TMD:

operator definition, factorization and renormalization

Chapter I. light-cone gauge; one-gluon exchanges

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1. **Chapter I.**

- Meet TMD: introduction; theoretical, phenomenological and experimental challenges
- TMD: generalization of the integrated collinear PDFs (**?**)
- QCD factorization within the TMD approach: state-of-the-art
- Operator definition of TMDs: renormalization properties; extra divergences; gauge invariance
- EXAMPLE: calculation of the LO anomalous dimensions of TMDs in the light-cone gauge
- Evolution equations for TMDs

2. **Chapter II.**

- Open problems

SEMI – INCLUSIVE PROCESSES

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \to \mathbf{H_2}(\mathbf{P}') + \mathcal{X}$$

Sensitivity to the intrinsic transverse momentum of partons

$\longrightarrow$ `Collins: PRL 42 (1979) 291; Soper: PRL 43 (1979) 1847`

$$P_{a/A}(x,\mu) \to \mathcal{P}_{a/A}(x,\boldsymbol{k},\mu,\zeta)$$

- depends on how fast the hadrons is moving (rapidity): ζ!
- collinear PDFs (are expected to) restore after $\boldsymbol{k}_\perp$-integration

$$P_{a/A}(x,\mu) \sim \int dk_\perp \, \mathcal{P}_{a/A}(x,\boldsymbol{k},\mu,\zeta)$$

$\longrightarrow$ Talks by Mulders, Collins

CURRENT and PLANNED "TMD" EXPERIMENTS

- **SIDIS process** $lH^{\uparrow} \to l'hX$**:** HERMES, COMPASS, JLab, EIC. To be studied: Sivers, Collins, transversity, Boer-Mulders, unpolarized X-sections, etc.

- **DY process** $H_1^{(\uparrow)} H_2^{\uparrow} \to l^+ l^- X$ **:** COMPASS, PAX, GSI, RHIC. To be studied: distribution functions, transversity.

- **Hadron collisions** $H_1^{(\uparrow)} H_2^{\uparrow} \to l^+ l^- X$ **:** RHIC

- $e^+ e^- \to h_1 h_2 X$**:** BELLE, BaBar

QCD FACTORIZATION within the TMD approach

$\longrightarrow$ `Collins, Soper: NPB (1981);`
$\longrightarrow$ `Collins, Metz: PRL (2004);`
$\longrightarrow$ `Ji, Ma, Yuan: PRD (2005);`
$\longrightarrow$ `Bacchetta, et al: PRD (2005), EPJC (2006)`

Standard factorization expected:

$$F(x_B, z_h, \boldsymbol{P}_{h\perp}, Q^2) = \sum_i e_i^2 \cdot H \otimes \mathcal{F}_D \otimes \mathcal{F}_F \otimes S$$

- Extra (rapidity) divergences;

- Complicated structure of gauge links: non-universality (generalized factorization):
  $\longrightarrow$ `Bacchetta, Bomhof, Mulders, Pijlman`

- Even generalized factorization may fail: counter-examples have been given
  $\longrightarrow$ `Collins, Qiu; Mulders, Rogers`

PROBLEMS of OPERATOR DEFINITION of TMD

"Naive" definition:

$$\mathcal{Q}_i(x, \boldsymbol{k}_\perp) = \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \; \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} .$$

$$\cdot \langle P, S | \bar{\psi}_i(\xi^-, \xi_\perp) [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp;]^\dagger_{[n]} [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger_{\boldsymbol{l}} \cdot$$

$$\cdot \, \gamma^+ \, \cdot$$

$$\cdot [\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{\boldsymbol{l}} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_{[n]} \psi_i(0^-, 0_\perp) | P, S \rangle \; |_{\xi^+ = 0}$$

Formally:

$$\int d^2k_\perp \mathcal{Q}_i(x, \boldsymbol{k}_\perp) = Q_i(x)$$

PROBLEMS of OPERATOR DEFINITION of TMD

1. gauge invariance: in the light-cone gauge, dependence on the pole prescription in the gluon propagator remains

2. extra (rapidity) divergences associated with the features of the light-cone gauge, or the light-like Wilson lines (*in the integrated case, these divergences cancel*)

3. reduction to the integrated case: formal integration doesn't produce correct result because of additional uncanceled UV divergences

CLASSIFICATION of SINGULARITIES

1. $\sim \frac{1}{\varepsilon}$ poles, usual UV-singularities: removed by the standard $R-$operation and are controlled by renormalization-group evolution equations (DGLAP in integrated case)

2. pure rapidity divergences: give rise to logarithmic and double-logarithmic terms of the form $\sim \ln \eta \, , \, \ln^2 \eta$; have to be resumed

3. overlapping divergences: contain both UV and soft singularities simultaneously $\sim \frac{1}{\varepsilon} \ln \eta$; highly undesirable—depend on the parameters of the chosen gauge; prevents the removal of *all* UV-singularities by the standard $R-$procedure; a special *generalized* renormalization procedure is needed

EXAMPLE: three different definitions of a unintegrated quark distribution:

A . *pure light-cone* $\mathcal{F}_{[n]}$: $\left[\, n^2 = 0 \;,\; n^+ = 0 \;,\; \boldsymbol{n}_\perp = 0 \right]$

$$\mathcal{F}_{[n]}\,(x, \boldsymbol{k}_\perp; \mu, \eta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \cdot \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]_n^\dagger [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]_{\boldsymbol{l}}^\dagger \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \mathbf{0}_\perp]_{\boldsymbol{l}} [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp]_n\, \psi(0^-, \mathbf{0}_\perp)\,|\, h \rangle$$

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A. *pure light-cone* $\mathcal{F}_{[n]}$: $\left[\, n^2 = 0 \;,\; n^+ = 0 \;,\; \boldsymbol{n}_\perp = 0 \right]$

B. *off-light-cone* $\mathcal{F}_{[v]}$: $\left[\, v^2 > 0 \;,\; v^- \gg v^+ \;,\; \boldsymbol{v}_\perp = 0 \right]$, $\zeta = \frac{4(P\cdot v)^2}{v^2}$

$$\mathcal{F}_{[v]}\,(x, \boldsymbol{k}_\perp; \mu, \zeta) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h\, |\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\; [\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]_v^\dagger [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]_{\boldsymbol{l}}^\dagger \gamma^+$$

$$[\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \boldsymbol{0}_\perp]_{\boldsymbol{l}} [\infty^-, \boldsymbol{0}_\perp; 0^-, \boldsymbol{0}_\perp]_v\, \psi(0^-, \boldsymbol{0}_\perp)\,|\, h\rangle$$

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Γ. *direct link* $\mathcal{F}_{[v_0]}$: $\left[\, v_0^2 = \boldsymbol{v}_0^2 < 0 \ , \ v^+ = 0 \ , \ , \ \zeta = \frac{4(P\cdot v_0)^2}{v_0^2} \right]$

$$\mathcal{F}_{[v_0]}\,(x, \boldsymbol{k}_\perp; \mu, \zeta_0) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2}\, \mathrm{e}^{-ik\xi} \langle\, h \,|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\, \gamma^+ \, [\xi^-, \boldsymbol{\xi}_\perp; 0^-, \mathbf{0}_\perp]_{v_0}\, \psi(0^-, \mathbf{0}_\perp)\,|\, h\rangle$$

ONE – GLUON ORDER ANOMALOUS DIMENSIONS

→ `ICh, Stefanis:  2008,2009`

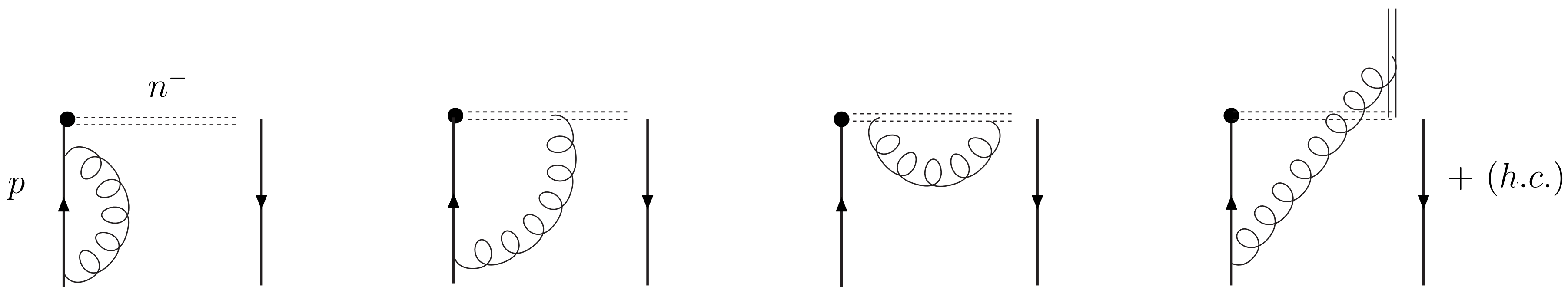
Tree approximation (distribution of *quark in a quark*):

$$\mathcal{F}^{(0)}(x, \boldsymbol{k}_\perp) =$$

$$= \frac{1}{2} \int \frac{d\xi^- d^2\xi_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle p | \bar{\psi}(\xi^-, \xi_\perp) \gamma^+ \psi(0^-, 0_\perp) | p \rangle =$$

$$= \delta(1-x)\delta^{(2)}(\boldsymbol{k}_\perp)$$

One-gluon exchanges, contributing to the UV-divergences:



(a) (b) (c) (d)

Source of the uncertainties and extra divergences: pole in the gluon propagator

$$D^{\mu\nu}_{\rm LC}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

Possible pole prescriptions:

$$d^{\mu\nu}_{\rm PV}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

$$d^{\mu\nu}_{\rm Adv/Ret}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

Mandelstam-Leibbrand prescription

$$\frac{1}{[q^+]_{\rm ML}} = \left\{ \begin{array}{c} \frac{1}{q^+ + i0q^-} \\ \\ \frac{q^-}{q^+q^- + i0} \end{array} \right.$$

UV divergent part reads:

$$\Sigma^{UV}_{\rm left}(p,\alpha_s;\epsilon) =$$

$$= -\frac{\alpha_s}{\pi} C_{\rm F}\,\frac{1}{\epsilon}\left[-\frac{3}{4} - \ln\frac{\eta}{p^+} + \frac{i\pi}{2} + i\pi\,C_\infty\right] + \alpha_s C_{\rm F}\;\frac{1}{\epsilon}\left[iC_\infty\right] =$$

$$= -\frac{\alpha_s}{\pi} C_{\rm F}\,\frac{1}{\epsilon}\left[-\frac{3}{4} - \ln\frac{\eta}{p^+} + \frac{i\pi}{2}\right]$$

prescription dependence is canceled

$$C_\infty = \begin{cases} 0\,, & {\rm Advanced}: \frac{1}{[q^+]} = \frac{1}{q^+ - i\eta} \\ -1\,, & {\rm Retarded}: \frac{1}{[q^+]} = \frac{1}{q^+ + i\eta} \\ -\frac{1}{2}\,, & {\rm PrincipalValue}: \frac{1}{[q^+]} = \frac{1}{2}\left(\frac{1}{q^+ - i\eta} + \frac{1}{q^+ + i\eta}\right) \end{cases}$$

Full UV divergent part:

$$\Sigma_{\rm tot}(p, \alpha_s(\mu); \epsilon) = \Sigma_{\rm left} + \Sigma_{\rm right} =$$

$$= -\frac{\alpha_s}{4\pi} C_{\rm F} \, \frac{2}{\epsilon} \left(-3 - 4 \ln \frac{\eta}{p^+} \right)$$

Dependence on η remains:

- gauge invariance is not complete
- AD doesn´t coincide with AD_{2q}

LO anomalous dimension is defined via the renormalization constant

$$\gamma = \frac{1}{2}\,\frac{1}{Z^{(1)}}\,\mu\frac{\partial\alpha_s(\mu)}{\partial\mu}\,\frac{\partial Z^{(1)}(\mu,\alpha_s(\mu);\epsilon)}{\partial\alpha_s}$$

$$\gamma_{\rm LC} = \gamma_{\rm smooth} - \delta\gamma \quad , \quad \gamma_{\rm smooth} = \frac{3}{4}\frac{\alpha_s}{\pi}C_{\rm F} + O(\alpha_s^2)$$

Defect of anomalous dimension

$$\delta\gamma = -\frac{\alpha_s}{\pi}C_{\rm F}\,\ln\frac{\eta}{p^+}$$

contains undesirable p^+-dependent term: must be removed by a consistent procedure.

$\delta\gamma$ is nothing else, but the cusp anomalous dimension:

$$p^+ = (p \cdot n^-) \sim \cosh \chi$$

defines an angle χ between the direction of the quark momentum p_μ and the light-like vector n^-. In the large χ limit:

$$\ln p^+ \to \chi, \ \chi \to \infty$$

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$$\ln p^+ \to \chi, \; \chi \to \infty$$

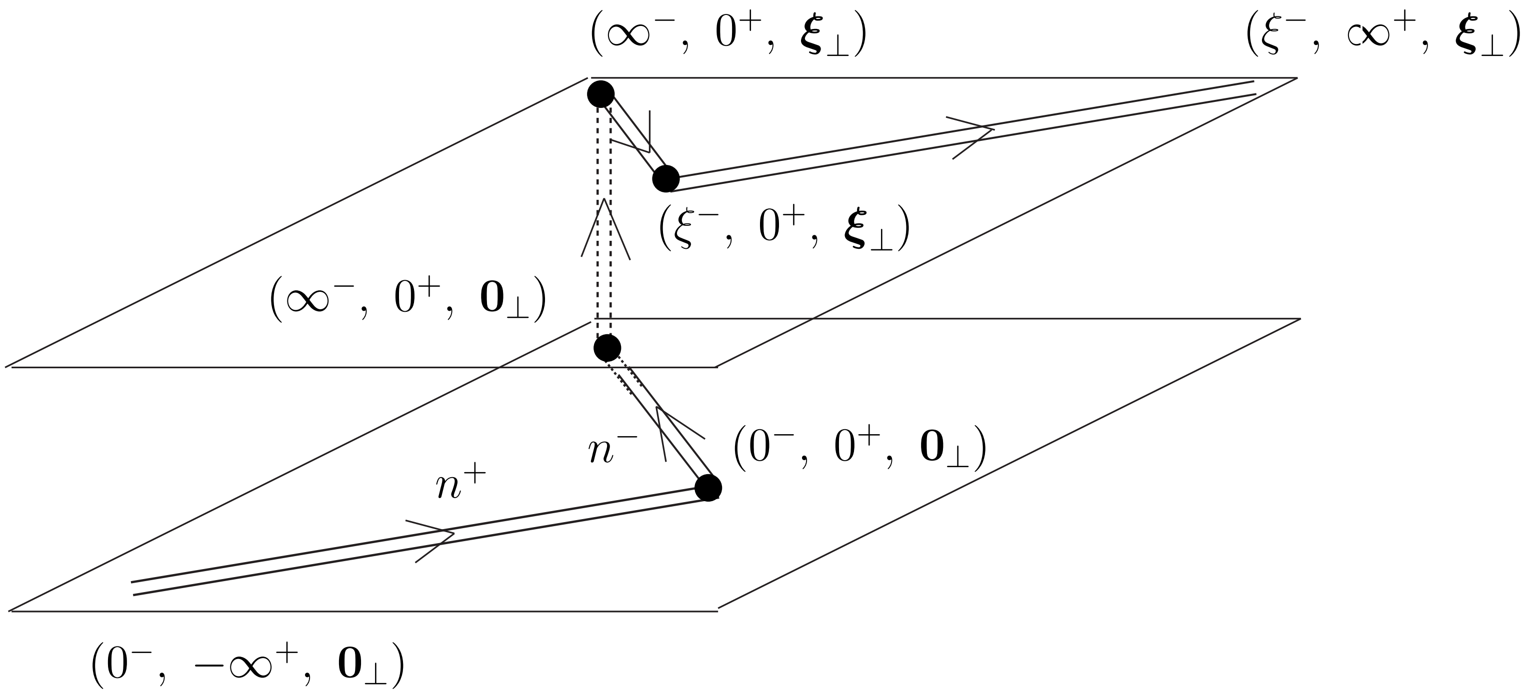
Renormalization of the Wilson operators with obstructions requires extra renormalization factor
$\to$ `Korchemsky, Radyushkin:  1987`

$$Z_\chi = \left[\langle 0 | \mathcal{P} \exp \left[ig \int_\chi d\zeta^\mu \; \hat{A}^a_\mu(\zeta) \right] | 0 \rangle \right]^{-1}$$

Generalized renormalization

$$\mathcal{O}_{\rm ren}(\chi, \ldots) = Z_\chi Z_{\rm R} \, \mathcal{O}(\chi, \ldots)$$

Space-time picture: integration trajectory for the additional cusp-dependent renormalization factor



One-gluon exchanges for the generalized multiplicative renormalization factor

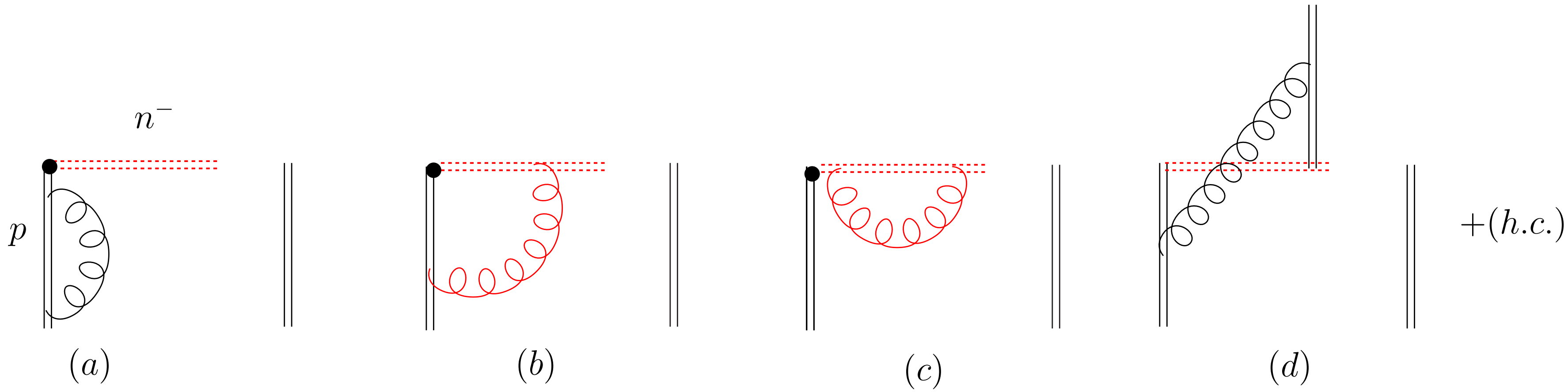
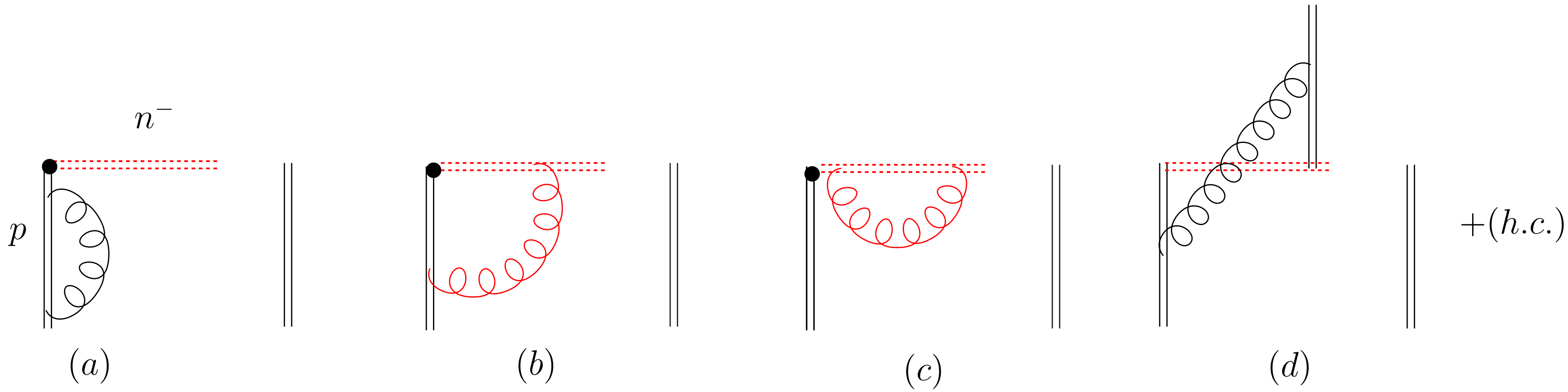


Figure 1: One-gluon exchanges for the generalized multiplicative renormalization factor

One-gluon exchanges for the generalized multiplicative renormalization factor



Generalized renormalization constant reads

$$\hat{Z}_{\rm mod} = 1 + \frac{\alpha_s}{4\pi} C_{\rm F} \, \frac{2}{\epsilon} \left(-3 - 4 \ln \frac{\eta}{p^+} + 4 \ln \frac{\eta}{p^+} \right) = 1 - \frac{3\alpha_s}{4\pi} C_{\rm F} \, \frac{2}{\epsilon}$$

so that

$$\frac{1}{2} \mu \frac{d}{d\mu} \ln \hat{Z}_{\rm mod}(\mu, \alpha_s, p^+) = \frac{3}{4} \frac{\alpha_s}{\pi} C_{\rm F} + O(\alpha_s^2)$$

Renormalization group equations

$\mathbf{B}, \mathbf{\Gamma}$. *off-light-cone* and *direct link*

$$\mu \frac{d}{d\mu} \mathcal{F}_{[v\,,\,v_0]} = \gamma_0\, \mathcal{F}_{[v\,,\,v_0]} \quad , \quad \gamma_0 = \frac{3}{4}\, \frac{\alpha_s C_{\rm F}}{\pi} + O(\alpha_s^2)$$

extraction of the soft factor

$$\mathcal{F}_{[\,v\,]} \to \mathcal{F}_{[\,v\,]} \cdot R_v^{-1}$$

$$\mu \frac{d}{d\mu} \left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right] = (\gamma_0 - \gamma_{\rm R}) \left[\mathcal{F}_{[\,v\,]} \cdot R_v^{-1}\right]$$

$\gamma_{\rm R}$—anomalous dimension of the soft factor

A. *pure light-cone*

$$\mu\frac{d}{d\mu}\,\mathcal{F}_{[\,n\,]} = (\gamma_0 - \gamma_{\rm cusp})\,\mathcal{F}_{[\,n\,]}$$

generalized renormalization “restores” the anomalous dimension

$$\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta)\cdot R^{-1}(\eta)$$

$$\mu\frac{d}{d\mu}\,\left[\mathcal{F}_{[\,n\,]}\cdot R_n^{-1}\right] = \gamma_0\,\left[\mathcal{F}_{[\,n\,]}\cdot R_n^{-1}\right]$$

A. *pure light-cone* with Mandelstam-Leibbrandt prescription

$$\mu\frac{d}{d\mu}\left[\mathcal{F}^{\mathrm{ML}}_{[n]} \cdot R_n^{-1}\right] = \mu\frac{d}{d\mu}\,\mathcal{F}^{\mathrm{ML}}_{[n]} =$$

$$\gamma_0\left[\mathcal{F}^{\mathrm{ML}}_{[n]} \cdot R_n^{-1}\right] = \gamma_0\,\mathcal{F}^{\mathrm{ML}}_{[n]}$$

anomalous dimension without light-cone artifacts from the very beginning!

Generalized definition of TMD:

$$\mathcal{F}_{[n]}\,(x, \boldsymbol{k}_\perp) \cdot R_n^{-1} =$$

$$\frac{1}{2} \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2\pi(2\pi)^2} \mathrm{e}^{-ik^+\xi^- + ik_\perp \cdot \xi_\perp} \langle h|\bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)[\xi^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\xi}_\perp]^\dagger_{[n]}$$

$$\times [\infty^-, \boldsymbol{\xi}_\perp; \infty^-, \boldsymbol{\infty}_\perp]^\dagger_{[\boldsymbol{l}]}\, \gamma^+\, [\infty^-, \boldsymbol{\infty}_\perp; \infty^-, \mathbf{0}_\perp]_{[\boldsymbol{l}]}\, [\infty^-, \mathbf{0}_\perp; 0^-, \mathbf{0}_\perp]_{[n]} \psi(0^-, \mathbf{0}_\perp)|P\rangle \times$$

$$\times \left[\text{SOFT FACTOR}\right]^{-1}$$

SOFT FACTOR =

$$\langle 0|\, \mathcal{P} \exp\left[ig \int_{\mathcal{C}_{\rm cusp}} d\zeta^\mu\, t^a A^a_\mu(\zeta)\right] \cdot \mathcal{P}^{-1} \exp\left[-\, ig \int_{\mathcal{C'}_{\rm cusp}} d\zeta^\mu\, t^a A^a_\mu(\xi + \zeta)\right] |0\rangle$$

- Collinear PDF from TMDs:

  Definition **A** reproduces the DGLAP evolution after integration:

$$\tilde{\int} d^2\boldsymbol{k}_\perp \, \mathcal{F}_{[n]}(x, \boldsymbol{k}_\perp, \mu) = F_{[n]}(x, \mu)$$

$$\mu \frac{d}{d\mu} F_{[n]} = \mathcal{K}_{\text{DGLAP}} \otimes F_{[n]}$$

  Definition **B** fails to reproduce the DGLAP evolution after integration:

$$\tilde{\int} d^2\boldsymbol{k}_\perp \, \mathcal{F}_{[v]}(x, \boldsymbol{k}_\perp, \mu) = F_{[v]}(x, \mu)$$

$$\mu \frac{d}{d\mu} F_{[v]} = \mathcal{K}_v \otimes F_{[v]} \quad , \quad \mathcal{K}_v \neq \mathcal{K}_{\text{DGLAP}}$$

$(NOT)\ FULL\ SET\ of\ the\ EVOLUTION\ EQUATIONS\ for\ TMD$

- UV-evolution (in the integrated case—DGLAP)

$$\mu\frac{d}{d\mu}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm UV}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

- rapidity evolution (Collins-Soper)

$$\zeta\frac{d}{d\zeta}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm CS}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

- BFKL evolution $\to$ Collins-Soper?

$$x\frac{d}{dx}\,\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta) = \mathcal{K}_{\rm BFKL}\otimes\mathcal{P}(x,\boldsymbol{k}_\perp,\mu,\zeta)$$

- ... ?

OPEN PROBLEMS

1. Status of the TMD Factorization:
   - No complete proof so far (all-order factorization proof for off-the-light-cone gauge links $\longrightarrow$ `Ji, Ma, Yuan: (2004)`; no explicit proof of a factorization theorem for definition $\mathbf{A}$!)
   - Several counter-examples have been given $\longrightarrow$ `Collins, Qiu: PRD (2007); Rogers, Mulders: (2010)`
2. Relationship between (unintegrated) TMDs and collinear (integrated) PDFs
   - Questionable with off-light-cone gauge links
   - Satisfactory in the light-cone gauge (LO)
3. Role of the soft factor: different within different frameworks?
4. Complete set of evolution equations: not known! *(UV, CS, BFKL...?)*

Field-theoretic analysis of

T M D

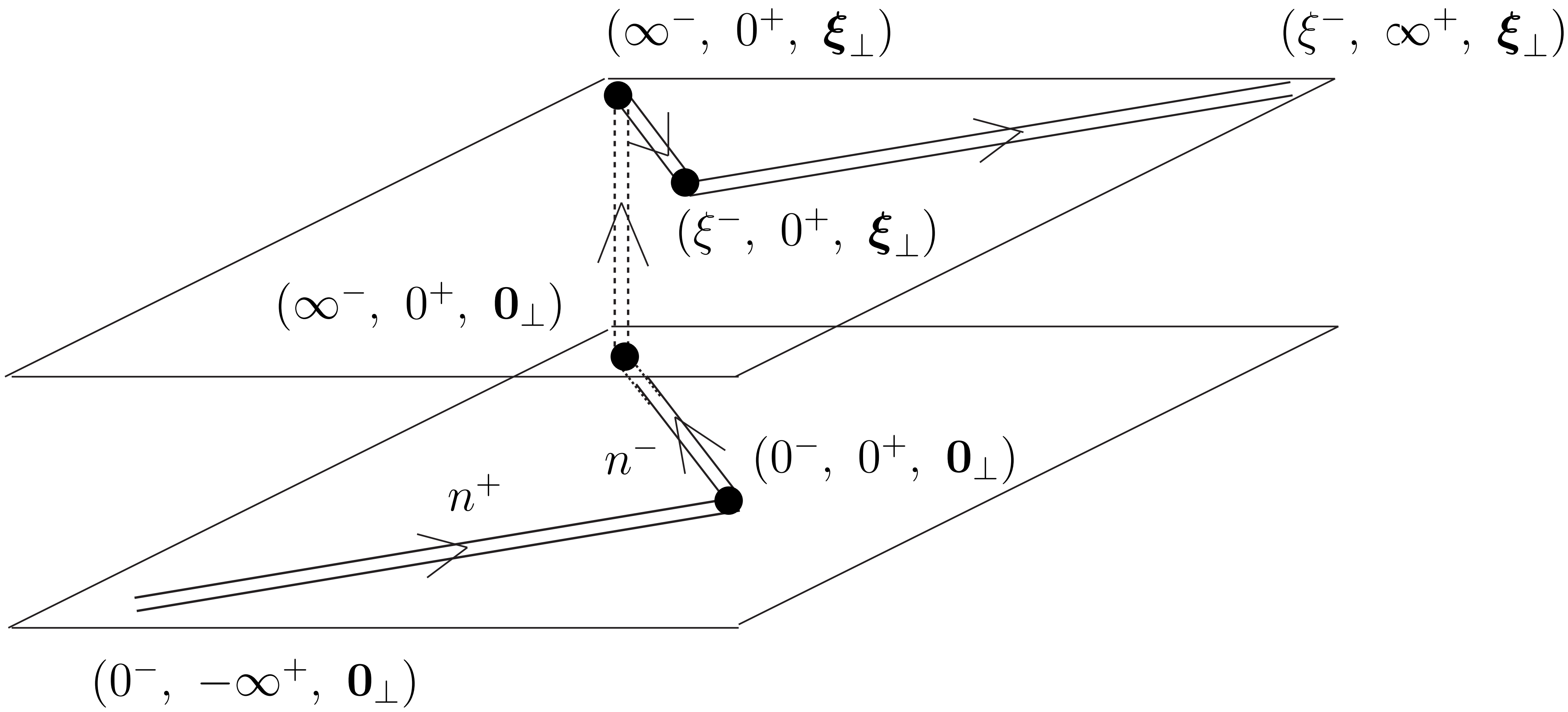
is in progress!

ICh., N. Stefanis:

- **AIP Conf. Proc.** 1105 (2009) 327
- **Mod. Phys. Lett A** 24 (2009) 2913
- **Phys. Rev. D** 80 (2009) 054008
- **Nucl. Phys. B** 802 (2008) 146
- **Phys. Rev. D** 77 (2008) 094001
- `arXiv: 0911.1031 [hep-ph]`
- `arXiv: 0811.4357 [hep-ph]`
- `arXiv: 0809.5235 [hep-ph]`
- `arXiv: 0809.1315 [hep-ph]`
- `arXiv: 0808.3390 [hep-ph]`

APPENDIX

integration contour for the soft factor



KINEMATICS

$$l^\mu = (l^+, l^-, \boldsymbol{l}_\perp)\ ,\ l^\pm = (l^0 \pm l^3)/\sqrt{2}\ ,\ l^2 = 2l^+l^- - \boldsymbol{l}_\perp^2$$

$$n^{*\mu} = \Omega(1, 1, \mathbf{0}_\perp)\ ,\ \ n^\mu = \frac{1}{2\Omega}(1, -1, \mathbf{0}_\perp)\ \ ,\ \ n^{*+} = \sqrt{2}\Omega$$

$$n^{*-} = 0\ ,\ n^+ = 0\ ,\ n^- = \frac{1}{\sqrt{2}\Omega}\ ,\ \ n^* n = 1\ ,\ (n^*)^2 = n^2 = 0$$

$$P^\mu = n^{*\mu} + \frac{M^2}{2} n^\mu\ ,\ \ P^2 = M^2$$

$$q^\mu = -x_N n^{*\mu} + \frac{Q^2}{2x_N} n^\mu \ \ \rightarrow\ \ q^+ = -\sqrt{2} x_N \Omega\ ,\ \ q^- = \frac{Q^2}{2\sqrt{2} x_N \Omega}$$

x_N — Nachtmann variable

$x_B = Q^2/2(Pq)$ — Bjorken variable

$$\sqrt{2}\Omega = P^+ \to x_B = \frac{x_N}{1 - \frac{M^2}{Q^2} x_N^2} = x_N + O\left(\frac{M^2}{Q^2}\right)$$

kinematical approximations are important!

`Collins, Rogers, Stasto: PRD (2008)`

→ **fully unintegrated** parton correlation functions

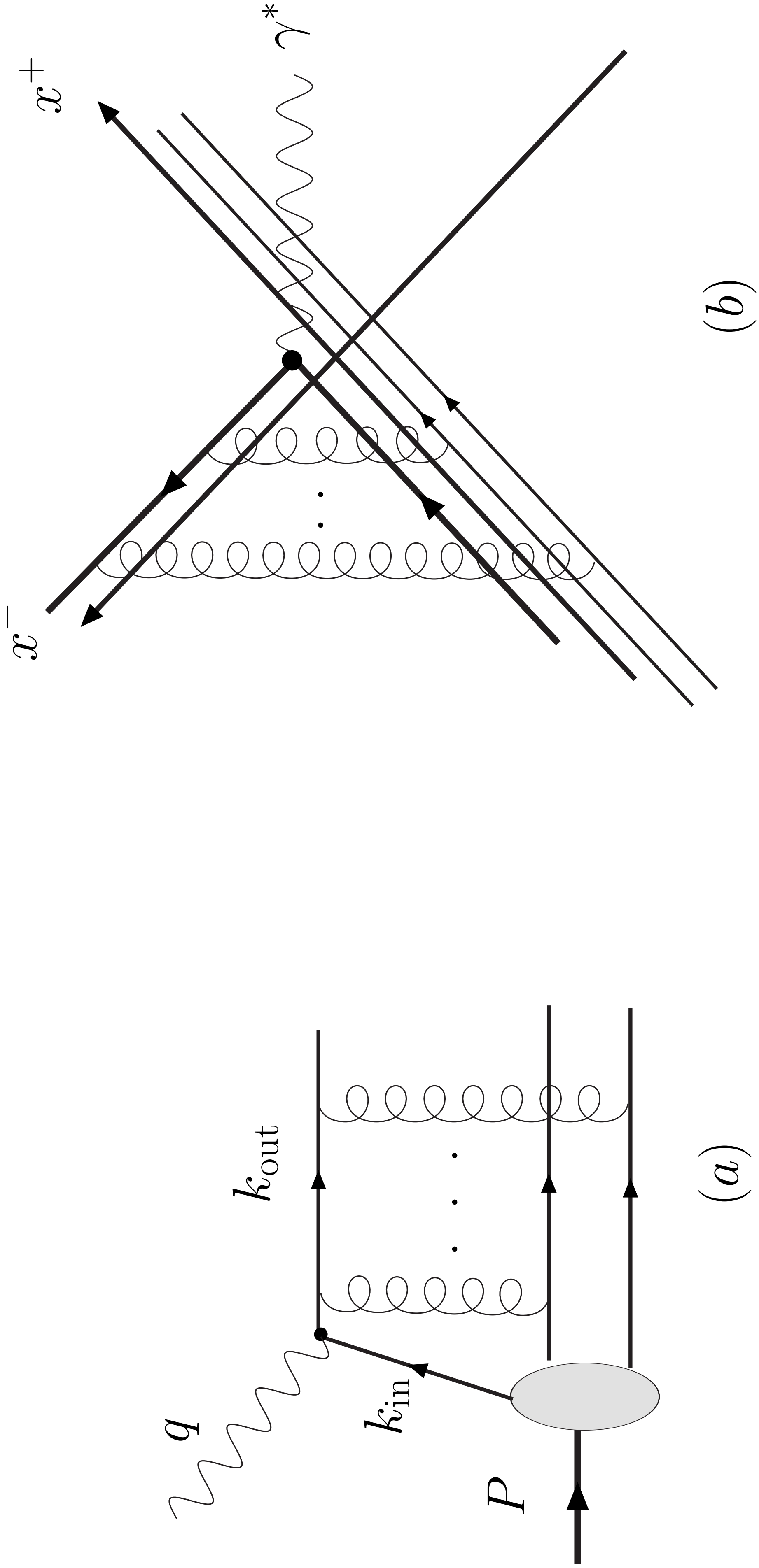
M^2/Q^2 corrections neglected

$$x_B \approx x_N$$

$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}_\perp\right) \quad , \quad q^\mu = \left(-x_B P^+, \frac{Q^2}{2x_B P^+}, \mathbf{0}_\perp\right)$$

$$P^+ \sim E_P = \text{hadron energy}$$

$$s \sim \frac{Q^2}{x_B}$$



x^+
γ^*
x^-
(b)
$k_{\rm out}$
q
$k_{\rm in}$
P
(a)

source of extra divergences: pole in the gluon propagator

$$D^{\mu\nu}_{\mathrm{LC}}(q) = \frac{1}{q^2}\left[g^{\mu\nu} - \frac{q^\mu n^{-\nu}}{[q^+]} - \frac{q^\nu n^{-\mu}}{[q^+]}\right]$$

q^--independent pole prescriptions:

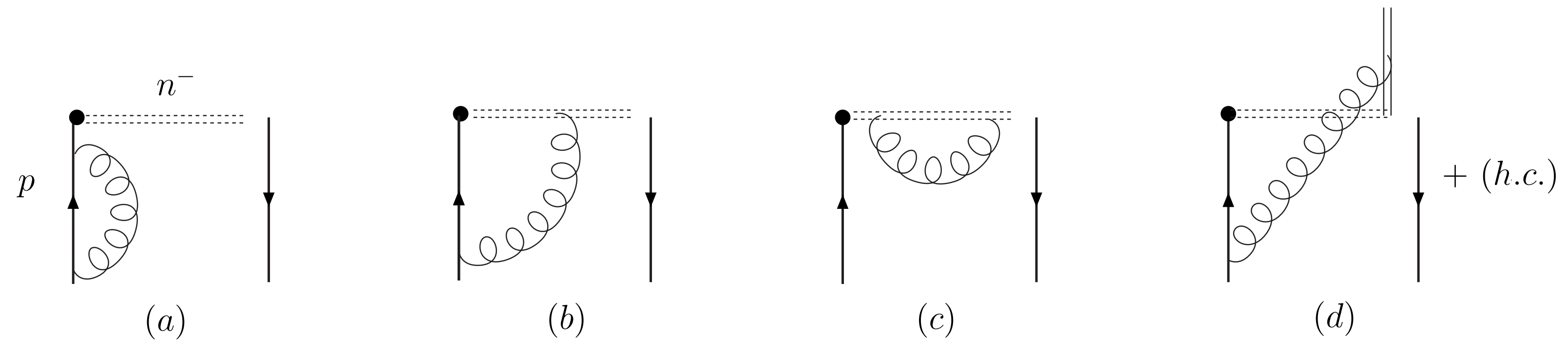
$$d^{\mu\nu}_{\mathrm{PV}}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{2}\left(\frac{1}{q^+ + i\eta} + \frac{1}{q^+ - i\eta}\right)$$

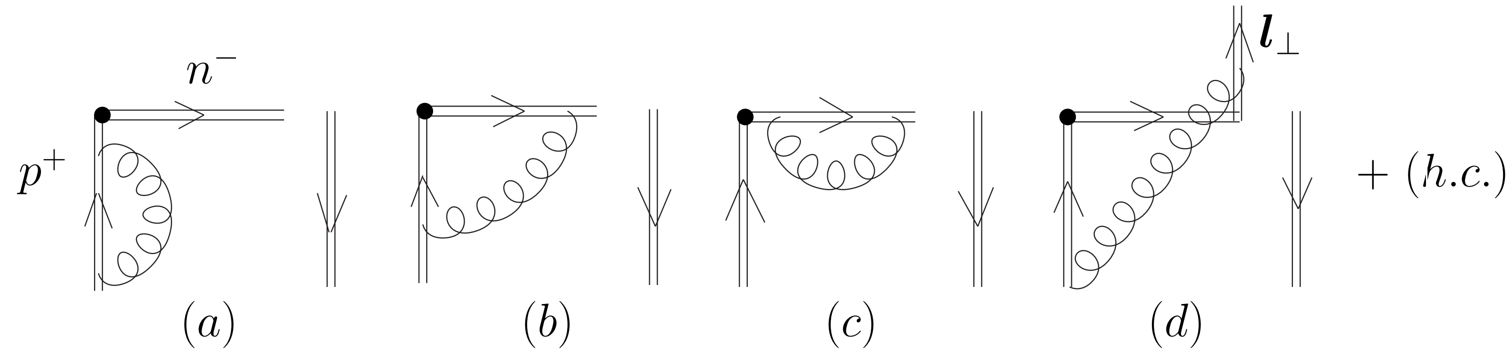
$$d^{\mu\nu}_{\mathrm{Adv/Ret}}(q) = -(q^\mu n^{-\nu} + q^\nu n^{-\mu})\frac{1}{q^+ \mp i\eta}$$

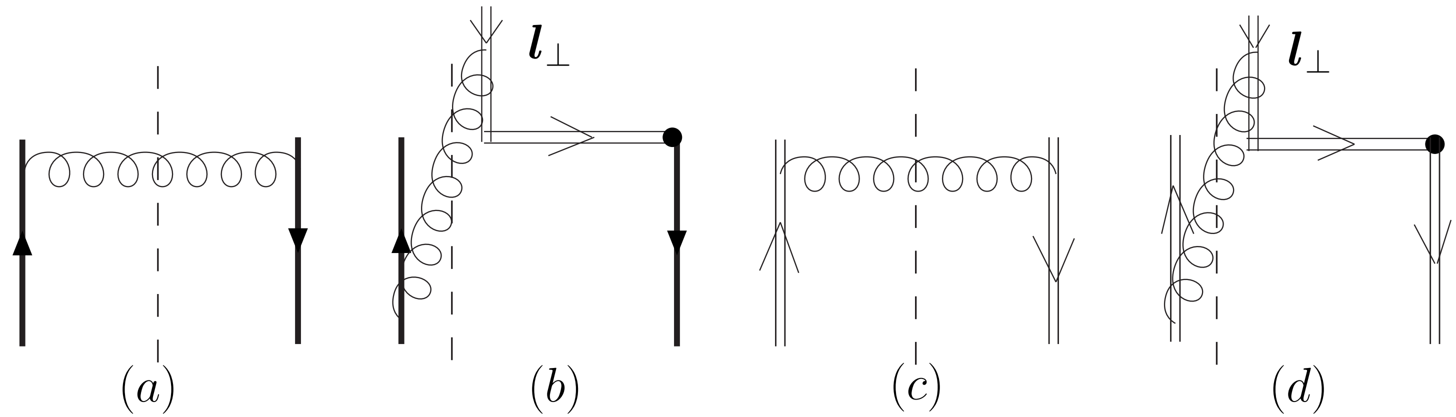
Mandelstam-Leibbrandt pole prescriptions:

$$\frac{1}{[q^+]_{\mathrm{ML}}} = \begin{cases} \frac{1}{q^+ + i0q^-} \\ \frac{q^-}{q^+q^- + i0} \end{cases}$$

calculation of the **one-gluon** diagrams







renormalization of the Wilson operators with obstructions (cusps, self-intersections) requires additional renormalization factor depending on the cusp angle (Korchemsky, Radyushkin)

$$Z_\chi = \left[\langle 0 | \mathcal{P} \exp \left[ig \int_\chi d\zeta^\mu \ \hat{A}^a_\mu(\zeta) \right] |0\rangle \right]^{-1}$$

generalized renormalization:

$$\mathcal{O}_{\rm ren}(\chi, ...) = Z_\chi Z_{\rm R} \ \mathcal{O}(\chi, ...)$$

approaches to **semi-inclusive DIS**

large $\boldsymbol{P}_\perp$ $\qquad$ $\boldsymbol{P}_\perp \sim Q$
large Q^2 $\qquad$ $Q^2 \gg \Lambda^2_{\rm QCD}$

perturbative calculations with **integrated** densities

Meng, Olness, Soper: NPB (1992)

moderate $\boldsymbol{P}_\perp$ $\qquad$ $\Lambda_{\rm QCD} \ll \boldsymbol{P}_\perp \ll Q$
large Q^2 $\qquad$ $Q^2 \gg \Lambda^2_{\rm QCD}$

perturbative calculations with **integrated** densities plus resummation of large double logs $\alpha_s \ln^2 \boldsymbol{P}_\perp/Q$

Collins, Soper: NPB (1981, 1982)

Dokshitzer, Diakonov, Troian: PR (1980) *et al.*

CURRENT STATUS of **SOLUTIONS**

1. Gauge invariance: transverse gauge link at light-cone infinity cancels the pole-prescription dependence (Belitsky, Ji, Yuan; Boer, Mulders, Pijlman)

   → SOLVED

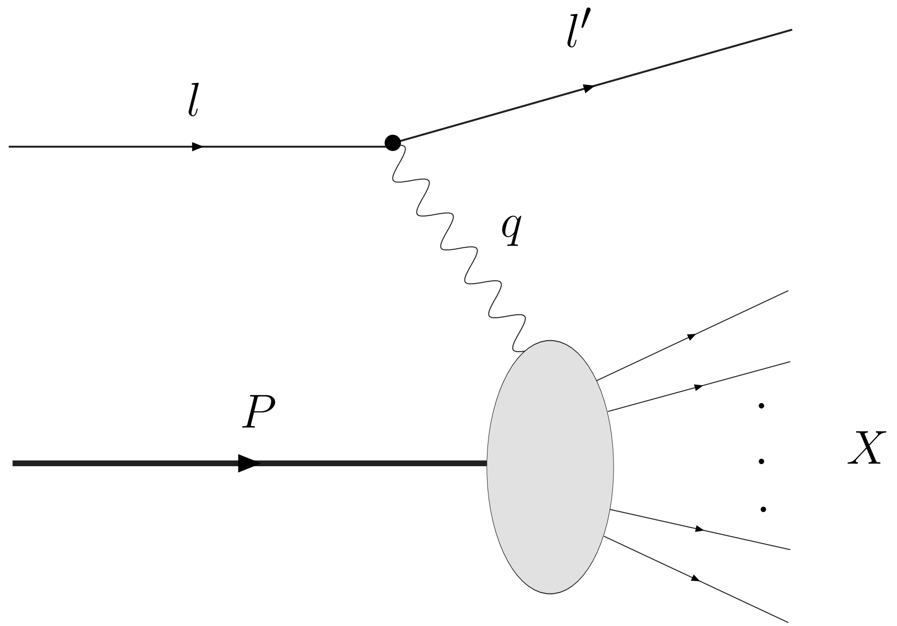
2. Extra divergences:

   - non-light-like gauge links in covariant gauges, or an axial gauge off the light cone (Collins, Soper)
   - subtractive method: for the light-like Wilson lines (Collins, Hautmann)
   - generalized renormalization procedure in the light-cone gauge (ICh, Stefanis)

   → REASONABLE, but not finally

3. Collinear PDF from TMDs: solved within generalized renormalization on the light-cone (LO); (at least) questionable in other cases

INCLUSIVE PROCESSES (DIS)



hadronic tensor

$$W_{\mu\nu} = \frac{1}{2\pi}\,\Im m\,\left[i\int d^4\xi\; \mathrm{e}^{iq\xi}\langle P|T\{J_\mu(\xi)J_\nu(0)\}|P\rangle\right]$$

$$= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)\, F_1(x_B, Q^2) + \frac{1}{P\cdot q}\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)\, F_2(x_B, Q^2)$$

QCD FACTORIZATION in the DIS

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \to \mathcal{X}$$

$$F(x_B, Q^2) = H(x_B, Q^2/\mu^2) \otimes F_D(\mu^2) = \sum_i \int_{x_B}^1 \frac{d\xi}{\xi}\, C_i\left(\frac{x}{\xi}, \frac{Q^2}{\mu^2}\right)\, F_D^i(\xi, \mu^2)$$

$$F_1(x_{\mathrm{B}}, Q^2) = \frac{1}{2x_{\mathrm{B}}}\, F_2\left(x_{\mathrm{B}}, Q^2\right) = \frac{1}{2}\sum_i e_i^2 \left[Q_i(x_{\mathrm{B}}, Q^2) + \bar{Q}_i(x_{\mathrm{B}}, Q^2)\right]$$

Renormalization Group properties: DGLAP

$$\mu \frac{d}{d\mu} Q_{i/h}(x,\mu) = \sum_j \int_x^1 \frac{dz}{z} \, P_{ij}\left(\frac{x}{z}\right) Q_{j/h}(x,\mu)$$

Moments of collinear PDFs are related to matrix elements of the local **twist-2** operators in OPE:

$$M_i(n) = \int_0^1 dx \; x^{n-1} \; Q_{i/h}(x,\mu) + (-)^n \int_0^1 dx \; x^{n-1} \; \bar{Q}_{i/h}(x,\mu)$$

$$\mathcal{O}_i^{\mu_1 \cdots \mu_n} = \bar{\psi}(0) \gamma^{\mu_1} \, iD^{\mu_2} ... iD^{\mu_n} \psi(0)$$

$$M_i(n) = \frac{1}{(P^+)^n} \langle P | \mathcal{O}_i^{+\cdots+} | P \rangle$$

Completely **gauge invariant** (quark) density:

$$Q_{i/h}(x,\mu) = \frac{1}{2}\int \frac{d\xi^-}{2\pi}\, e^{-ik^+\xi^-} \langle h(P)|\bar{\psi}_i(\xi^-,\mathbf{0}_\perp)[\xi^-,0^-]\gamma^+\psi_i(0^-,\mathbf{0}_\perp)|h(P)\rangle$$

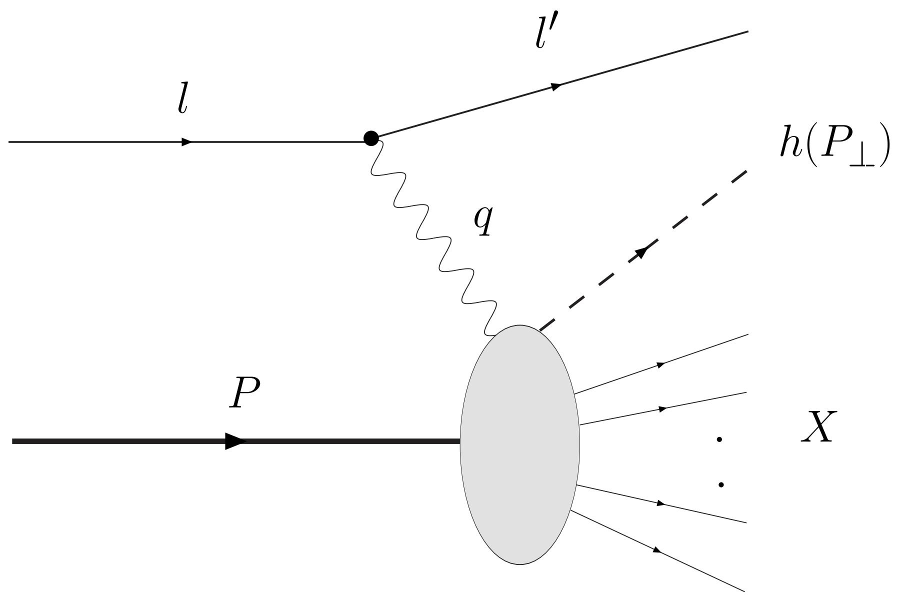
Gauge invariance is saved by the insertion of the gauge link

$$[y,x]_r = \mathcal{P}\exp\left[-ig\int_{\tau_1}^{\tau_2} d\tau r^\mu A_\mu^a(r\tau)t^a\right] \quad r^\mu\tau_1 = x\ ,\ r^\mu\tau_2 = y$$

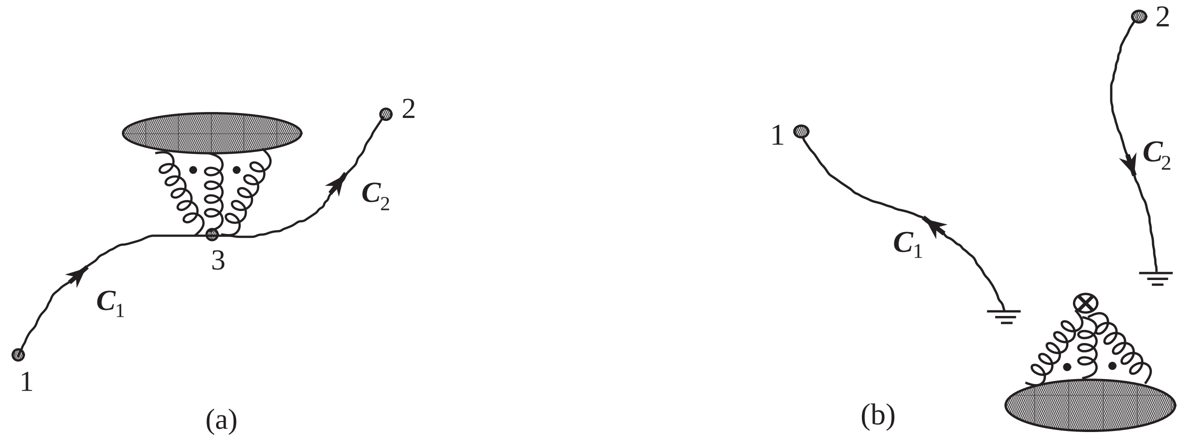
Note: distinguish between **longitudinal** $[\ ,\ ]_{[n,v,v_0]}$ and **transversal** $[\ ,\ ]_{[\boldsymbol{l}]}$ gauge links!

$SEMI - INCLUSIVE\ PROCESSES$

$$\gamma^*(q) + \mathbf{H_1}(\mathbf{P}) \rightarrow \mathbf{H_2}(\mathbf{P'}) + \mathcal{X}$$



Smooth (a) connector and gauge link going via infinity (b) with obstruction



(a) (b)

WHY TMD?

- Nonperturbative domain of QCD: quark models, lattice, nonperturbative vacuum, instantons;
- Factorization and evolution;
- Gauge invariance;
- Structure of nucleon;
- Spin-related observables;
- Flavor

EXAMPLE: three different definitions of a unintegrated quark distribution:

- definition **B**: in the covariant gauges, the gauge links shifted off the light-cone $v^2 > 0$, $v^+ \ll v^-$; or use the non-light-like axial gauge $(v \cdot A) = 0$, $v^2 > 0$ (Collins, Soper, Ji, Ma, Yuan);

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- definition **A** + soft factor: stay on the light-cone, but subtract soft factor R, which cancels the extra divergences: $\mathcal{F}_{[n]} \to \mathcal{F}_{[n]} \cdot R^{-1}$ (Collins, Hautmann):

TMD @ JLab

- **HEP-EX:** JLab@12Gev, EIC (anticipated) ($eP^{\uparrow}$) at $\sqrt{s} > 20 GeV$

- **HEP-PH:** Analysis of data (`Prokudin et al.`) requires correct $\boldsymbol{k}_\perp$-**evolution**—not known yet! Properties of **Wilson lines** (`Balitsky`) (gauge links)—non-trivial in TMD! **Factorization** (`Bacchetta et al.`)—proof wanted! **Model** calculations (`Bacchetta, Gamberg, Schlegel et al`)—correct definition needed!

- **HEP-LAT:** Lattice calculations of **TMD** has been started recently (`Haegler, Musch et al.`)—consistent **operator definition** of TMDs needed!

Imaginary part the UV singularity— gluons in the Glauber regime (`Idilbi, Majumder`)

$$\mathrm{Im}\ \Sigma_{\mathrm{left}}^{UV} = -\frac{\alpha_s}{2\epsilon} C_{\mathrm{F}}$$

In the covariant gauge, it stems from the Wilson line (Collins, Soper):

$$\int_0^\infty d\xi^- A^+(\xi^-, 0^+, \mathbf{0}_\perp) = \int d^4q \tilde{A}^+(q) \frac{-i}{q^+ - i\eta}$$

$T-$odd functions responsible for the single-spin asymmetries in SIDIS and DY (e.g, Sivers mechanism of SSA).

Imaginary parts of the anomalous dimensions *could* be responsible for the evolution of the $T-$odd PDFs

EXAMPLE: three different definitions of a unintegrated quark distribution:

- definition **A** + soft factor: direct regularization of the light-cone singularities in the gluon propagator

$$\frac{1}{q^+} \to \frac{1}{[q^+](\eta)}$$

—generalized renormalization procedure; $\mathcal{F}_{[n]}(\eta) \to \mathcal{F}_{[n]}(\eta) \cdot R^{-1}(\eta)$; keeps the overlapping singularities under control and treats the extra term in the UV-divergent part by means of the *cusp anomalous dimension*— specific form of the gauge contour in the soft factor (`ICh, Stefanis`)

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- definition **A** + soft factor: light-cone gauge with the Mandelstam-Leibbrandt pole prescription

$$\frac{1}{q^+} \to \frac{1}{q^+ + i0q^-} \text{ or } \frac{q^-}{q^+q^- + i0}$$

—**overlapping singularities do not appear** at all (ICh, Stefanis)