Testing Vacuum Wavefunctionals on the Lattice

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"Confining Flux Tubes & Strings"

Trento, Italia, July 2010

Confinement and Casimir Scaling from Dimensional Reduction

A long time ago it was argued that at large distance scales, the pure Yang-Mills vacuum state in temporal gauge, satisfying

$$H\Psi_0 = E_0\Psi_0$$

looks like this:

$$\Psi_0^{eff}[A] \approx \exp\left[-\mu \int d^3x \; F_{ij}^a(x) F_{ij}^a(x)\right]$$ **J.G. (1979)**

This vacuum state has the property of ***dimensional reduction***:

Computation of a spacelike loop in 3+1 dimensions reduces to the calculation of a Wilson loop in Yang-Mills theory in 3 Euclidean dimensions.

Dimensional reduction from D = 4 → 3 → 2

$$
\begin{aligned}
W(C) &= \langle \mathrm{Tr}[U(C)] \rangle^{D=4} = \langle \Psi_0^{(3)} | \mathrm{Tr}[U(C)] | \Psi_0^{(3)} \rangle \\
&\sim \langle \mathrm{Tr}[U(C)] \rangle^{D=3} = \langle \Psi_0^{(2)} | \mathrm{Tr}[U(C)] | \Psi_0^{(2)} \rangle \\
&\sim \langle \mathrm{Tr}[U(C)] \rangle^{D=2}
\end{aligned}
$$

In D=2 dimensions the Wilson loop can be calculated analytically, and we know there is an area-law falloff, and ***Casimir-scaling*** string tensions.

- Is this the right explanation for Casimir scaling?
  Can we test it?

On the other hand, dimensional reduction can't be exactly right:

- no color screening, wrong N-ality properties
- wrong high momentum behavior

So what does the vacuum state look like at all scales, not just large scales?

A Proposal

arXiv: 0707.2860,

Stefan Olejnik and I claim that the ground state solution in D=2+1 dimensions, in temporal gauge, at all scales, is approximated by

$$\Psi_0[A] = \exp\left[-\frac{1}{2}\int d^2x d^2y \; B^a(x) \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}}\right)^{ab}_{xy} B^b(y)\right]$$

where

$B^a = F^a_{12}$ is the color magnetic field strength

$D^2 = D_k D_k$ is the covariant Laplacian in adjoint color representation,

λ_0 is the lowest eigenvalue of $-D^2$

m is a constant proportional to g^2

Generalization to D=3+1 dimensions is straightforward.

In support of this claim, we have shown that this expression for Ψ_0

- is a solution of the YM Schrodinger equation in the $g \to 0$ limit;
- solves the YM Schrodinger equation in the strong field, zero-mode limit;
- confines if *m > 0*, and that *m > 0* seems energetically preferred;
- results in the numerically correct values for the
  - mass gap
  - ghost propagator
  - Coulomb potential

derived from numerical simulation of our wavefunctional. (The string tension is an input.)

Dimensional Reduction: Expand B(x) in eigenmodes of the covariant Laplacian:

$$
\begin{aligned}
(-D^2)^{ab}\phi_n^b(x) &= \lambda_n \phi^a(x) \\
B^a(x) &= \sum_{n=0}^{\infty} b_n \phi_n^a(x) \\
B^{a,\text{slow}}(x) &= \sum_{n=0}^{n_{max}} b_n \phi_n^a(x)
\end{aligned}
$$

The cutoff mode sum defines the "slowly varying" B-field. Choosing n_{max} such that $\lambda_{n_{max}} - \lambda_0 \ll m^2$

$$
\int d^2x d^2y \; B^{a,\text{slow}}(x) \left(\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} \right)_{xy}^{ab} B^{b,\text{slow}}(y)
$$

$$
\approx \quad \frac{1}{m} \int d^2x \; B^{a,\text{slow}}(x) B^{a,\text{slow}}(x)
$$

So the part of the squared wavefunctional that involves B^{slow} is

$$|\Psi_0|^2 = \exp\left[-\frac{1}{m}\int d^2x\ B^{slow}B^{slow}\right]$$

which is the probability distribution of D=2 dimensional Yang-Mills (i.e. dimensional reduction). The string tension σ can be calculated analytically; in lattice units it is

$$\sigma = \frac{3}{4}\frac{m}{\beta}$$

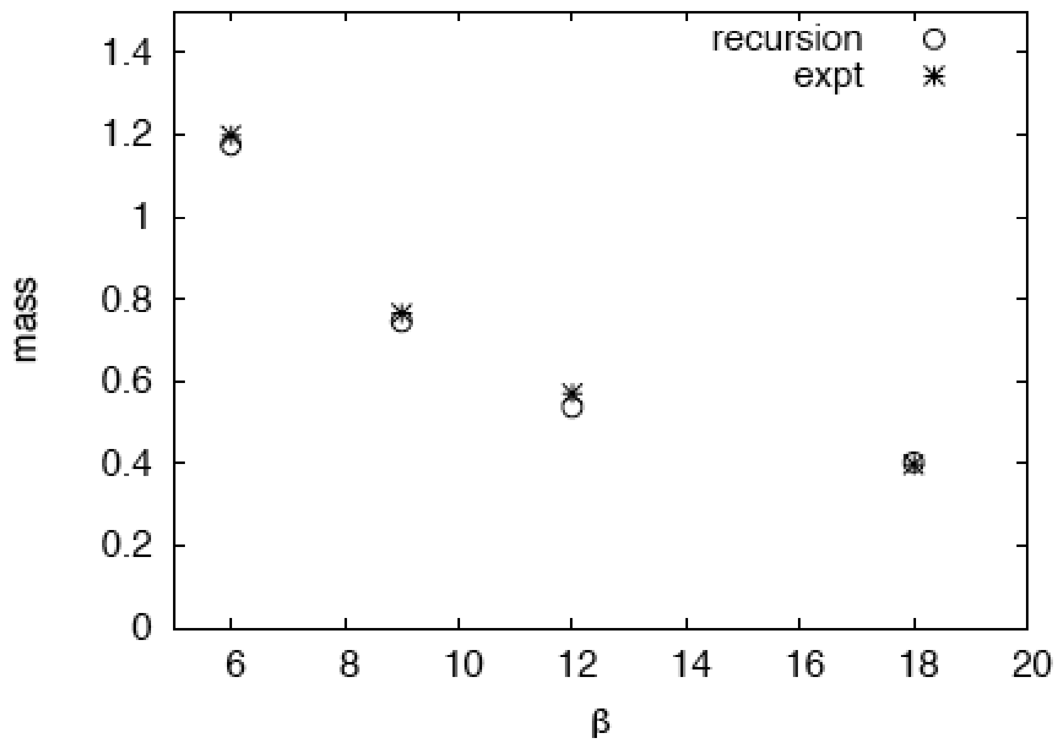
Suppose we turn this around, and fix $m = \frac{4}{3}\beta\sigma$, with σ taken from the Monte Carlo data. Then the full vacuum wavefunctional

$$\Psi_0[A] = \exp\left[-\frac{1}{2}\int d^2xd^2y\ B^a(x)\left(\frac{1}{\sqrt{-D^2-\lambda_0+m^2}}\right)^{ab}_{xy}B^b(y)\right]$$

must imply a definite value for the mass gap. What is it?

Results for the mass gap

- "recursion" is our result.
- "expt" is the Monte Carlo result for the 0^+ glueball, obtained by Meyer and Teper.

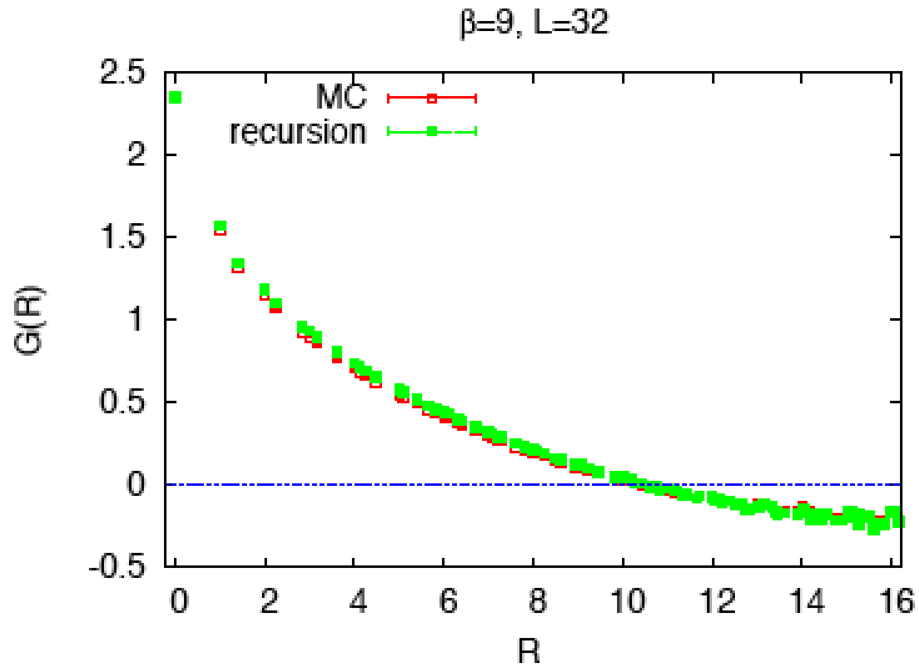


Given string tension σ, we have determined fairly accurately the 0^+ glueball mass.

Another observable we have looked at **(arXiv:1002.1189)** is the Coulomb gauge ghost propagator. This is evaluated by transforming each (MC or recursion) lattice to Coulomb gauge, and evaluating

$$G_{ghost}(\mathbf{x}-\mathbf{y}) = \left\langle \left(\frac{1}{-\nabla \cdot D} \right)^{aa}_{\mathbf{xy}} \right\rangle$$

with the result



The Coulomb potential is related to the ghost propagator

$$V_C(|x-y|) \propto \left\langle \left(\frac{1}{\nabla \cdot D}\right)_{xz} (-\nabla^2)_z \left(\frac{1}{\nabla \cdot D}\right)_{zy} \right\rangle$$

The Coulomb potential is very sensitive to “exceptional” configurations with very small λ_0 ; these lead to huge errorbars. To compare recursion and MC results, we impose cuts on the data, throwing away these rare configurations.

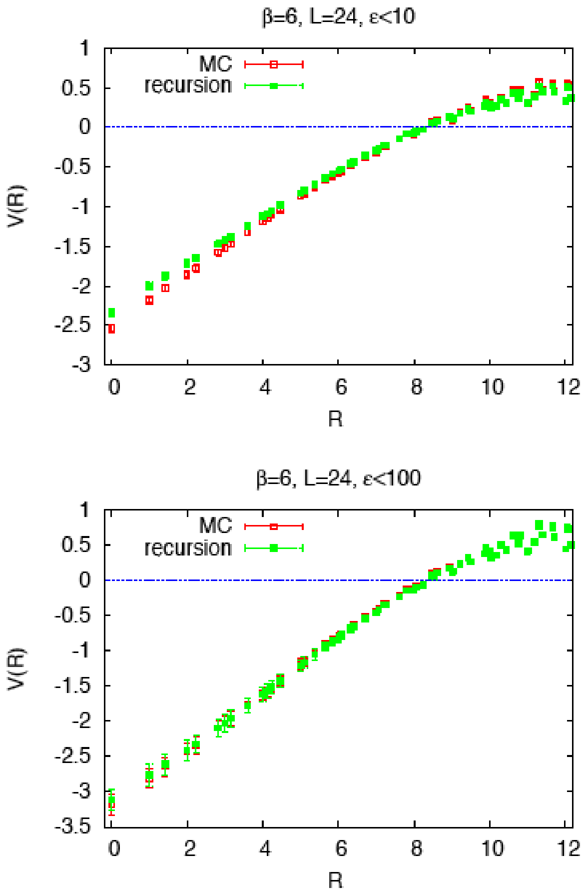


Figure 3: Color-Coulomb potential, $\beta = 6, L = 24$, computed from configurations with $\epsilon = |V(0)| < 10, 20, 100$ and 200.

Direct Measurement of $|\Psi_0[A]|^2$

A modified Monte Carlo: on time-slice t=0, restrict the available configurations to some subset

$$\{U_\mu^{(m)}, \quad m = 1, 2, ..., M\}$$

and select them, at t=0, via the Metropolis algorithm.

Let N_m be the number of times the m-th configuration is generated. In the limit that $N_{m,n} \rightarrow \infty$

$$\left|\frac{\Psi_0[U^{(m)}]}{\Psi_0[U^{(n)}]}\right|^2 = \frac{N_m}{N_n}$$

Reason: start with the identity

$$\Psi_0^2[U^{(n)}] = \frac{1}{Z}\int DU\ \delta(U_0)\delta[U(\mathbf{x},0) - U^{(n)}(\mathbf{x})]e^{-S}$$

Rescale

$$\begin{aligned}\widetilde{\Psi}_0^2[U^{(n)}] &= \frac{\Psi_0^2[U^{(n)}]}{\sum_{m=1}^{M}\Psi_0^2[U^{(m)}]} \\ &= \frac{\int DU\ \delta(U_0)\delta\Big[U(\mathbf{x},0) - U^{(n)}(\mathbf{x})\Big]e^{-S}}{\sum_{m=1}^{M}\int DU\ \delta(U_0)\delta\Big[U(\mathbf{x},0) - U^{(m)}(\mathbf{x})\Big]e^{-S}}\end{aligned}$$

This is a statistical system with the t=0 timeslice restricted to a set of M lattices. The system can be simulated numerically, and

$$\widetilde{\Psi}_0^2[U^{(n)}] = \lim_{N_{tot}\to\infty}\frac{N_n}{N_{tot}}$$

Since $\widetilde{\Psi}_0[U]$ is just a rescaling of $\Psi_0[U]$

$$\frac{\Psi_0^2[U^{(n)}]}{\Psi_0^2[U^{(m)}]} = \frac{\widetilde{\Psi}_0^2[U^{(n)}]}{\widetilde{\Psi}_0^2[U^{(m)}]} = \lim_{N_{tot}\to\infty} \frac{N_n}{N_m}$$

So, given

$$\mathbf{\Psi}_0[A] = \mathcal{N} e^{-R[A]}$$
$$= e^{-R[A]-R_0}$$

with *anybody's* proposal for **R[A]** , all we have to do is to plot

$$-\log\left[\frac{\mathbf{N_m}}{\mathbf{N_{tot}}}\right] \quad \text{vs.} \quad \mathbf{R}[\mathbf{U}^{(\mathbf{m})}]$$

If the proposal is right, then the data should fall on a straight line, with **slope = 1**.

Other Proposals

1. **Coulomb Gauge** (Reinhardt et al., Szczepaniak et al.)

This is basically a Gaussian ansatz

$$\Psi_0[A] = (\det[-\nabla \cdot D])^{-1/2} \exp\left[-\int A\omega A\right]$$

where the Faddeev-Popov factor cancels against the integration measure. Determine ***ω(p)*** by minimizing ***<H>*** ; it is found that ***ω(0)>0***.

I think this ansatz fails to give an area law for spacelike string tensions; I also think it is ruled out by the data (to be shown). Adam and Hugo may not (yet) agree.

2. New Variables D=2+1 (Karabali, Kim & Nair)

Change variables from A_μ^a to gauge-invariant J^a, the tradeoff is local gauge invariance for local holomorphic invariance under

$$\bar{\partial} J \rightarrow h(z)\bar{\partial} J h^{-1}(z)$$

KKN wavefunction in new (J-field) and old (A-field) variables:

$$\begin{aligned}\Psi_0^2 &= \exp\left[-\int \bar{\partial} J \frac{1}{\sqrt{-\nabla^2+m^2}+m} \bar{\partial} J\right] + \ldots \\ &= \exp\left[-\int B \frac{1}{\sqrt{-\nabla^2+m^2}+m} B\right] + \ldots\end{aligned}$$

where $m = \dfrac{g^2 C_A}{2\pi}$.

In new variables the bilinear is not holomorphic invariant, in old variables it is not gauge-invariant, and is therefore ***not a physical state***, and so impossible to test as it stands. *Some gauge-invariant completion is required.*

Options

- Assume that: gauge-inv KKN wf = the given bilinear form for abelian configurations, and test only on those.
- Assume that the gauge-inv KKN wf is obtained by replacing $-\nabla^2 \Rightarrow -D^2$.
- A hybrid approach: let $-\nabla^2 \Rightarrow -D^2 - \lambda_0$

We have so far tried out the first two options.

The second option probably doesn't work, because the spectrum of $-D^2$ is divergent in the continuum, leading (by diml reduction) to an infinite string tension. This tendency was already seen in arXiv:0707.2860, where the KKN string tension was off by 45% at β=12.

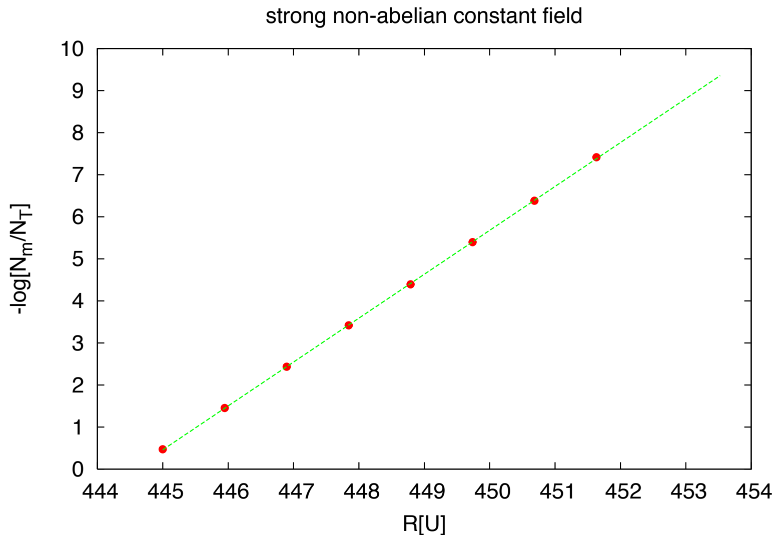
But we can try it anyway.

Some Results

I. Non-abelian constant configurations

$$U_1^{(m)}(n_1,n_2) = \sqrt{1-a_m^2}\,\mathbb{1} + i a_m \sigma_1$$
$$U_2^{(m)}(n_1,n_2) = \sqrt{1-a_m^2}\,\mathbb{1} + i a_m \sigma_2$$
$$a_m = \left(\frac{\alpha + m}{20L^2}\right)^{1/4}$$

parameters *β=6, L=24, α=400* ⇒



The slope is 1.04 .

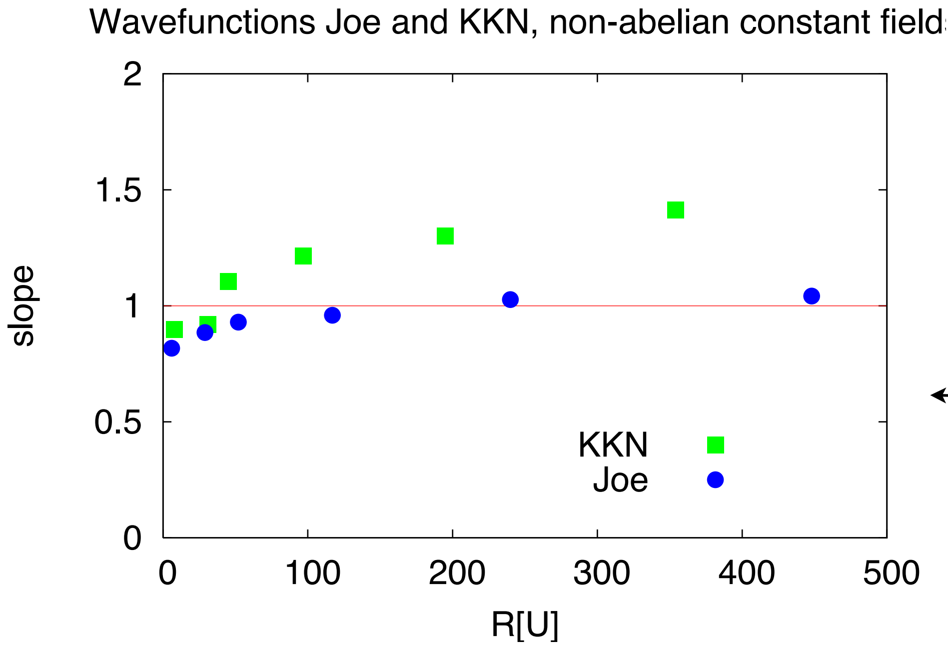
In any one run we can only look at a small range of R[U]

$$R_0 - \frac{1}{2}\Delta R < R[U] < R_0 + \frac{1}{2}\Delta R$$

but we can calculate the slope of

$$-\log\left[\frac{\mathbf{N_m}}{\mathbf{N_{tot}}}\right] \quad \text{vs.} \quad \mathbf{R}[\mathbf{U}^{(\mathbf{m})}]$$

in a sequence of runs, over a large range of R_0 .



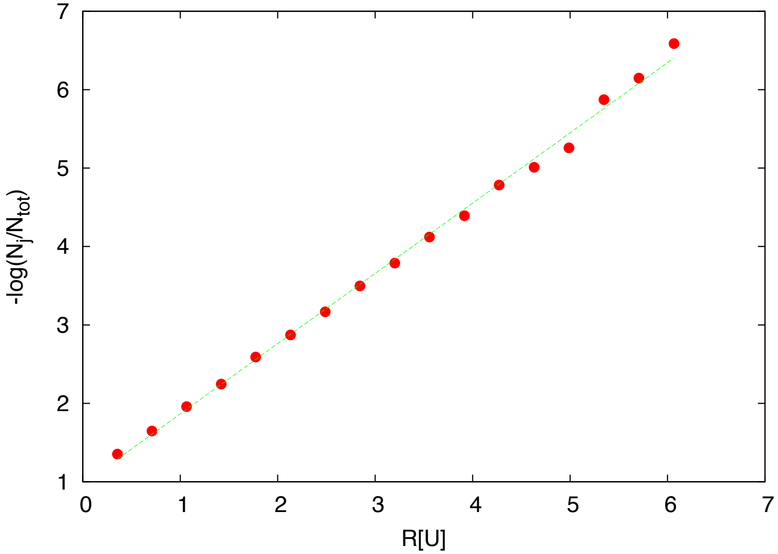
"Joe" = our proposal

"KKN" = Karabali, Kim, Nair
(gauge-inv)

This is at β=6 and L=24.

IIa. Abelian plane-wave configurations (same λ, different amplitudes)

$$U_1^{(j)}(n_1,n_2) = \sqrt{1-a_j^2(n_2)}\,\mathbb{1}_2 + i a_j(n_2)\sigma_3$$
$$U_2^{(j)}(n_1,n_2) = \mathbb{1}_2$$
$$a_j(n_2) = \frac{1}{L}\sqrt{\alpha+\gamma j}\cos(\widetilde{k} n_2)$$
$$\widetilde{k} = \frac{2\pi}{L}$$
$$\widetilde{k}^2 = 2(1-\cos(\widetilde{k}))$$



parameters α=0 , γ=0.5, β=6, L=24

the wavelength **λ = L** is the lattice extension. In a plot of

$$-\log\left[\frac{\mathbf{N_m}}{\mathbf{N_{tot}}}\right] \quad \text{vs.} \quad \mathbf{R}[\mathbf{U}^{(\mathbf{m})}]$$

the slope is 0.90 .

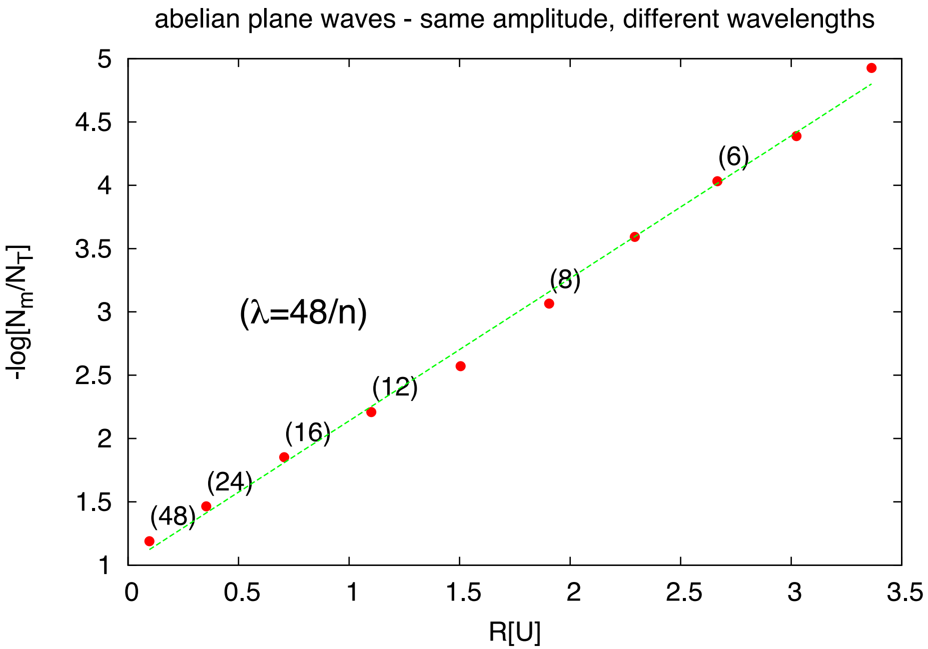
IIb. Abelian plane-wave configurations (same amplitude, different λ)

$$U_1^{(m)}(n_1,n_2) = \sqrt{1-a_m^2(n_1,n_2)}\,\mathbb{1}_2 + i a_m(n_1,n_2)\sigma_3$$
$$U_2^{(m)}(n_1,n_2) = \mathbb{1}_2$$
$$a_m(n_1,n_2) = \sqrt{\frac{\kappa}{L^2}}\cos\left(\frac{2\pi m}{L}n_2\right)$$

parameters κ=1 , β=6, L=48, m=1-10

wavelengths λ = 4.8 - 48

the slope is 1.08 ± 0.02 .



Combined Abelian Plane-Wave Data

The previous data was at $\beta=6$. Now we want to look at abelian plane waves at a variety of β and $\lambda=L$. To show all of the data together, we must express the dependence of R[U] on λ in physical units.

For the abelian plane waves used in these tests, the amplitude is varied while $\lambda=L$ is held fixed. We have, in physical units,

$$R[U^{(j)}] = 2(\alpha + j\gamma)\omega^{th}(k)$$

$$\omega^{th}(k) = \frac{k^2}{\sqrt{k^2 + m^2}}$$

where

$$k^2 = \frac{2}{a^2}\left(1 - \cos\left[\frac{2\pi}{L}\right]\right)$$

and the lattice spacing in physical units is

$$a = \sqrt{\frac{\sigma_{lattice}}{\sigma}} \quad , \quad \sigma = (440 \text{ MeV})^2$$

Extract ω_{MC} , from the MC lattices with a particular $\sigma_{lattice}$, from a best fit of the $\log(N_m/N_{th})$ data to a straight line

$$-\log(N_j/N_{tot}) = 2(\alpha + \gamma j)\omega_{MC} + \text{const.}$$

compare to

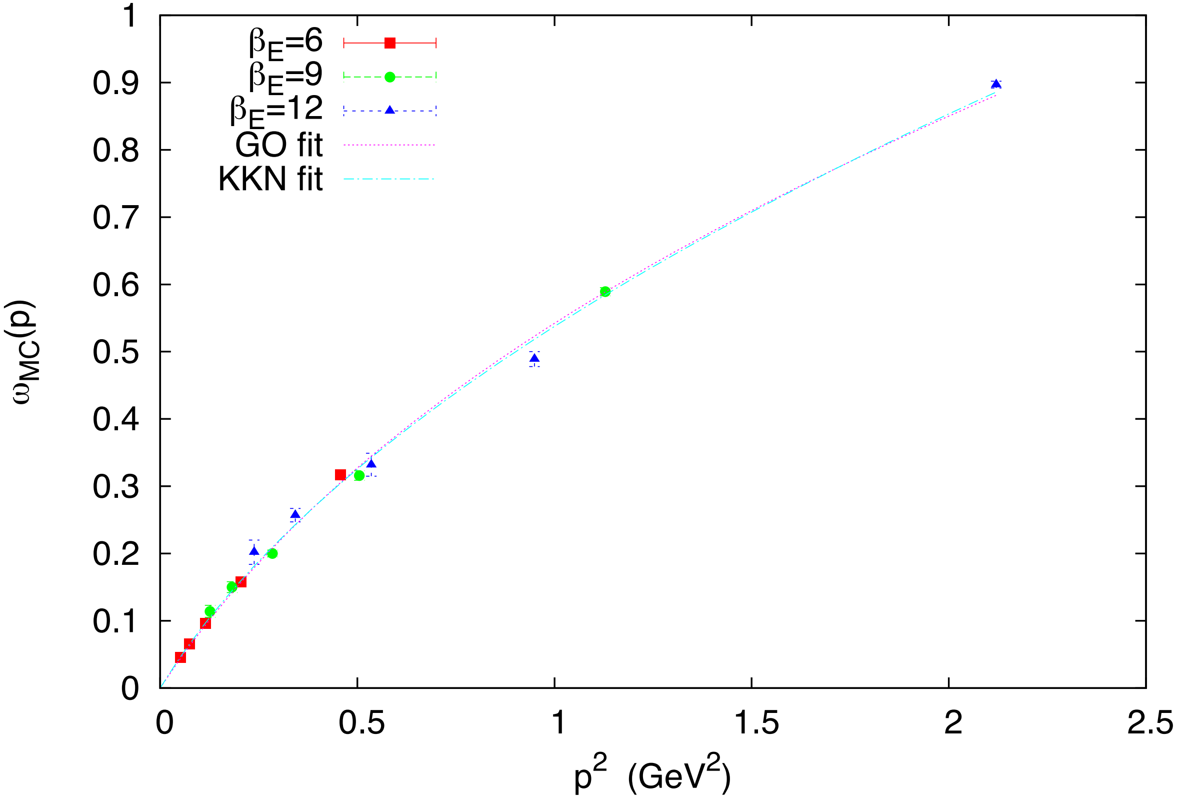
$$R[U^{(j)}] = 2(\alpha + j\gamma)\omega^{th}(k)$$

Plot $\omega_{MC}(k)$ vs. k.   If $\omega^{th}(k) \approx \omega_{MC}(k)$ then

- all data points should fall on the same curve, no matter what β was used to derive each point; and
- that curve should be $\omega_{MC}(k) = \frac{1}{g^2}\frac{k^2}{\sqrt{k^2+m^2}}$ for some g^2, m .

combined abelian plane-wave data at
β = 6, 9, 12 and **λ = 16, 24, 32, 40, 48**

Fitting parameters for GO and KKN are ***g^2 , m***



Cannot easily be reconciled with ***$\omega_{MC}(0) > 0$*** , as in the Reinhardt/ Szczepaniak proposals. Both GO and KKN work fine, and in this momentum range are indistinguishable.

What about N-ality?

Ψ_0 has the dimensional reduction form at large scales

$$\Psi_0[A] = \exp\left[-\frac{1}{2m}\int d^2x\ B^2\right]$$

2D Yang-Mills --> Casimir scaling.

However, Casimir scaling in D=2+1 and 3+1 does not hold asymptotically for all group representations. At large enough scales, the string tension should depend only on the N-ality of the static color charges, due to color screening by gluons.

So - what about color screening?

How is this problem solved at strong couplings, where we can solve for the ground state analytically? *(JG, 1980)*

The vacuum is

$$\Psi_0[U] = \exp[R(U)]$$

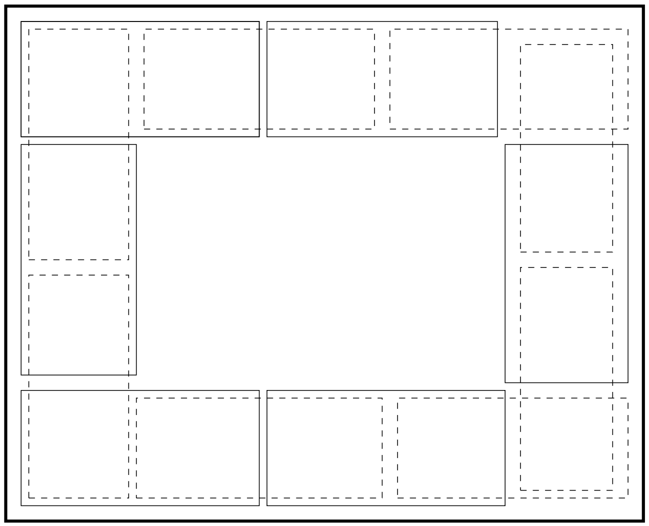
where, up to O(β^4)

$$R[U] \;=$$

$$\sum_{\text{contours}} c_0 \,\square \;+\; c_1 \,\boxed{} \;+\; c_2 \,\square\square \;+\; c_3 \,\boxed{\square}$$

$$+ \;\; \text{larger contours}$$

The 1X2 rectangle c_1  screens adjoint loops.



It also gives the leading correction to dimensional reduction.

Expansion in powers of lattice spacing:

$$\Psi_0[U] = \exp\left[-\frac{2}{\beta}\int d^2x\ \left(a\kappa_0 B^2 - a^3\kappa_2 B(-D^2)B + \ldots\right)\right]$$

where *(Guo et al, 94)*

$$\begin{aligned}\kappa_0 &= \frac{1}{2}c_0 + 2(c_1 + c_2 + c_3) \\ \kappa_2 &= \frac{1}{4}c_1\end{aligned}$$

Note! Leading correction comes from the rectangle ($\propto c_1$)

That's strong coupling. Returning to our proposal, expand the kernel in powers of $1/m^2$

$$\frac{1}{\sqrt{-D^2 - \lambda_0 + m^2}} = \frac{1}{m}\left(1 - \frac{-D^2 - \lambda_0}{2m^2} + \ldots\right)$$

Then the part of the vacuum that depends on B^{slow} is

$$\exp\left[-\frac{1}{m}\int d^2x \ \left(B^{slow}B^{slow} - B^{slow}\frac{-D^2 - \lambda_0}{2m^2}B^{slow} + \ldots\right)\right]$$

Leading correction is similar (and has the same sign) to the leading correction to diml reduction produced by the rectangle term.

Maybe this correction is responsible for color screening?

This term allows B (gluons) to "propagate", and, perhaps, break adjoint strings.

Conclusions

With the string tension as input, our proposal for the YM ground state wavefunctional has passed a number of very non-trivial tests:

- mass gap
- Coulomb gauge ghost propagator
- Coulomb potential
- Direct measurement of the wavefunctional, for non-abelian constant and abelian plane wave lattice configurations

all come out about right.

Wavefunctionals bilinear in the A rather than the B-field do not seem consistent with the data.

Next Steps

- To proceed further with the KKN proposal, *we need to know what it is!*
- Repeat in 3+1 dimensions.
- Understand color screening better.
- Determine ***m*** analytically, via a variational calculation??