Twist three distributions in light-front hamiltonian approach

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- Transverse momentum distributions
- Light front Hamiltonian approach
- Calculation of $f_\perp(x, k^\perp)$ and $e(x)$

Twist three distribution $f_\perp(x, k^\perp)$

Defined as

$$\frac{k^i}{P^+} f_\perp(x, k^\perp) = \int \frac{dy^- d^2y^\perp}{4(2\pi)^3} e^{\frac{i}{2} P^+ y^- x} e^{-ik^\perp \cdot y^\perp} \langle P, S \mid \bar{\psi}(0) U(0, y) \gamma^i \psi(y^-, y^\perp) \mid P, S \rangle \mid_{y^+=0}.$$

$U(0, y)$: path ordered exponential (link) required for color gauge invariance; plays important role in time reversal odd distributions

$f_\perp(x, k^\perp)$: T-even. We choose light front gauge and do not consider the transverse link at infinity

Plays an important role in $cos\ \phi_h$ asymmetry in unpolarized semi inclusive deep inelastic scattering (SIDIS) , the so-called Cahn effect

Cahn, 1978,1989

Twist three distribution $f_\perp(x, k^\perp)$

Cahn Effect : Unpolarized SIDIS cross section depends on the azimuthal angle ϕ_h between the lepton plane and the hadron production plane, and on the transverse momentum of the detected hadron : observed experimentally

European Muon Collaboration (1987); CLAS Collaboration (2009), HERMES Collaboration (2009), COMPASS Collaboration (2009)

At $\frac{1}{Q}$ level $f_\perp(x, k^\perp)$ contributes to this dependence. If one neglects the explicit quark gluon interaction terms in the distribution and fragmentation functions as well as contributions from T-odd functions, then this $cos\ \phi_h$ dependence of the cross section is given in terms of the unpolarized distribution and fragmentation functions

Has been investigated in a parton model approach by neglecting quark-gluon interaction terms in the distribution and fragmentation functions and introducing a phenomenologically motivated intrinsic $k^\perp$ dependence (Anselmino *et al* (2005))

Spectator model calculation of $f_\perp(x, k_\perp)$: Jakob, Mulders, Rodrigues (1997); Bag model : Avakian, Efremov, Schweitzer, Yuan (2010)

Light-front Fock representation

- P^- is the light-front Hamiltonian, generates x^+ (light-front time) evolution.
$H_{LC} = P^+ P^- - (P^\perp)^2$

- Proton state satisfies $H_{LC} \mid \psi_p\rangle = M^2 \mid \psi_p\rangle$

$$\left|\psi_p(P^+, \vec{P}_\perp)\right\rangle = \sum_n \prod_{i=1}^n \frac{\mathrm{d}x_i\, \mathrm{d}^2\vec{k}_{\perp i}}{\sqrt{x_i}\, 16\pi^3}\; 16\pi^3 \delta\left(1 - \sum_{i=1}^n x_i\right)\; \delta^{(2)}\left(\sum_{i=1}^n \vec{k}_{\perp i}\right)$$

$$\times\, \psi_n(x_i, \vec{k}_{\perp i}, \lambda_i) \left|n;\, x_i P^+, x_i \vec{P}_\perp + \vec{k}_{\perp i}, \lambda_i\right\rangle .$$

- n-particle states are normalized as

$$\left\langle n;\, p_i'^{+}, \vec{p}'_{\perp i}, \lambda_i' \,\middle|\, n;\, p_i^+, \vec{p}_{\perp i}, \lambda_i\right\rangle = \prod_{i=1}^n 16\pi^3 p_i^+ \delta(p_i'^{+} - p_i^+)\; \delta^{(2)}(\vec{p}'_{\perp i} - \vec{p}_{\perp i})\; \delta_{\lambda_i' \lambda_i}$$

Light-front Hamiltonian approach

Using the light-front projection operators $\Lambda^{\pm} = \frac{1}{2}\gamma^0\gamma^{\pm}$, the operator can be written as

$$\bar{\psi}(0)\gamma^i\psi(y^-, y^\perp) = \psi^{(-)\dagger}(0)\alpha^i\psi^{(+)}(y^-, y^\perp) + \psi^{(+)\dagger}(0)\alpha^i\psi^{(-)}(y^-, y^\perp).$$

Involves the 'bad'component $\psi^{(-)}$: twist three
Equation of constraint is given by

$$\psi^-(y) = \frac{1}{i\partial^+}(i\alpha^\perp \cdot \partial^\perp + g\alpha^\perp \cdot A^\perp + \beta m)\psi^+(y)$$

$\frac{1}{\partial^+}$ is defined as

$$\frac{1}{\partial^+}f(x^-) = \frac{1}{4}\int_{-\infty}^{\infty} dy^- \epsilon(x^- - y^-)f(y^-)$$

The antisymmetric step function is given by

$$\epsilon(x^-) = -\frac{i}{\pi}\mathcal{P}\int \frac{d\omega}{\omega} e^{\frac{i}{2}\omega x^-}$$

Using the equation of constraint $\psi^{(-)}$ can be eliminated

$f_\perp(x,k^\perp)$ *in Light-front Hamiltonian approach (AM 2010)*

The operator becomes

$$O_{k^\perp} = \psi^{(+)\dagger}(0)\Big[(\alpha^\perp\cdot \overleftarrow{\partial^\perp})(\overleftarrow{\frac{1}{\partial^+}})\alpha^1 + \alpha^1(\overrightarrow{\frac{1}{\partial^+}})(\alpha^\perp\cdot \overrightarrow{\partial^\perp})\Big]\psi^{(+)}(y)$$

$$O_g = g\psi^{(+)\dagger}(0)\Big[(\alpha^\perp\cdot A^\perp)(\overleftarrow{\frac{1}{-i\partial^+}})\alpha^1 + \alpha^1(\overrightarrow{\frac{1}{i\partial^+}})(\alpha^\perp\cdot A^\perp)\Big]\psi^{(+)}(y)$$

$$O_m = m\psi^{(+)\dagger}(0)\gamma^1\Big[(\overleftarrow{\frac{1}{-i\partial^+}}) - (\overrightarrow{\frac{1}{i\partial^+}})\Big]\psi^{(+)}(y)$$

For the dynamical field ψ^+ we use two component formalism

$$\psi^+ = \begin{pmatrix}\xi\\ 0\end{pmatrix}$$

Fock Space expansion of the state

Instead of proton we take the state to be a dressed quark; Fock space expansion of such a state can be written as

$$
\begin{aligned}
\mid P,\sigma\rangle \quad &= \quad \phi_1 b^\dagger(P,\sigma)\mid 0\rangle \\
&+ \sum_{\sigma_1,\lambda_2}\int \frac{dk_1^+ d^2k_1^\perp}{\sqrt{2(2\pi)^3 k_1^+}}\int \frac{dk_2^+ d^2k_2^\perp}{\sqrt{2(2\pi)^3 k_2^+}}\sqrt{2(2\pi)^3 P^+}\delta^3(P-k_1-k_2) \\
&\quad \phi_2(P,\sigma\mid k_1,\sigma_1;k_2,\lambda_2) b^\dagger(k_1,\sigma_1) a^\dagger(k_2,\lambda_2)\mid 0\rangle
\end{aligned}
$$

We introduce Jacobi momenta x_i, $q_i{}^\perp$ such that $\sum_i x_i = 1$ and $\sum_i q_i{}^\perp = 0$. They are defined as

$$
x_i = \frac{k_i^+}{P^+}, \qquad q_i^\perp = k_i^\perp - x_i P^\perp
$$

$$
\psi_1 = \phi_1, \qquad \psi_2(x_i, q_i^\perp) = \sqrt{P^+}\phi_2(k_i^+, k_i{}^\perp);
$$

These are boost invariant wave functions

Fock Space expansion of the state

Using the eigenvalue equation for the light-cone Hamiltonian, the two-particle light front wave function can be written as

$$\psi^{\sigma}_{2\sigma_1,\lambda}(x,q^{\perp}) = \frac{x(1-x)}{(q^{\perp})^2+m^2(1-x)^2}\frac{1}{\sqrt{(1-x)}}\frac{g}{\sqrt{2(2\pi)^3}}T^a \chi^{\dagger}_{\sigma_1}\Big[-2\frac{q^{\perp}}{1-x}-\frac{\tilde{\sigma}^{\perp}\cdot q^{\perp}}{x}\tilde{\sigma}^{\perp} + im\tilde{\sigma}^{\perp}\frac{(1-x)}{x}\Big]\chi_{\sigma}\epsilon^{\perp *}_{\lambda}\psi_1 .$$

m is the bare mass of the quark, $\tilde{\sigma}_1=\sigma_2$, $\tilde{\sigma}_2=-\sigma_1$

Normalization of the state gives

$$\mid \psi_1 \mid^2 = 1-\frac{\alpha_s}{2\pi}C_f\ log\frac{Q^2}{\mu^2}\int_{\epsilon}^{1-\epsilon}dx\frac{1+x^2}{1-x},$$

Calculation of $f_\perp(x, k^\perp)$

$O_{k^\perp}$ and O_m will have contribution from single particle sector as well as two particle sector of the state; O_g will get contribution from an overlap of a single particle and a two-particle light-front wave functions

Contribution from $O_{k^\perp}$:

$$\int \frac{dy^- d^2y^\perp}{4(2\pi)^3} e^{\frac{i}{2}P^+y^-x} e^{-ik^\perp \cdot y^\perp} \langle P, S \mid O_{k^\perp} \mid P, S\rangle = \delta(1-x)\frac{P^1}{P^+} \mid \psi_1 \mid^2 + \frac{q^1 + xP^1}{xP^+} \mid \psi^s_{2,s_1,\lambda}(x, q^\perp) \mid^2$$

O_m gives zero contribution

Contribution from O_g

$$\int \frac{dy^- d^2y^\perp}{4(2\pi)^3} e^{\frac{i}{2}P^+y^-x} e^{-ik^\perp \cdot y^\perp} \langle P, S \mid O_g \mid P, S\rangle = -\frac{\alpha_s}{2\pi^2} C_f \frac{q^1}{q^{\perp 2}} \frac{1}{x(1-x)} \mid \psi_1 \mid^2$$

Here we have used explicit form of $\psi^s_{2,s_1,\lambda}(x, q^\perp)$

Calculation of $f_\perp(x, k^\perp)$

One sees from the above expressions that the equation of motion relation

Bacchetta, Diehl, Goeke, Metz, Mulders, Scheghel (2007)

$$x f_\perp = x\tilde{f}_\perp + f_1$$

is satisfied, as $k^\perp = q^\perp + xP^\perp$; for the single particle contribution $k^1 = P^1$, and $\tilde{f}_\perp$ is the genuine twist three quark-gluon interaction part which in our calculation, comes from O_g

Integrated distribution has an integration over $k^\perp/q^\perp$: operator is bilocal only in minus direction

Operator can still be separated into three parts, O_m, $O_{k\perp}$ and O_g using the equation of constraint for ψ^- O_m, as before gives zero contribution as well as O_g

We have

$$\langle P, 1/2 \mid O_{k\perp} \mid P, 1/2\rangle = \frac{P^1}{P^+}\left[\delta(1-x) + \frac{\alpha_s}{2\pi} log\frac{Q^2}{\mu^2} C_f \int dx \frac{1+x^2}{1-x}_+\right]$$

where we have used the normalization of the state, which cancels the singularity at $x = 1$

Total contribution comes from $O_{k\perp}$ and is the same as the twist two unpolarized distribution function $f_1(x, Q^2)$: nonzero result only when $P^\perp$ is nonzero

Twist Three Distribution e(x)

Twist three distribution $e(x)$ is given by

$$e(x) = \frac{P^+}{M} \int \frac{dy^-}{8\pi} e^{\frac{i}{2} x P^+ y^-} \langle P, S \mid \bar{\psi}(0)\psi(y^-) \mid P, S \rangle .$$

$e(x)$ is spin independent and chiral odd

Enters together with the chirally odd Collin's fragmentation function $H_1^\perp$ in the azimuthal asymmetry in semi-inclusive deep inelastic scattering (DIS) of longitudinally polarized electrons off unpolarized nucleons : measured experimentally

Apart from $e(x)$, several other distribution and fragmentation functions also appear in this asymmetry

Bag model calculation of $e(x)$ (Jaffe, Ji, 1992); chiral quark soliton model (Schweitzer 2003), spectator model (Jakob, Mulders, Rodrigues 1997) and perturbative one loop model (Burkardt & Koike 2002)

Sum Rules

First moment of $e(x)$ obeys the sum rule

$$\int_{-1}^{1} dx\ e(x) = \frac{1}{2M}\langle P,S \mid \bar{\psi}(0)\psi(0) \mid P,S\rangle.$$

Second moment of $e(x)$ obeys:

$$\int_{-1}^{1} dx\ x\ e(x) = \frac{m}{M}N_q;$$

m is the mass of the quark and M is the mass of the proton. N_q is quarks of a given flavor

These sum rules are not satisfied in the bag model or in the spectator model

In some of the models a $\delta(x)$ singularity has been found in $e(x)$; as $x = 0$ cannot be experimentally reached there is no experimental evidence

Again, in some models the first sum rule is saurated by the $\delta(x)$ term and in some other

models only part of the contribution comes from it

$e(x)$ in light-front Hamiltonian approach (AM, PLB 2010)

$$e(x) = \frac{P^+}{M} \int \frac{dy^-}{8\pi} e^{\frac{i}{2} x P^+ y^-} \langle P, S \mid \bar{\psi}(0)\psi(y^-) \mid P, S \rangle.$$

In light-front gauge and using the equation of constraint from $\psi^{(-)}$ the operator becomes

$$O_e = \bar{\psi}(0)\psi(y^-) = \psi^{(-)\dagger}(0)\gamma_0\psi^{(+)}(y^-) + \psi^{(+)\dagger}(0)\gamma_0\psi^{(-)}(y^-).$$

$$O_e = O_m + O_g + O_k$$

$$O_m = m\psi^{(+)\dagger}(0)\Big[(-\frac{\overleftarrow{1}}{i\partial^+}) + (\frac{\overrightarrow{1}}{i\partial^+})\Big]\psi^{(+)}(y^-);$$

$$O_k = \psi^{(+)\dagger}(0)\left[\frac{\overrightarrow{\not\partial^\perp}}{\partial^+} - \frac{\overleftarrow{\not\partial^\perp}}{\partial^+}\right]\psi^{(+)}(y^-);$$

$e(x)$ in light-front Hamiltonian approach (continued)

$$O_g = g\psi^{(+)\dagger}(0)\Big[\not{A}_T(\overleftarrow{\frac{1}{i\partial^+}}) + (\overrightarrow{\frac{1}{i\partial^+}})\not{A}_T\Big]\psi^{(+)}(y^-).$$

For a dressed quark state of momentum P and helicity σ:

$$x\ e_k(x,Q^2) = 0;$$

$$x\ e_g(x,Q^2) = \frac{m}{M}\frac{\alpha_s}{2\pi}\ C_f\ log\frac{Q^2}{\mu^2}\Big[\frac{x}{2}\delta(1-x) - 1 + x\Big];$$

$$x\ e_m(x,Q^2) = \frac{m}{M}\Big[\delta(1-x) + \frac{\alpha_s}{2\pi}\ C_f\ log\frac{Q^2}{\mu^2}\Big\{\frac{1+x^2}{1-x} - \delta(1-x)\int_\epsilon^{1-\epsilon} dy\frac{1+y^2}{1-y}\Big\}\Big];$$

In the above, we have used the normalization condition of the state

$$\frac{1+x^2}{1-x} - \delta(1-x)\int_\epsilon^{1-\epsilon} dy\frac{1+y^2}{1-y} = \frac{1+x^2}{1-x}_+ + \frac{3}{2}\delta(1-x).$$

$e(x)$ *in light-front Hamiltonian approach (continued)*

For a dressed quark state, M is the renormalized mass of the quark. The bare mass m of the quark is given in terms of the renormalized mass

$$m = M\left(1 - \frac{3\alpha_s}{4\pi} C_f log \frac{Q^2}{\mu^2}\right).$$

Using this, we can write

$$xe(x,Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_f log \frac{Q^2}{\mu^2}\left[\frac{2x}{1-x_+} + \frac{1}{2} x\delta(1-x)\right].$$

In the above result, the divergence at $x \to 1$ gets canceled by the contribution from the normalization of the state and we get the plus prescription; result agrees with another calculation (Burkardt & Koike 2002)

No $\delta(x)$ contribution in $xe(x)$

$xe(x)$ is zero if we take the quark mass to be zero

Equation of Motion

$xe(x)$ can be related to the twist two unpolarized quark distribution through the equation of motion relation:

$$xe(x) = x\tilde{e}(x) + \frac{m}{M} f_1(x)$$

Where $f_1(x)$ is the twist two unpolarized distribution function :

$$f_1(x) = \int \frac{dy^-}{8\pi} e^{\frac{i}{2} x P^+ y^-} \langle P, S \mid \bar{\psi}(0)\gamma^+\psi(y^-) \mid P, S \rangle .$$

The above relation is unaffected by the presence of the gauge link. $\tilde{e}(x)$ is the genuine twist three quark-gluon interaction part

For a dressed quark

$$f_1(x, Q^2) = \delta(1-x) + \frac{\alpha_s}{2\pi} C_f log \frac{Q^2}{\mu^2} \left[\frac{1+x^2}{1-x_+} + \frac{3}{2}\delta(1-x) \right]$$

Equation of motion relation is satisfied with $x\ \tilde{e}(x, Q^2) = \frac{\alpha_s}{2\pi}\ C_f\ log \frac{Q^2}{\mu^2} \left[\frac{1}{2} x \delta(1-x) - 1 + x \right]$

Sum Rules

The first moment of $e_m(x)$ and $e_g(x)$ are given by

$$\int_\epsilon^1 e_m(x,Q^2)\,dx = 1 + \frac{\alpha_s}{2\pi}\,C_f\,log\frac{Q^2}{\mu^2}\,(-\log\epsilon - 1);$$

$$\int_\epsilon^1 e_g(x,Q^2)\,dx = \frac{\alpha_s}{2\pi}\,C_f\,log\frac{Q^2}{\mu^2}\,(log\epsilon + \frac{3}{2}).$$

Each part has divergence as $x \to 0$. However, their total contribution is free of divergence :

$$\begin{aligned}\int_\epsilon^1 e(x,Q^2)\,dx &= \int_\epsilon^1 (e_m(x,Q^2) + e_g(x,Q^2)\,dx = 1 + \frac{\alpha_s}{2\pi}\,C_f\,log\frac{Q^2}{\mu^2}\,\frac{1}{2}\\ &= \frac{1}{2M}\langle P,S \mid \bar{\psi}(0)\psi(0) \mid P,S\rangle.\end{aligned}$$

Sum rule is satisfied ; x region is limited by 0 and 1 as this is physically allowed. This implies that the sum is not saturated by a $\delta(x)$ contribution

Sum Rules

The second moment of $e(x)$ becomes

$$\int_0^1 \; x \; e(x,Q^2)dx = 1 - \frac{\alpha_s}{2\pi} \; C_f \; log \frac{Q^2}{\mu^2} \; \frac{3}{2} = \frac{m}{M};$$

with

$$\int_0^1 dx \; x \; \tilde{e}(x,Q^2) = \int_0^1 \; dx \left[\frac{x}{2} \; \delta(1-x) - 1 + x \right] = 0$$

The RHS vanishes in the chiral limit $m = 0$. Delta function (if present) does not contribute to this sum rule

Moment Relations

n-th moment of $e(x)$ defined by $[e]_n = \int_0^1 dx\, x^{n-1}\, e(x)$

$$[e]_n = [\tilde{e}]_n + \frac{m}{M}[f_1]_{n-1}.$$

The n-th moment is calculated using the expression above :

$$\begin{aligned} [e]_n \quad &= \quad 1 + \frac{\alpha_s}{2\pi}\, C_f\, log\frac{Q^2}{\mu^2} \int_0^1 \, dx\, x^{n-2} \left[\frac{2x}{1-x}_+ + \frac{1}{2} x\delta(1-x) \right] \\ &= 1 + \frac{\alpha_s}{2\pi}\, C_f\, log\frac{Q^2}{\mu^2} \left[-2 \sum_{j=1}^{n-1} \frac{1}{j} + \frac{1}{2} \right]; \end{aligned}$$

where we have used $\int_0^1 \, dx\, x^{n-1}\, \frac{1}{1-x}_+ = -\sum_{j=1}^{n-1} \frac{1}{j}$

$$\begin{aligned} [\tilde{e}]_n \quad &= \quad \frac{\alpha_s}{2\pi}\, C_f\, log\frac{Q^2}{\mu^2} \int_0^1 dx x^{n-2} \left[\frac{x}{2}\delta(1-x) - 1 + x \right] \\ &= \frac{\alpha_s}{2\pi}\, C_f\, log\frac{Q^2}{\mu^2} \left[\frac{1}{2} - \frac{1}{n-1} + \frac{1}{n} \right]; \end{aligned}$$

Moment Relations

$$\frac{1}{M}[mf_1]_{n-1} \quad = \quad 1 + \frac{\alpha_s}{2\pi}\; C_f \; log\frac{Q^2}{\mu^2}\int_0^1 \; dx\; x^{n-2}\left[\frac{1+x^2}{1-x}_+ + \frac{3}{2} - \frac{3}{2}\right]$$

$$= \frac{\alpha_s}{2\pi}\; C_f \; log\frac{Q^2}{\mu^2}\left[-2\sum_{j=1}^{n-1}\frac{1}{j} - \frac{1}{n} + \frac{1}{n-1}\right];$$

So the moment relation is satisfied

Summary

• We presented a calculation of twist three distributions $f_\perp(x, k^\perp)$ in light-front Hamiltonian approach. This distribution is known to play an important role in the observed Cahn effect in semi inclusive deep inelastic scattering.

• We took the state to be a quark dressed with a gluon. Higher Fock component of the LFWF is analytically known : important for higher twist as there is explicit quark-gluon interaction dependence in the operator (genuine twist three)

• A 'formal' study with a field theory inspired parton model where the partons are interacting and have intrinsic transverse momenta : intuitive picture

• Also presented a calculation of $e(x)$: sum rules and equation of motion relations are investigated and compared with other model calculations.