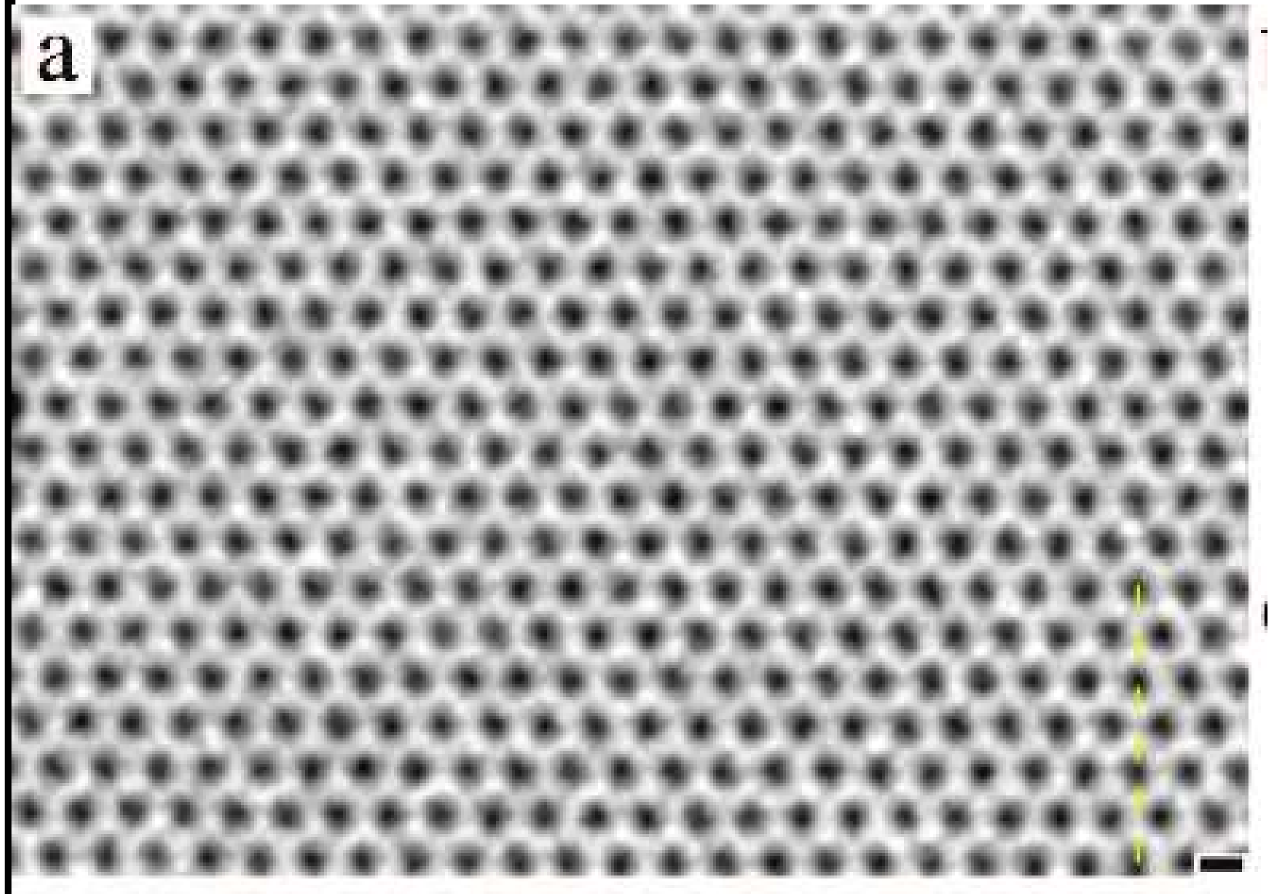
Many-body physics and the Anomalous Hall Effect in Graphene

Gordon W. Semenoff

University of British Columbia

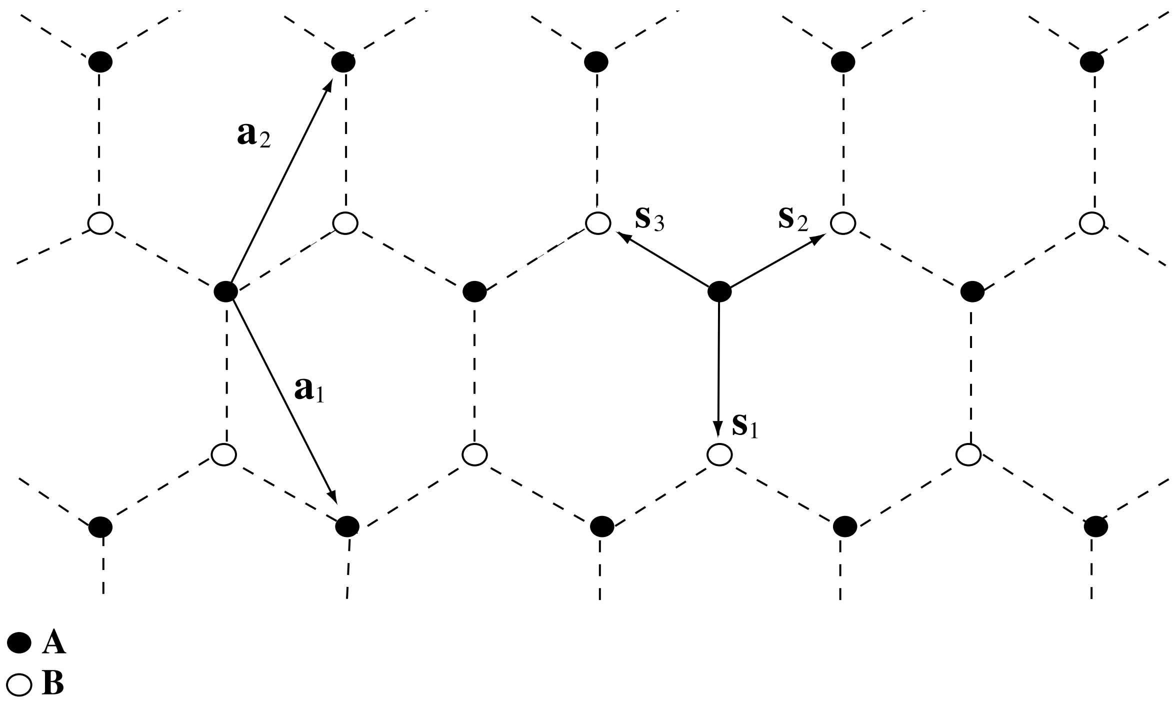
New frontiers in graphene physics, ECT Trento, April 13, 2010

Graphene is a 2-dimensional array of carbon atoms



Jannik C. Meyer, C. Kisielowski, R. Erni, Marta D. Rossell, M. F. Crommie, and A. Zettl, Nano Letters 8, 3582 (2008).

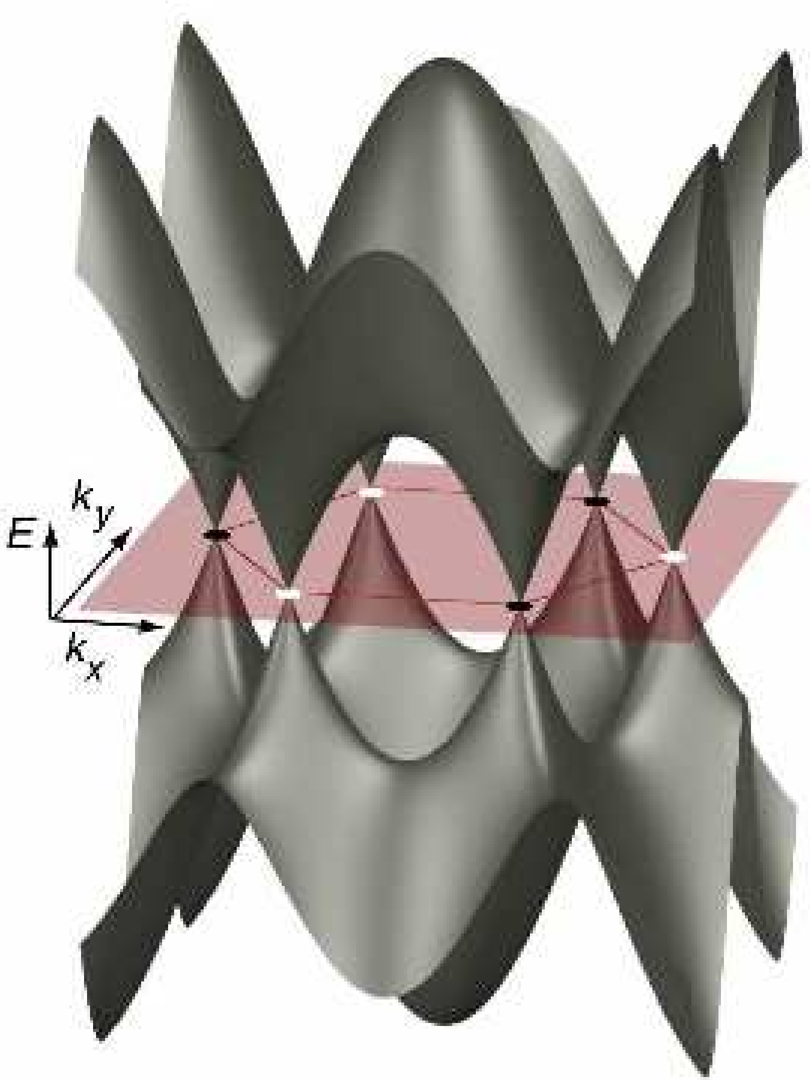
Tight-binding model



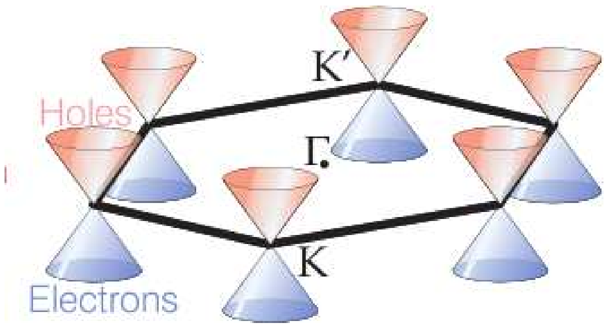
hexagonal lattice = two triangular sub-lattices $\vec{A}$ and $\vec{B}$ connected by vectors $\vec{s}_1, \vec{s}_2, \vec{s}_3$.

$$H = \sum_{\vec{A},i} \left(t \ b^\dagger_{\vec{A}+\vec{s}_i} a_{\vec{A}} + t^* \ a^\dagger_{\vec{A}} b_{\vec{A}+\vec{s}_i} \right) \quad , \quad t \sim 2.7ev \quad |\vec{s}_i| \sim 1.4\mathring{A}$$

Band structure of graphene



Linearize spectrum near degeneracy points



$$E(k) = \hbar v_F |\vec{k}|$$

$v_F \sim 10^6 m/s \sim c/300$, good up to $\sim 1ev$

$$H_{\text{Dirac}} = \hbar v_F \begin{bmatrix} \begin{matrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & k_x + ik_y \\ k_x - ik_y & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} \psi_A(k-K) \\ \psi_B(k-K) \\ \psi_A(k-K') \\ \psi_B(k-K') \end{bmatrix}$$

Massless electrons seen experimentally

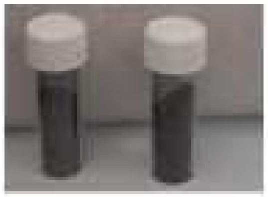
Shubnikov-de-Haas oscillations

K. S. Novoselov et. al. *Nature 438, 197 (2005)*

Graphene was produced and identified in the laboratory in 2004

- Micromechanical cleavage of bulk graphite up to 100 micrometer in size via adhesive tapes !

Novoselov et al, Science **306**, 666 (2004)



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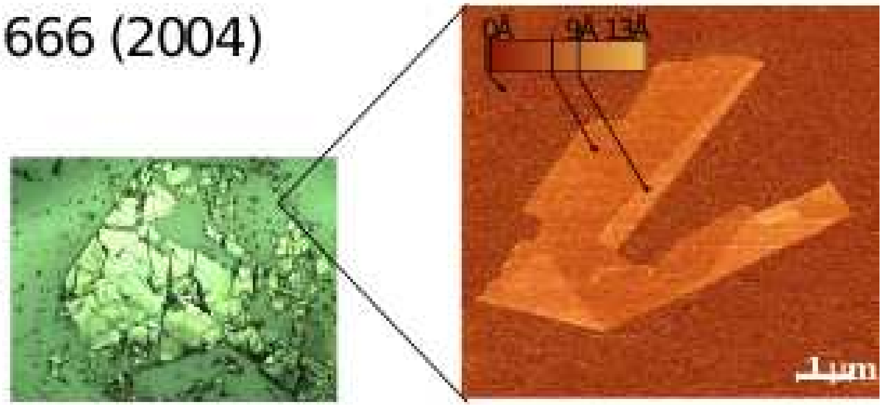

Kostya Novoselov

Andre Geim

Emergent Dirac Equation

$$\left[i\hbar\frac{\partial}{\partial t} + H\right]\psi(t,\vec{x}) = 0$$

$$H_{\text{Dirac}} = \hbar v_F \begin{bmatrix} 0 & -i\partial_x - \partial_y & 0 & 0 \\ -i\partial_x + \partial_y & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\partial_x + \partial_y \\ 0 & 0 & -i\partial_x - \partial_y & 0 \end{bmatrix}$$

$v_F \approx \frac{c}{300}$

2 valleys $\times$ 2 spin states

Emergent SU(4) symmetry

Consequences of the Dirac equation

$$i\frac{\partial}{\partial t}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} = \begin{bmatrix} 0 & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & 0 & 0 \\ -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\frac{\partial}{\partial x}+\frac{\partial}{\partial y} \\ 0 & 0 & -i\frac{\partial}{\partial x}-\frac{\partial}{\partial y} & 0 \end{bmatrix}\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}$$

Minimal coupling to magnetic field: $B = \vec{\partial} \times \vec{A}$

$$\vec{\partial} \;\rightarrow\; \vec{D} = \vec{\partial} + i\vec{A}$$

$$H_{\rm Dirac} = \left[\begin{array}{cc} \begin{array}{cc} 0 & -iD_x - D_y \\ -iD_x + D_y & 0 \end{array} & 0 \\ 0 & \begin{array}{cc} 0 & -iD_x + D_y \\ -iD_x - D_y & 0 \end{array} \end{array}\right]$$

Index Theorem

$\#$ zero modes $= 2(2)\left|\frac{1}{2\pi}\int d^2x B(x)\right|$ solutions of $H_{\rm Dirac}\psi_0(x) = 0$

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In the neutral ground state of graphene, half of zero modes are filled. **G. W. S.,** *Phys. Rev. Lett. 53, 2449 (1984)*

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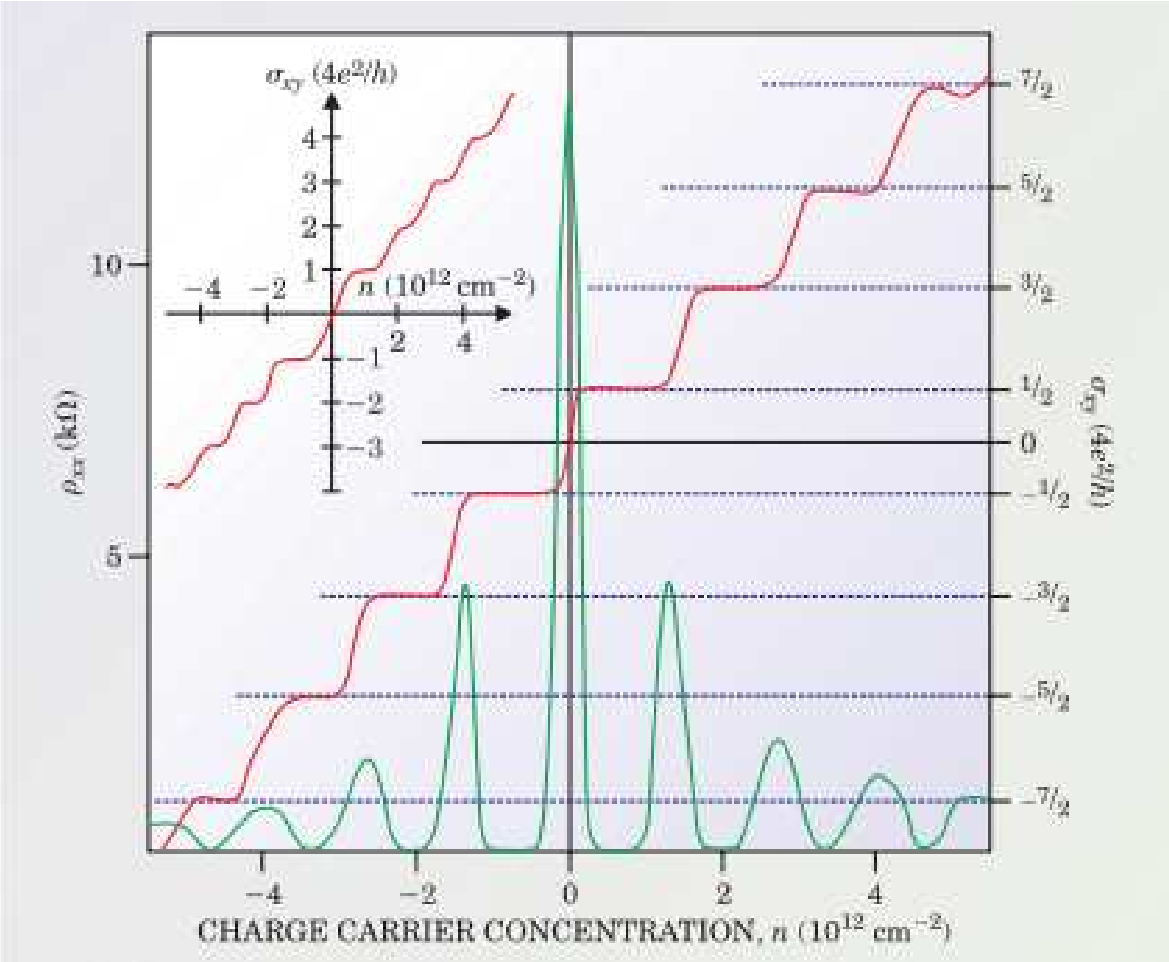
Confirmed by the quantum Hall effect

K. Novoselov et. al. *Nature 438, 197 (2005)*

Y. Zhang et. al. *Nature 438, 201 (2005)*



$$\sigma_{xy} = 4\frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

Atiyah-Patodi-Singer Index Theorem:

Hamiltonian for one valley and one spin:

$$h = \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{bmatrix} \quad , \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad \begin{bmatrix} u \in A \\ v \in B \end{bmatrix}$$

Positive and negative energy states are paired

$$h\psi_E = E\psi_E \quad , \quad H(\Gamma\psi_E) = -E(\Gamma\psi_E)$$

There can also be zero energy states

Atiyah-Patodi-Singer Index Theorem:

Hamiltonian for one valley and one spin:

$$h = \begin{bmatrix} 0 & \mathcal{D} \\ \mathcal{D}^\dagger & 0 \end{bmatrix} \ , \ \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ , \ \begin{bmatrix} u \in A \\ v \in B \end{bmatrix}$$

Zero energy states satisfy

$$h \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0 \ , \ \beta \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \pm \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

$$\mathcal{D} v_0 = 0 \ \to \ h \begin{bmatrix} 0 \\ v_0 \end{bmatrix} = 0 \ , \ \beta \begin{bmatrix} 0 \\ v_0 \end{bmatrix} = - \begin{bmatrix} 0 \\ v_0 \end{bmatrix}$$

$$\mathcal{D}^\dagger u_0 = 0 \ \to \ h \begin{bmatrix} u_0 \\ 0 \end{bmatrix} = 0 \ , \ \beta \begin{bmatrix} u_0 \\ 0 \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$$

$$\text{Index}(h) = \dim\ker \mathcal{D}^\dagger - \dim\ker \mathcal{D}$$

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The index depends only on topological data.

$$\text{Index}(h) = \frac{1}{2\pi}\int d^2x B(x) + \eta(h_{\delta R^2})$$

$$\eta(h_{\delta R^2}) = \ \text{TR}\ \text{sign}(h_{\delta R^2}) = \ \text{TR}\ \text{sign}(\partial_\theta - A_\theta)$$

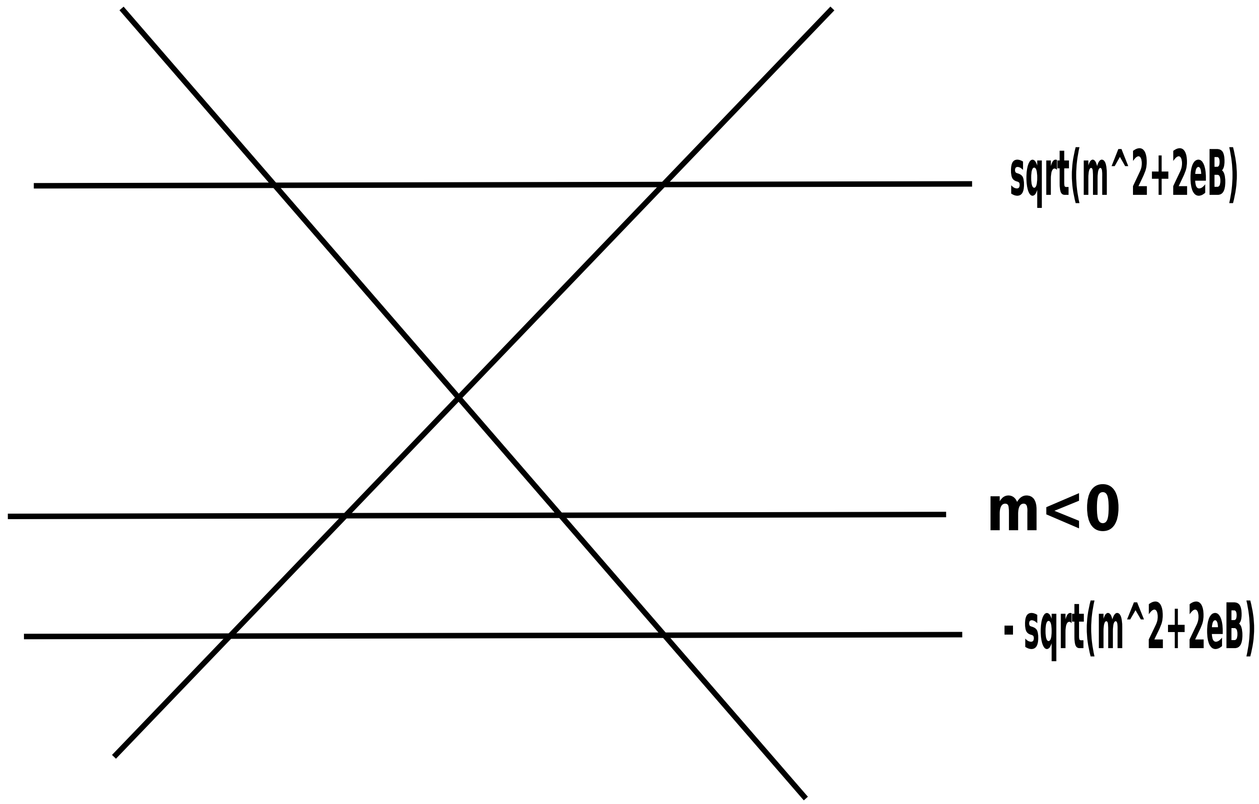
$$\eta(h_{\delta R^2}) = \left[\frac{1}{2\pi}\int d^2x B(x)\right] - \frac{1}{2\pi}\int d^2x B(x)$$

But "charge" takes continuum as well as discrete modes into account and is given by

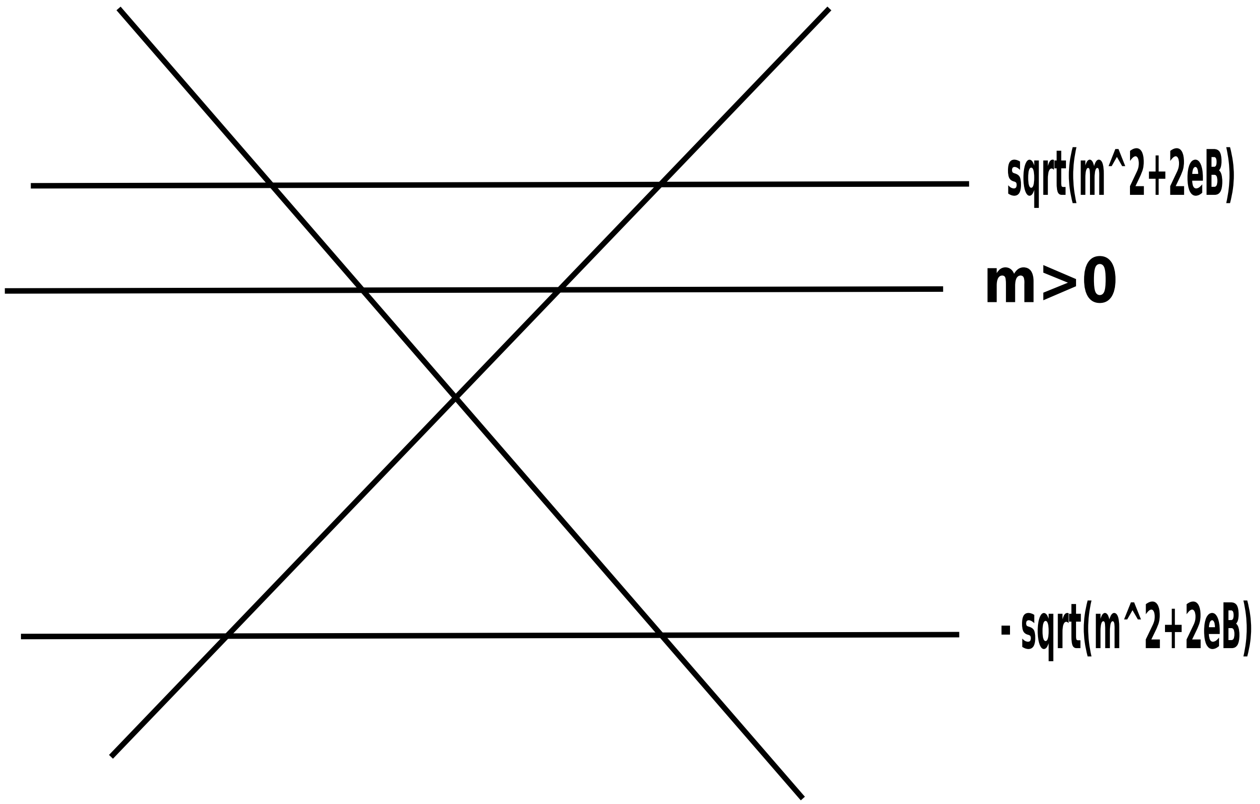
$$< Q >= -\frac{1}{2}\eta(h+\beta m) = -\frac{1}{2}\text{sign}(m)\cdot\frac{1}{2\pi}\int d^2x B(x)$$

Finite flux: bound states plus continuum states

Infinite flux: Landau level

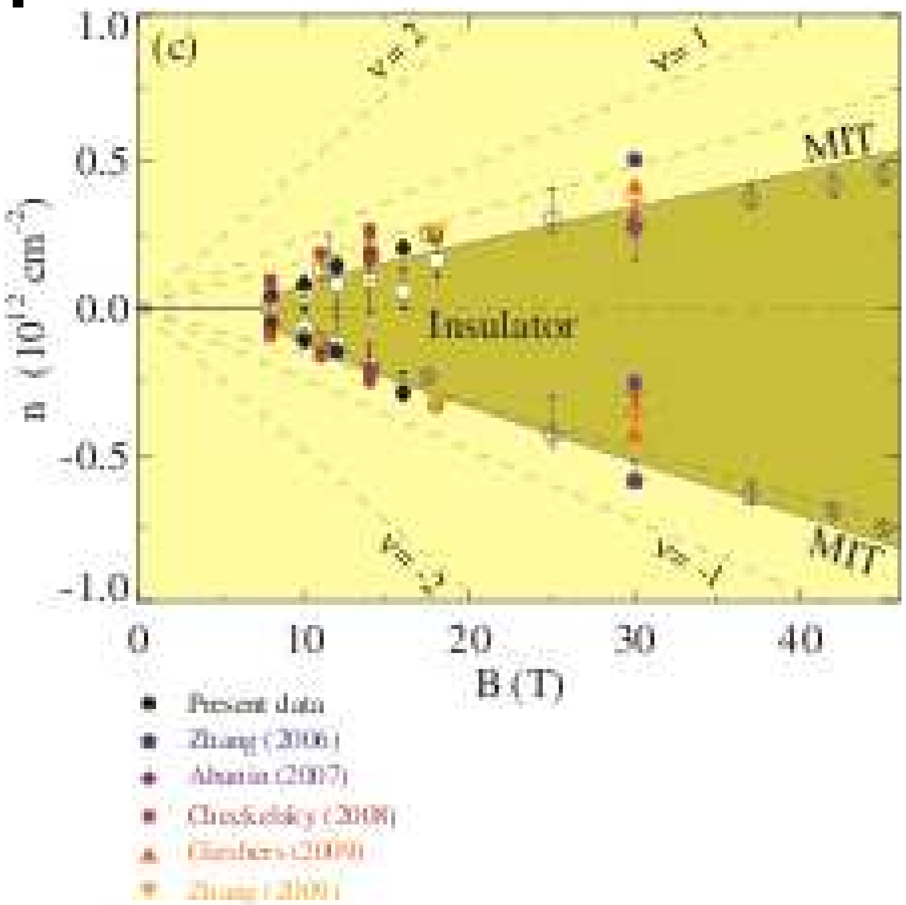


Depending on the sign of the mass, m, and the sign of $\int B$, the unpaired Laudau level can be on the negative threshold



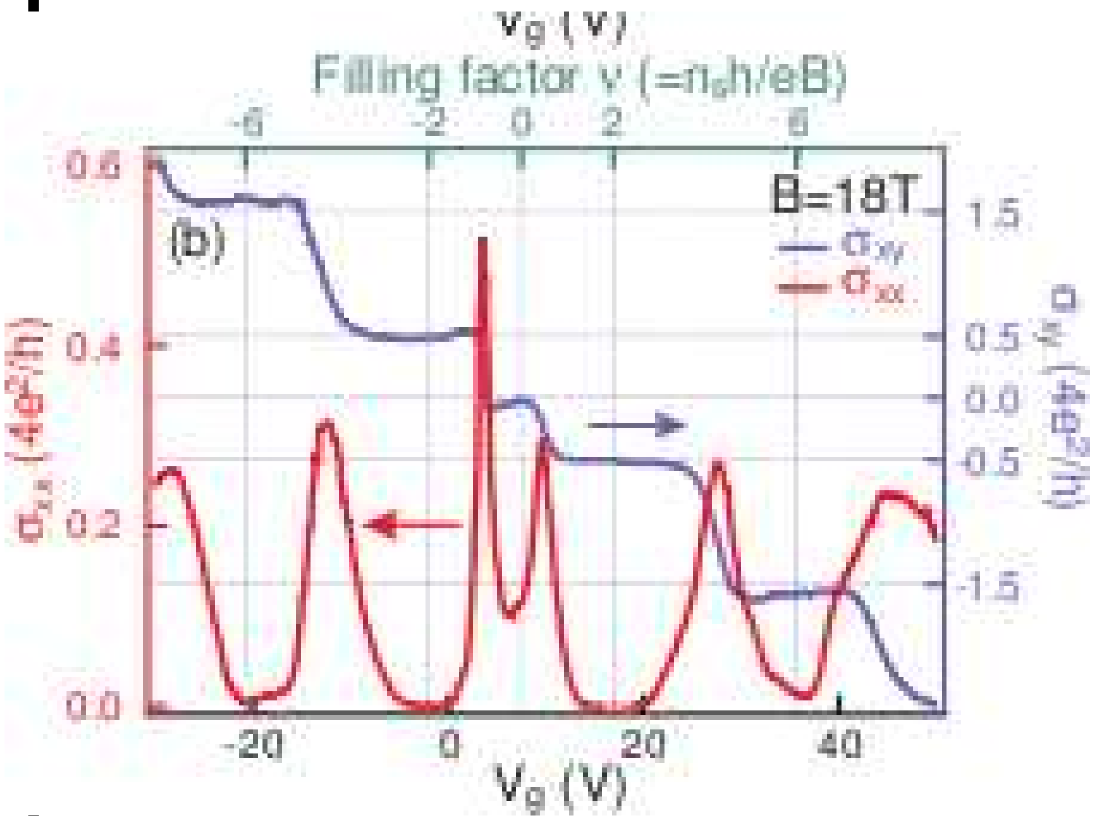
or the positive threshold

Splitting of $\nu = 0$ Landau level   Zhang et.al. arXiv:1003.2738



Quantum Hall Metal-Insulator transition

Splitting of $\nu = 0$ Landau level



QHE data as a function of the gate voltage V_g, for B = 18 T at T = 0.25 K

σ_{xx} and σ_{xy} , top axis shows LL filling factor ν.

With masses m_i $i = 1...4$ for spins and valleys

$$< Q >= -\frac{1}{2}\eta(h + \beta m) = -\frac{1}{2}\sum_i \text{sign}(m_i) \cdot \frac{1}{2\pi}\int d^2 x B(x)$$

$\psi^\dagger \beta M \psi$ M =Mass matrix 4×4, Hermitian, diagonalized by SU(4):

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}$$

Vafa-Witten theorem – no spontaneous parity breaking

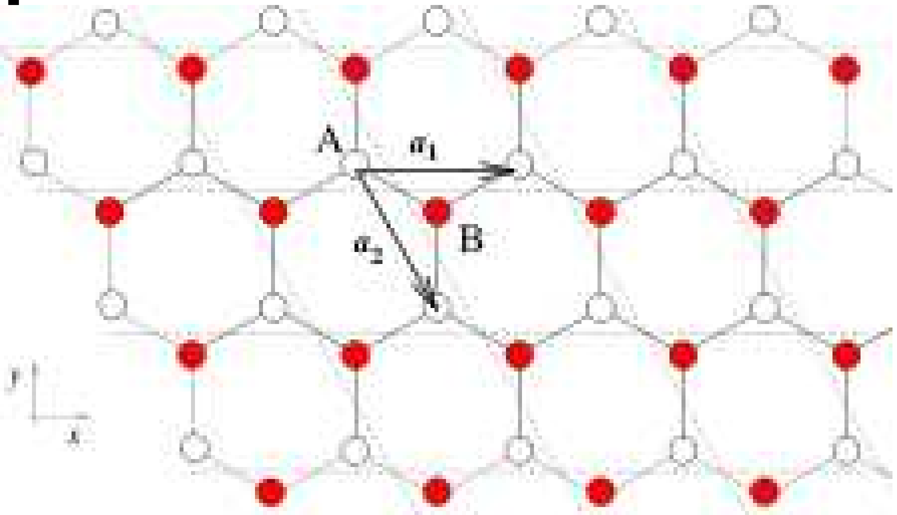
Parity and T: only if $\exists \Gamma \in SU(4)$: $\Gamma M + M\Gamma = 0$

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & -m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & -m_2 \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & -m_1 & 0 \\ 0 & 0 & 0 & -m_1 \end{bmatrix}$$

$$SU(4) \to U(1)^3 \qquad \text{or} \qquad SU(4) \to SU(2) \otimes SU(2) \otimes U(1)$$

Mass term from staggered on-site energy:

G. W. S. *Phys. Rev. Lett. 53, 2449 (1984)*



$$H = \sum_{A,i} \left(t b^{\dagger}_{A+\vec{b}_i} a_A + t^* a^{\dagger}_A b_{A+\vec{b}_i} \right) + \mu \sum_A a^{\dagger}_A a_A - \mu \sum_B b^{\dagger}_B b_B$$

$$H = \begin{bmatrix} i\vec{\sigma}\cdot\vec{D} + \sigma^3\mu & 0 \\ 0 & i\vec{\sigma}\cdot\vec{D} - \sigma^3\mu \end{bmatrix}$$

Chiral Symmetry breaking

In particle physics, *strong nuclear interactions (QCD)*

- break approximate chiral symmetry
- Quarks get mass,
- Pions are the (pseudo-) Goldstone bosons.
- Could this happen in graphene?

Continuum 2+1-dimensional quantum field theory with SU(4) symmetry:

$$\mathcal{L}(x) = i\bar{\psi}_a(x)\gamma^\mu D_\mu \psi_a(x) + \text{interactions} \quad a = 1, ..., 4$$

Chiral Symmetry breaking

- Consider graphene with Coulomb interaction

$$H = t\sum_{A,b_i}\left(\psi^\dagger_{A+b_i}\psi_A + \psi^\dagger_A\psi_{A+b_i}\right) + \frac{1}{2}\sum_{n,n'\in A,B}\rho_n\frac{e^2}{4\pi|n-n'|}\rho_{n'}$$

$$\rho_n = \sum_{\sigma=\uparrow\downarrow}\left(\psi^\dagger_{\sigma n}\psi_{\sigma n} - \frac{1}{2}\right)$$

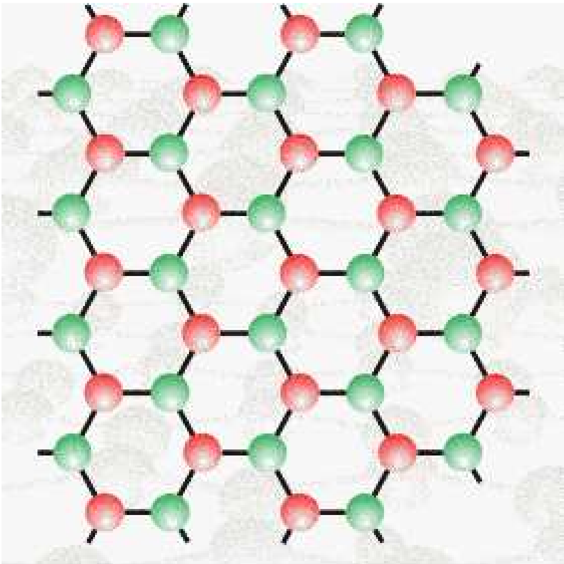
- Interaction is strong $\alpha_{\rm graphene} = \frac{e^2}{4\pi\hbar v_F} = \frac{c}{v_F}\frac{e^2}{4\pi\hbar c} \approx \frac{300}{137} \approx 2$ (P.Abbamonte 2010) $\alpha_{\rm eff} \approx 1/7$

- Coulomb interaction $\sim 10ev$. Hopping energy $\sim 2.7ev$.

- Analyze Coulomb interacting graphene using a strong coupling expansion: Neglect hopping term. Interaction Hamiltonian and charge density can be simultaneously diagonal.

$$H_{\text{Coulomb}} = \frac{1}{2} \sum_{n,n' \in A,B} \left(\psi^\dagger_{\sigma n}\psi_{\sigma n} - 1\right) \frac{e^2}{4\pi|n-n'|} \left(\psi^\dagger_{\sigma' n'}\psi_{\sigma' n'} - 1\right)$$

- Minimum of energy has one electron on each site (Coulomb energy vanishes).



$$H_{\text{Coulomb}} = \frac{1}{2} \sum_{n,n' \in A,B} \left(\psi^\dagger_{\sigma n} \psi_{\sigma n} - 1\right) \frac{e^2}{4\pi |n - n'|} \left(\psi^\dagger_{\sigma' n'} \psi_{\sigma' n'} - 1\right)$$

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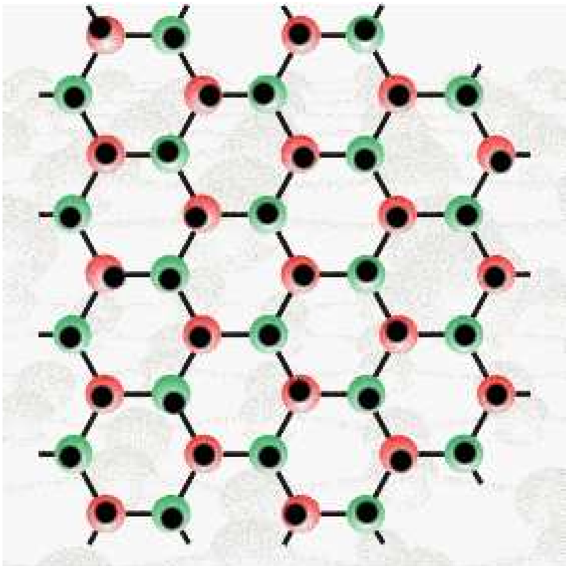
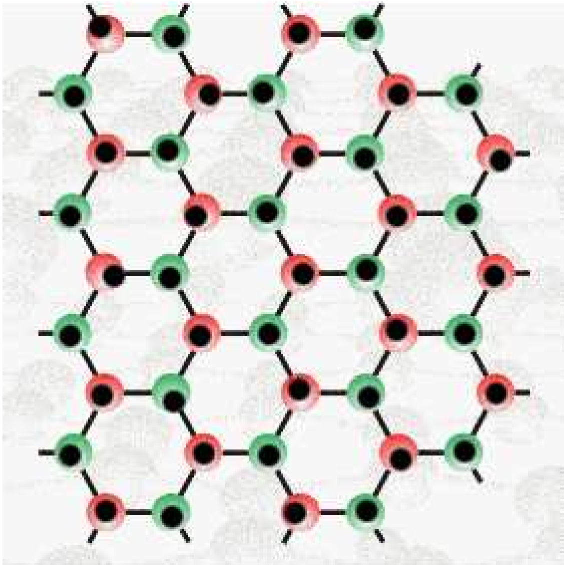
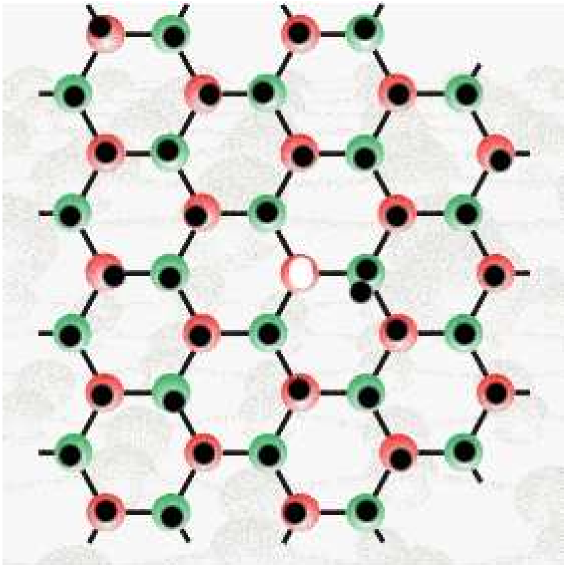




- Energy penalty for double occupation $\sim 10ev$
- Effective Hubbard model with $\frac{1}{2}$-filling:

$$H = t \sum_{A,b_i} \left(\psi^\dagger_{A+b_i} \psi_A + \psi^\dagger_A \psi_{A+b_i} \right) + U \sum_{n \in A,B} \left(\sum_{\sigma=\uparrow\downarrow} \psi^\dagger_{\sigma n} \psi_{\sigma n} - 1 \right)^2$$

$U \approx 10ev$



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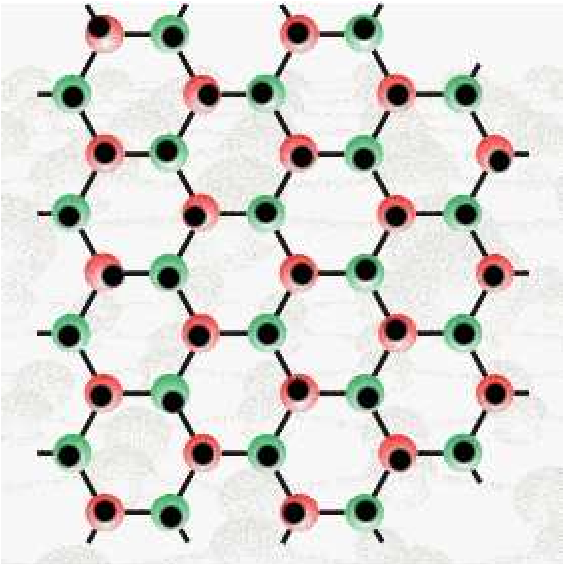
$U \approx 10ev$

- Degenerate state: 1 electron resides at each lattice site. All possible spin orientations have same energy.



- Resolve spin degeneracy using perturbation theory. Heisenberg antiferromagnet, $H \approx \frac{t^2}{2U} \sum_{A,i} \psi^\dagger_A \vec{\tau} \psi_A \cdot \psi^\dagger_{A+b_i} \vec{\tau} \psi_{A+b_i}$
- chiral symmetry breaking = antiferromagnetic order
  chiral condensate $= \psi^\dagger_A \tau_z \psi_A - \psi^\dagger_B \tau_z \psi_B = \bar{\psi} \tau_z \psi$

- Hubbard model on a hexagonal lattice has a critical $\frac{U}{t} \approx 5$ where antiferromagnetic order becomes stable.
T.Paiva, R.T.Scalettar, W.Zheng, R.R.P.Singh, J.Oitmaa, Phys.Rev.B 72, 085123 (2005)
S.Sorella, E.Tosatti 1992 Europhys.Lett.19 699

- Z.Y.Meng, T.C.Lang, S.Wessel, F.F.Assad, M.Muramatsu ArXiv:1003:5809
Semimetal $U/t < 3.4$, spin liquid $3.4 < U/t < 4.3$,
Neel $4.3 < U/t$.

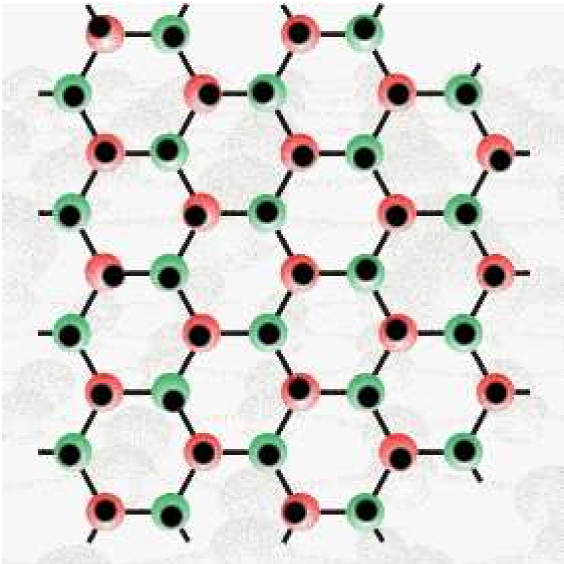
- for graphene $\frac{U}{t} \approx 4(??)$ is (maybe barely) subcritical.

- If $\frac{U}{t}$ could be increased, it could drive a quantum phase transition which breaks chiral symmetry.

- *Stretch graphene.* Increase lattice spacing. t decreases, U approximately constant as lattice spacing increases.

Spin Polarized Case

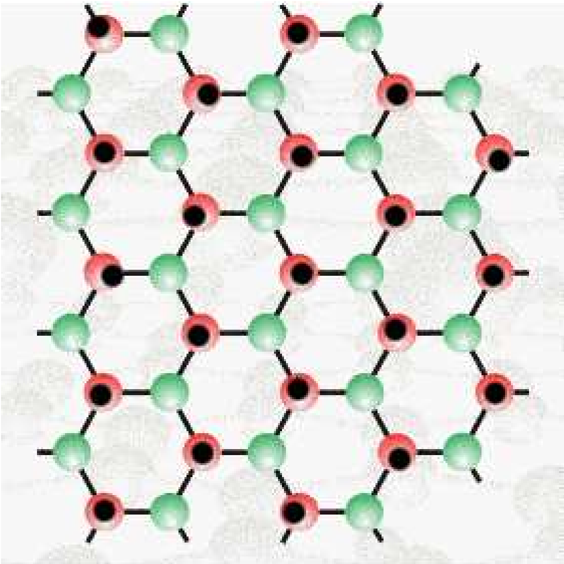
$$H_{\mathrm{Coul}} = \frac{1}{2}\sum_{xy}\left(\psi^\dagger(x)\psi(x) - \frac{1}{2}\right)\frac{e^2}{4\pi|x-y|}\left(\psi^\dagger(y)\psi(y) - \frac{1}{2}\right)$$



Half-filled lattice

Spin Polarized Case

$$H_{\rm Coul} = \frac{1}{2}\sum_{xy}\left(\psi^\dagger(x)\psi(x) - \frac{1}{2}\right)\frac{e^2}{4\pi|x-y|}\left(\psi^\dagger(y)\psi(y) - \frac{1}{2}\right)$$



Coulomb energy minimized by CDW

“Magnetic catalysis” by coulomb interaction $\nu = 0$

Gusynin et.al. Phys.Rev.Lett., 73 (1994) 3499; Phys.Rev.D, 52 (1995) 4718; Phys.Rev.B, 74 (2006) 195429
Khveshchenko PRL., 87 (2001) 206401; 87 (2001) 246802;
Khveshchenko and Leal , Nuc.Phys. 687 (2004) 323.
Herbut, PRL., 97 (2006) 146401; Phys.Rev.B, 75 (2007) 165411; 76 (2007) 085432

The tendency of Coulomb interactions to break chiral symmetry is enhanced by a magnetic field.

“Magnetic catalysis” by coulomb interaction $\nu = 0$

$$H_{\rm Coul} = \frac{1}{2}\int d^2x \int d^2y : \psi^\dagger(x)\psi(x) : \frac{e^2}{4\pi|x-y|} : \psi^\dagger(y)\psi(y) :$$

$$\psi(x) = \sum_{n=0}^{\int B/2\pi} \sum_{\sigma=\uparrow\downarrow} \psi_n(x) a_{n\sigma} + \sum_{E<0} \psi_E^{(+)}(x) a_E + \sum_{E<0} \psi_E^{(-)}(x) b_E^\dagger$$

$$: \psi^\dagger(x)\psi(x) := \sum_n \psi_m^\dagger(x)\psi_n(x) \sum_\sigma \left(a_{m\sigma}^\dagger a_{n\sigma} - \frac{\delta_{mn}}{2} \right) + ...$$

$$: \psi^\dagger(x)\psi(x) : |\text{singly occupied} >= \text{higher LL states}$$

Degenerate homogeneous state

Interactions (from mixing of Landau levels) break degeneracy →

Nëel state

“Magnetic catalysis” spin polarized $\nu = \pm 1$

$$H_{\rm Coul} = \frac{1}{2}\int d^2x \int d^2y : \psi^\dagger(x)\psi(x) : \frac{e^2}{4\pi|x-y|} : \psi^\dagger(y)\psi(y) :$$

$$\psi(x) = \sum_{n=0}^{\int B/2\pi} \psi_n(x) a_n + \sum_{E<0} (x)\psi_E a_E + \sum_{E<0} \psi_E(x) b_E^\dagger$$

$$: \psi^\dagger(x)\psi(x) := \sum_n \psi_m^\dagger(x)\psi_n(x)\left(a_{Am}^\dagger a_{An} + a_{Bm}^\dagger a_{Bn} - \delta_{mn}\right) + \ldots$$

$$: \psi^\dagger(x)\psi(x) : |\text{occupied sublattice A} >= \text{higher LL states}$$

CDW state

interactions from mixing of Landau levels

Antiferrmagnetic condensate

$$H = \begin{bmatrix} i\vec{\sigma}\cdot\vec{D} + \sigma^3\vec{m}\cdot\vec{\tau} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{D} - \sigma^3\vec{m}\cdot\vec{\tau} \end{bmatrix} \tag{1}$$

2×2 blocks are valleys

$\vec{D} = \vec{\nabla} - i\vec{A}$ electromagnetic $B_A = \vec{\nabla}\times\vec{A}$.

Néel order $\rightarrow$ CDW for each spin: $SU(4)\rightarrow SU(2)\times SU(2)\times U(1)$

"easy plane" (I.Herbut 2007) $\vec{m}\cdot\vec{\tau} = m_1\tau^2 + m_2\tau^2$

Vortex

$$\lim_{r\rightarrow\infty}\left[m_1(r,\theta) + im_2(r,\theta)\right] = \hat{m}e^{-in\theta} + \mathcal{O}\left(\frac{1}{r}\right)$$

Identical to Hamiltonian with mass from the Kekule distortion (Hou et.al. 2007, Jackiw et.al. 2007) interchange spin and valley labels. $\vec{\nabla} - i\vec{A} \rightarrow \vec{\nabla} - i\vec{A} - i\vec{V}\tau^3$

Zero modes from vortices (Herbut 2009)

$$h_0 = i\vec{\sigma}\cdot\vec{\nabla} + \vec{\sigma}\cdot\vec{A} + \vec{\sigma}\cdot\vec{V}\tau^3 + \vec{m}\cdot\vec{\tau}\sigma^3$$

gauge $A_r = 0$, $V_r = 0$

$A_\theta(r)$ = magnetic flux inside a disc of radius r

$$m(r)e^{in\tau^3\theta}\tau^1 u + ie^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta - \frac{A_\theta + \tau^3 V_\theta}{r}\right)v = 0$$

$$ie^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{A_\theta + V_\theta\tau^3}{r}\right)u - m(r)e^{in\tau^3\theta}\tau^1 v = 0$$

$u = e^{i\ell\theta}\tilde{u}(r)$, $v = e^{ik\theta}\tilde{u}(u)$, $\ell - k + n\tau^3 - 1 = 0$

$$m(r)\tau^1\tilde{u} + i\left(\partial_r + \frac{k - A_\theta - \tau^3 V_\theta}{r}\right)\tilde{v} \quad = \quad 0$$

$$i\left(\partial_r - \frac{\ell - A_\theta + V_\theta\tau^3}{r}\right)\tau^1\tilde{u} - m(r)\tilde{v} \quad = \quad 0 \quad , \ \tau^3 = \pm 1$$

near r=0, $\tau^1 \tilde{u}(r) \sim r^{-k}$, $\tilde{v}(r) \sim r^{\ell}$

Both solutions are normalizable at $r = 0$ if $\ell = 0, 1, 2, ..., |n| - 1$ and

$$\tau^3 \;=\; -\,\mathrm{sign}(n)$$

large r A_θ and V_θ grow slower than r

(Jackiw-Rossi 1981) $|n|$ zero modes

assume that at least one of A_θ V_θ grow faster than r

$$\tau^1 \tilde{u}(r) \;\sim\; e^{\int_0^r \frac{dr'}{r'}(A_\theta - \mathrm{sign}(n) V_\theta)}\; u_0$$

$$\tilde{v}(r) \;\sim\; e^{\int_0^r \frac{dr'}{r'}(-A_\theta - \mathrm{sign}(n) V_\theta)}\; v_0$$

$|A_\theta| > |V_\theta|$ one is normalizable $\rightarrow$ $|n|$ zero modes

$|A_\theta| < |V_\theta|$ $\mathrm{sign}(n) V_\theta < 0$ no sol'n

$|A_\theta| < |V_\theta|$ $\mathrm{sign}(n) V_\theta > 0$ 2 sol'ns mid-gap Landau level

Index theorem

$$\eta\left(h + \tau^3\sigma^3\epsilon\right) = \text{ Tr sign}\left(h + \tau^3\sigma^3\epsilon\right)$$

$\epsilon \to 0$ spectrum symmetric except for unpaired zero modes, +ve -ve

$$\langle Q \rangle = -\frac{e}{2}\eta\left(h + \tau^3\sigma^3\epsilon\right)$$

η is a topological invariant

total flux $\phi_V = \frac{1}{2\pi}\int d^2x B_V(x)$ finite,

electromagnetic flux $\phi_A = \frac{1}{2\pi}\int d^2x B_A(x)$ need not be

$$\eta(h + \epsilon\sigma^3\tau^3) = -\text{sign}(\epsilon)n - \frac{\epsilon}{\sqrt{\epsilon^2 + \hat{m}^2}}(n + \phi_V)$$

vorticity $n = \int_0^{2\pi} d\theta \epsilon^{ab} \frac{m^a \partial_\theta m^b}{2\pi m^2}$

does not depend on the electromagnetic gauge field $\vec{A}$ at all.

Conclusions

- Graphene provides a fascinating laboratory where some otherwise untestable field theory phenomena can be studied.
- Index theorem, related to the anomalous Hall effect.
- Graphene with a mass gap.
- Fractionally charged vortices.