ✓ Conjecture: all $R_n^{(4k)}$ are **exactly dual conformal**. Conformal anomaly in IR divergent MHV factor.
- ✓ Ordinary superconformal at tree level, but broken by IR divergences at loop level - **how?**

Dual $\mathcal{N} = 4$ superconformal symmetry

✓ Chiral dual superspace $(x_{\alpha\dot{\alpha}}, \theta_{\alpha}^A, \lambda_{\alpha})$:



$$\times p = \sum_{i=1}^n p_i = 0 \rightarrow p_i = x_i - x_{i+1}, \quad x_{n+1} = x_1$$

$$\times q = \sum_{i=1}^n \lambda_i \eta_i = 0 \rightarrow \lambda_{i\alpha} \eta_i^A = (\theta_i - \theta_{i+1})_{\alpha}^A, \quad \theta_{n+1} = \theta_1$$

✓ Dual $\mathcal{N} = 4$ superconformal symmetry in dual superspace

✗ $\mathcal{N} = 4$ super-Poincaré algebra

$$Q_{A\alpha} = \sum_{i=1}^n \frac{\partial}{\partial \theta_i^A \alpha}, \quad \bar{Q}_{\dot{\alpha}}^A = \sum_{i=1}^n \theta_i^A \alpha \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}, \quad P_{\alpha\dot{\alpha}} = \sum_{i=1}^n \frac{\partial}{\partial x_i^{\dot{\alpha}\alpha}}; \quad \{Q_{A\alpha}, \bar{Q}_{\dot{\alpha}}^B\} = \delta_A^B P_{\alpha\dot{\alpha}}$$

✗ Conformal inversion: $I[x_i] = x_i^{-1}$, $I[\theta_i] = \theta_i x_i^{-1}$, $I[\lambda_i] = \lambda_i x_i^{-1}$

✗ From Poincaré to conformal supersymmetry:

■ Conformal boosts: $K = IPI$

■ Special conformal supersymmetry: $S = I\bar{Q}I$, $\bar{S} = IQI \equiv \bar{q}$

Dual superconformal symmetry: MHV superamplitudes

- ✓ Impose cyclicity, $x_{n+1} = x_1$, $\theta_{n+1} = \theta_1$, through delta functions.
- ✓ MHV superamplitude in dual superspace

$$\mathcal{A}_n^{\text{MHV}}(x, \theta, \lambda) = \frac{\delta^{(4)}(x_1 - x_{n+1}) \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} [1 + aM_n^{(1)}(x_{ij}) + O(a^2)]$$

- ✗ **Tree** – manifestly dual **superconformal** covariant.
- ✗ **Loops** – IR divergent factor $M_n^{(1)}(x_{ij})$ satisfies anomalous dual conformal Ward identity
- ✗ **Loops** – dual supersymmetry \bar{Q} broken by $M_n(x_{ij})$
- ✓ Multi-particle discontinuity (branch cut)

$$\text{Disc}_{x_{1,j+1}^2} M_n^{(1)} = \ln \left[\frac{(1 - u_{1,n,j,j+1})(1 - u_{1,2,j+2,j+1})}{(1 - u_{1,2,j,j+1})(1 - u_{1,n,j+2,j+1})} \right], \quad u_{ijkl} = \frac{x_{il}^2 x_{jk}^2}{x_{ik}^2 x_{jl}^2}$$

IR finite and **dual conformal**, but not **dual supersymmetric**, why?

Dual superconformal symmetry: Holomorphic anomaly

- ✓ $\mathcal{A}_n^{\text{MHV}}$ is naively holomorphic in λ , but in reality

[Cachazo,Svrcek,Witten'04]

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{\alpha}}} \frac{1}{\langle \lambda \chi \rangle} = 2\pi \tilde{\chi}_{\dot{\alpha}} \delta(\langle \lambda, \chi \rangle) \delta([\tilde{\lambda}, \tilde{\chi}]) \rightarrow \text{holomorphic anomaly}$$

- ✓ Important when loops are made from trees via unitarity

[Cachazo,'04],[Bena,Bern,Kosower,Roiban'04]

$$\text{Disc}_{s_{1\dots j}} \mathcal{A}_n^{\text{MHV};1} = \mathcal{A}^{\text{MHV};0}(-\ell_1, 1, \dots, j, -\ell_2) \star \mathcal{A}^{\text{MHV};0}(\ell_2, j+1, \dots, n, \ell_1),$$

- ✓ Difference between K and \bar{Q}

$$K_{\alpha\dot{\alpha}} \frac{1}{\langle i i+1 \rangle} = \sum_k (x_{k+1})_{\alpha\dot{\beta}} \tilde{\lambda}_{k\dot{\alpha}} \partial_k^{\dot{\beta}} \frac{1}{\langle i i+1 \rangle} = 0$$

$$\bar{Q}_{\dot{\alpha}}^A \frac{1}{\langle i i+1 \rangle} = \sum_k \eta_k^A \partial_{k\dot{\alpha}} \frac{1}{\langle i i+1 \rangle} = 2\pi \delta(\langle i i+1 \rangle) \delta([i i+1]) \left(\eta_i^A \tilde{\lambda}_{i+1\dot{\alpha}} - \eta_{i+1}^A \tilde{\lambda}_{i\dot{\alpha}} \right)$$

- ✓ We have no way to control the \bar{Q} -anomaly:
 - ✗ The Wilson loop is only dual conformal, not **superconformal**
 - ✗ The Wilson loop knows nothing about helicity
 - ✗ If a dual supersymmetric model for amplitudes with helicity exists, can we imagine Poincaré SUSY breaking down ???

Dual superconformal symmetry: NMHV superamplitudes

✓ General superamplitude: $\mathcal{A}_n = \mathcal{A}_n^{\text{MHV}}(a, 1/\epsilon) \left[1 + R_n^{(4)} + \dots + R_n^{(4n-16)} + O(\epsilon) \right]$

✓ Conjecture: $R_n^{(4)}$ are finite dual (super)conformal invariants

[Drummond,Henn,Korchemsky,ES'08]

$$R_n^{(4)} = \sum_{p,q,r=1}^n c_{pqr} R_{pqr} [1 + aM_{pqr}^{(1)}(x_{ij}) + O(a^2)]$$

✗ dual conformal invariant with matching conformal and helicity weights

$$R_{pqr} = \frac{\langle q-1 q \rangle \langle r-1 r \rangle \delta^{(4)}(\langle p|x_{pq}x_{qr}|\theta_{rp}\rangle + \langle p|x_{pr}x_{rq}|\theta_{qp}\rangle)}{x_{qr}^2 \langle p|x_{pr}x_{r q-1}|q-1\rangle \langle p|x_{pr}x_{r q}|q\rangle \langle p|x_{pq}x_{q r-1}|r-1\rangle \langle p|x_{pq}x_{q r}|r\rangle}$$

✗ dual conformal invariant $M_{pqr}^{(1)}$, made of finite combinations of one-loop scalar box integrals

✗ All coefficients $c_{pqr} = 1$, why?

✓ Each term in the sum is both ordinary & dual superconformal invariant for arbitrary c_{pqr}

✓ Freedom fixed by the analytic properties of the amplitude

[Korchemsky,ES'09]

✗ Absence of spurious singularities at $\langle p|x_{pr}x_{r q}|q\rangle = 0$, etc.

✗ Collinear limit

$$\mathcal{A}_n^{\text{tree}}(\dots, i, i+1, \dots) \xrightarrow{i \parallel i+1} \frac{1}{\sqrt{z(1-z)} \langle i i+1 \rangle} \mathcal{A}_{n-1}^{\text{tree}}(\dots, \ell, \dots).$$

Conclusions and outlook

- ✓ Dual superconformal symmetry is a universal feature of $\mathcal{N} = 4$ scattering amplitudes
- ✓ Its field-theory origin is unknown (dynamical). Recent explanation from string theory.
[Berkovits, Maldacena'08], [Beisert,Ricci,Tseytlin'08]
- ✓ The closure of ordinary & dual superconformal symmetries is an infinite-dimensional Yangian \rightarrow integrability?
[Drummond,Henn,Plefka'09]
- ✓ Too early to say. We see that the symmetries do not completely fix even the tree. [Korchemsky,ES'09]
- ✓ Most of these symmetries are anomalous at loop level \rightarrow useless unless we can control the breaking.
- ✓ The MHV/Wilson loop duality does not see the helicity structure. Need to supersymmetrize the WL and test if it is dual to non-MHV superamplitudes.