Spectrum of (closed) flux tubes in SU(N) gauge theories
– results from the lattice

Mike Teper (Oxford)
+ Andreas Athenodorou (DESY, Zeuthen), Barak Bringoltz (UW, Seattle)

- closed flux tube spectrum : D=2+1

hep-th/0611286; arXiv:0709.0693; 0802.1490; 0812.0334

- closed flux tube spectrum : D=3+1

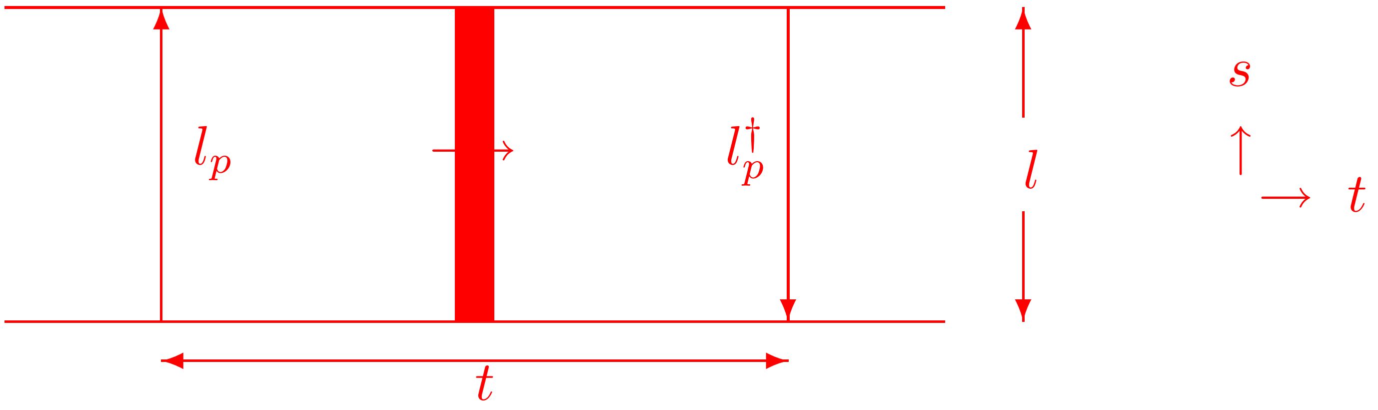
arXiv:0912.3238; draft paper

- What have we learned?

Calculate the mass of a confining flux tube winding around a spatial torus of length l, using correlators of Polyakov loops:

$$\langle l_p^\dagger(t) l_p(0)\rangle \stackrel{t\to\infty}{\propto} \exp\{-E_0(l)t\}$$

in pictures



where, for linear confinement,

$$E_0(l) \stackrel{l\to\infty}{=} \sigma l + O\left(\frac{1}{l}\right)$$

with a 'deconfining' phase transition at $l = l_c = 1/T_c$

where on general grounds:

$$
\begin{aligned}
E_0(l) \quad &\stackrel{l\to\infty}{=} \quad \sigma l && \text{linear confinement} \\
&- \quad \frac{\pi(D-2)}{6}\frac{1}{l} && \text{Luscher correction 1980} \\
&- \quad \frac{1}{2}\left(\frac{\pi(D-2)}{6}\right)^2 \frac{1}{\sigma l^3} && \text{Luscher-Weisz, Drummond 2004} \\
&- \quad \frac{1}{2}\left(\frac{\pi(D-2)}{6}\right)^3 \frac{1}{\sigma^2 l^5} && \text{Aharony-Karzbrun 2009} \\
&+ \quad O\left(\frac{1}{l^7}\right) && (1)
\end{aligned}
$$

and this is 'universal' – as long as only massless modes on the flux tube are from the transverse translations

Nambu-Goto in flat space-time :

– simplest example of a bosonic string theory

– a free string theory and 'sick' outside $D = 26$

but

diseases invisible in sector of states built on a single long string

e.g. P. Olesen, PLB160 (1985) 144; J. Polchinski, A. Strominger, PRL67 (1991) 1681.

$\Longrightarrow$

the gound state energy: J. Arvis, PLB 127 (1983) 127

$$E_0(l) = \sigma l \left(1 - \frac{\pi(D-2)}{3\sigma l^2}\right)^{\frac{1}{2}} \overset{l\to\infty}{=} \sigma l - \frac{\pi(D-2)}{6l} + \cdots$$

becomes tachyonic for

$$l \leq l_h = \sqrt{\tfrac{\pi(D-2)}{3\sigma}} \overset{!}{\sim} l_c = \tfrac{1}{T_c}$$

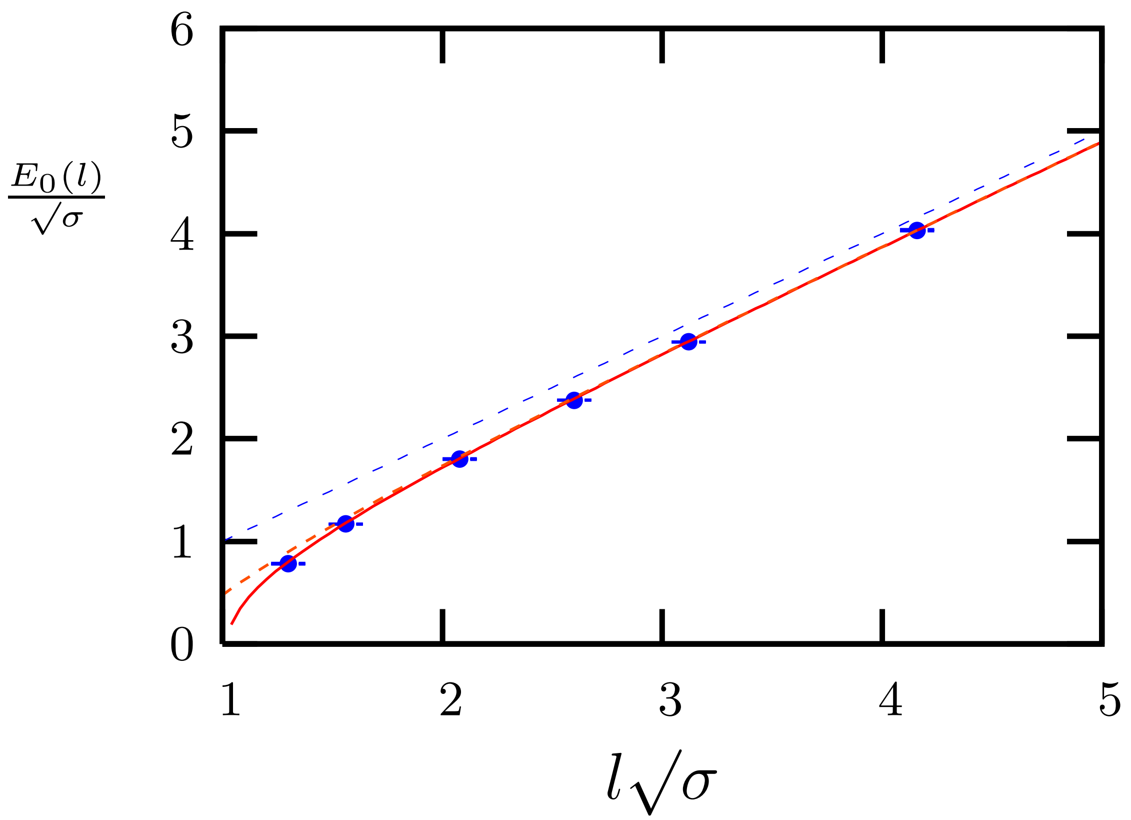
but $l_c > l_h$ for $N \to \infty \Rightarrow$ NG can, at larger N, be compared to the calculated energy for all $l \geq l_c$

$\longrightarrow$ D=2+1

- $g^2 \sim [m]^1$ sets scale ; instead of Λ_{MS} (dim trans)
- flux tubes $\rightarrow$ flux strips

SU(5) – ground state

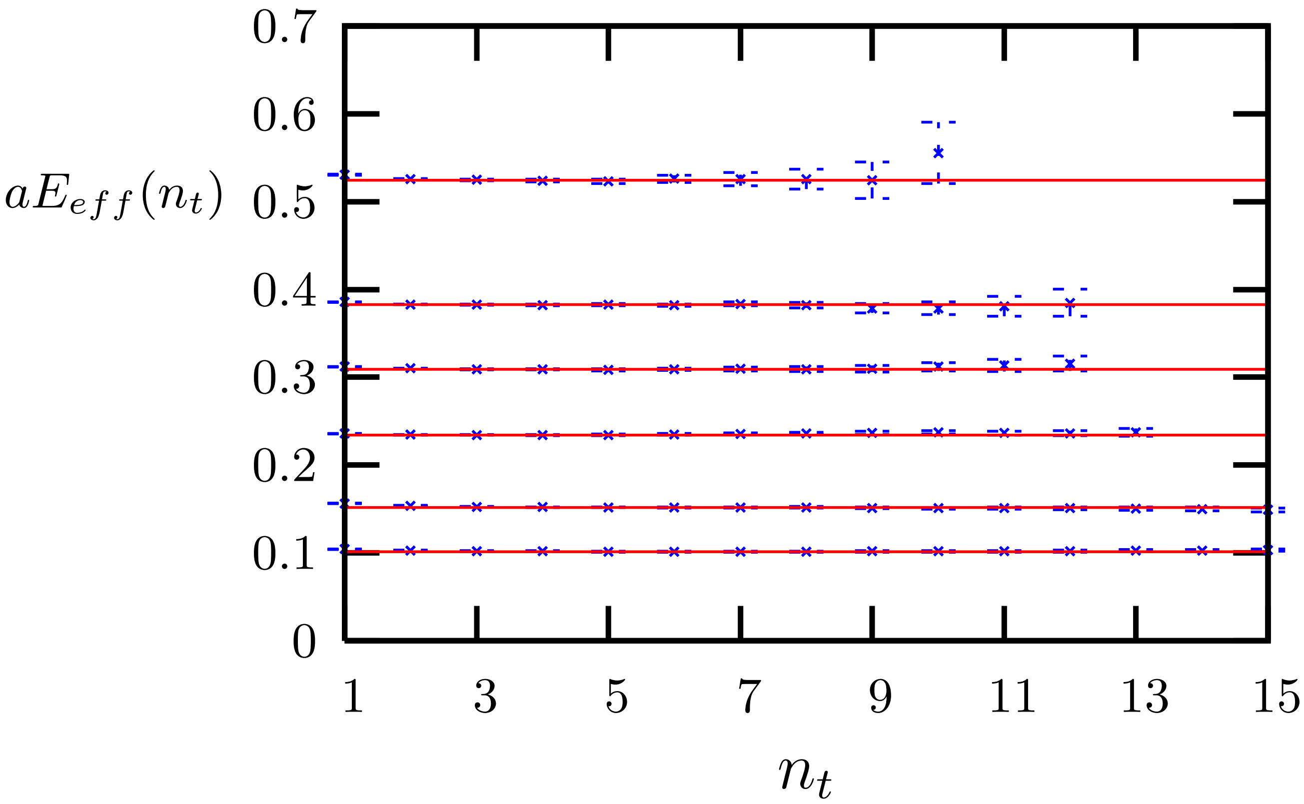
$a\sqrt{\sigma} \simeq 0.130 \quad ; \quad l_c\sqrt{\sigma} \simeq 1.07$



...Luscher: $E_0(l) = \sigma l - \frac{\pi}{6l}$; -Nambu-Goto:- $E_0(l) = \sigma l \left(1 - \frac{\pi}{3\sigma l^2}\right)^{\frac{1}{2}}$

how good are the exponential fits $C(t = n_t a) \propto e^{-aE_0(l)n_t}$?

$\longrightarrow$ local exponential fits: $aE_{eff}(t = n_t a) = -\ln C(n_t)/C(n_t - 1)$



$\longrightarrow$ good – but gets worse as l and $E \uparrow \longrightarrow$ very large l is inaccessible to MC

bosonic universality class?

fit to each pair of neighbouring values of l

- effective Luscher correction

$$E_0(l) = \sigma_{\text{eff}} l - c_{\text{eff}} \frac{\pi(D-2)}{6l}$$

- effective Nambu-Goto

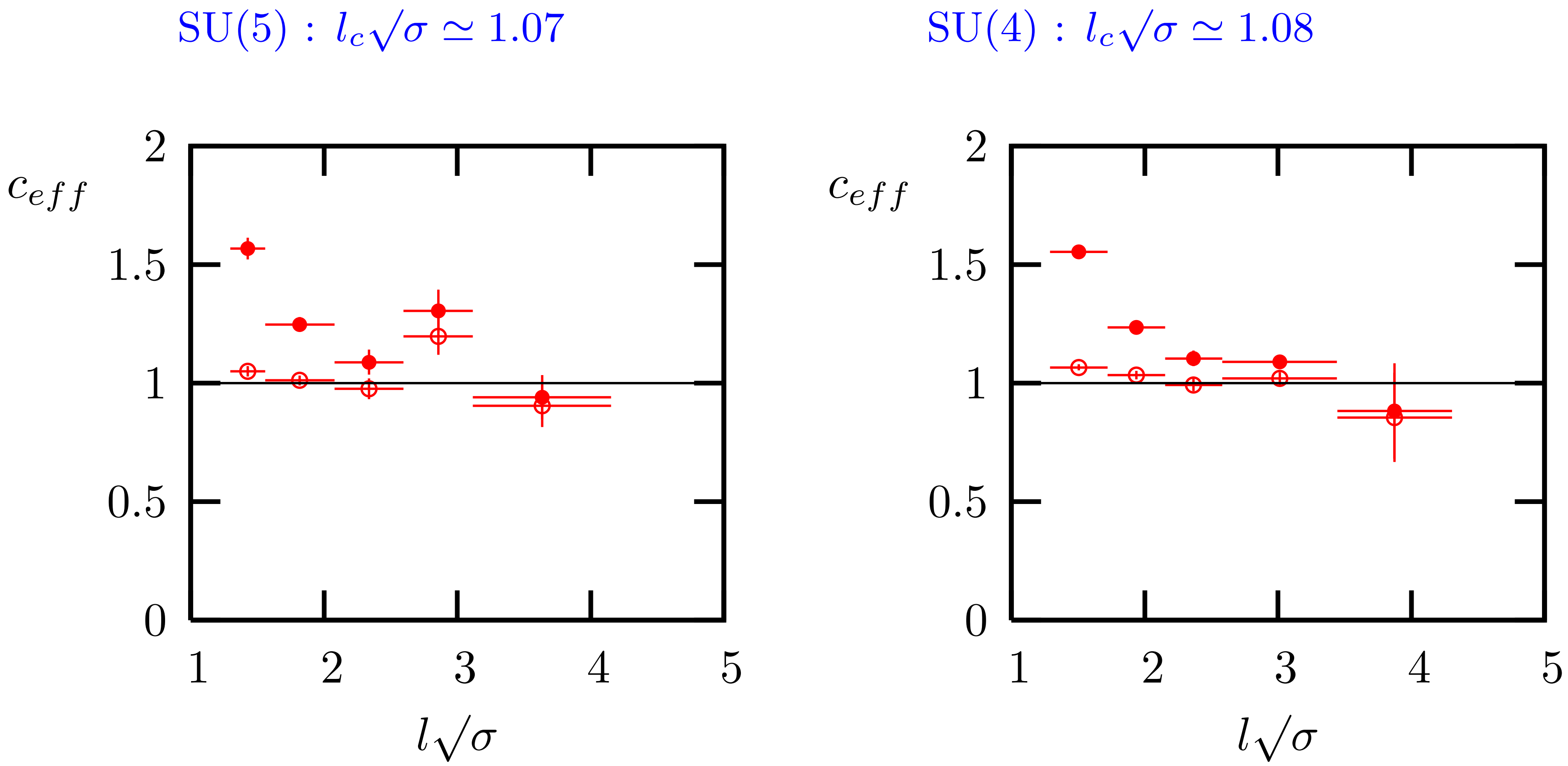
$$E_0(l) = \sigma_{\text{eff}} l \left(1 - c_{\text{eff}} \frac{\pi}{3\sigma_{\text{eff}} l^2}\right)^{\frac{1}{2}}$$

then if

$$c_{eff}(l) \stackrel{l\to\infty}{\longrightarrow} c = 1$$

the effective string action belongs to the universality class of a simple bosonic where the only massless modes of the string are the transverse translations

if $c \neq 1$ then there must be other massless modes on the string, e.g. $c = 1,\ 1.5,\ 0$ for bosonic, Neveu-Schwartz, Ramond strings respectively



c_{eff} : from Luscher ● , and from Nambu-Goto ○

$\Rightarrow$

- Luscher correction, good evidence that

$$c_{eff}(l \to \infty) \to 1$$

i.e. the only massless mode of confining flux tube is that associated with the spontaneous breaking of translations transverse to the string
but
significant deviation of $c_{eff}(l)$ from unity at smaller $l \to$ large higher order corrections to the Luscher term

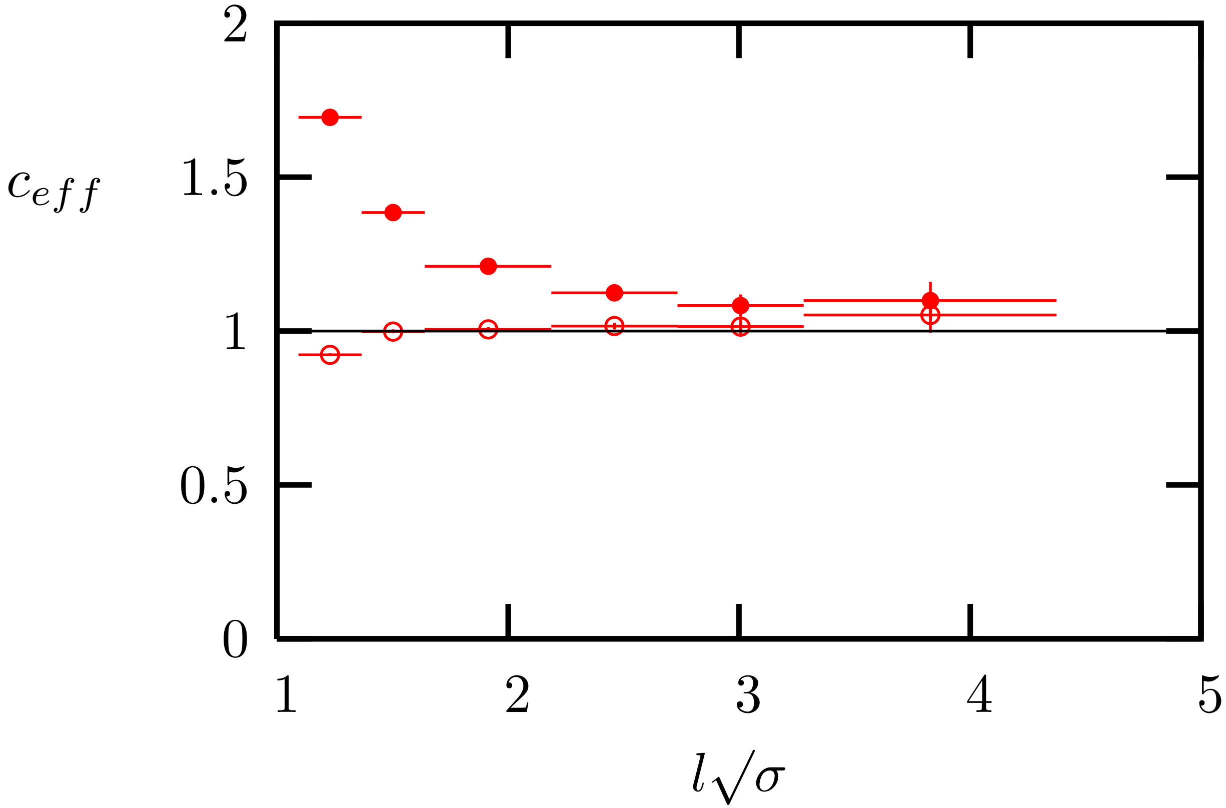
- by contrast: Nambu-Goto fits almost exactly all the way down to $l \sim l_c \sim 1/\sqrt{\sigma}$

$$c_{eff}^{NG}(l) \simeq 1 \quad \forall l$$

$\Rightarrow$ the confining flux tube behaves almost exactly like an ideal free bosonic string, even when it is hardly longer than it is wide

Is this a large N effect?

small N? $\longrightarrow$ SU(2) : $l_c\sqrt{\sigma} \simeq 0.95$



c_{eff} : from Luscher $\bullet$, and from Nambu-Goto $\circ$

$\Longrightarrow$ the ground state flux tube energy, $E_0(l)$, is very close to that of a free bosonic string theory $\forall N$.

∘ maybe not surprising since the next two corrections beyond the Luscher term are now known to be exactly what you get expanding Nambu-Goto in $1/\sigma l^2$?

∘ these corrections are very small and hard to see as they come from the zero-point energy contibutions of the oscillators: so they will be much larger in the excited states

∘ so what do the excited states look like?

Strings in D=2+1 : quantum numbers

- ∘ length of string, l

- ∘ non-zero momentum $p = 2\pi q/l$ along string

$\longrightarrow$ requires a deformation along the string

$\longrightarrow$ so need non-trivial phonon excitation: $q = N_L - N_R$

- ∘ parity: $h(x) \to -h(x) \quad \leftrightarrow \quad a_k \to -a_k, \tilde{a}_k \to -\tilde{a}_k$

$\longrightarrow P = (-1)^{number\,of\,phonons}$

- ∘ no rotations (2 space dimensions); transverse momentum uninteresting;

$C = \pm$ sectors degenerate, so charge conjugation uninteresting.

Nambu-Goto free string theory

$$\int \mathcal{D}X e^{-\frac{i}{\sigma}\times \text{Area}}$$

spectrum:

$$E_n^2(l;q) = (\sigma\, l)^2 + 8\pi\sigma \left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 .$$

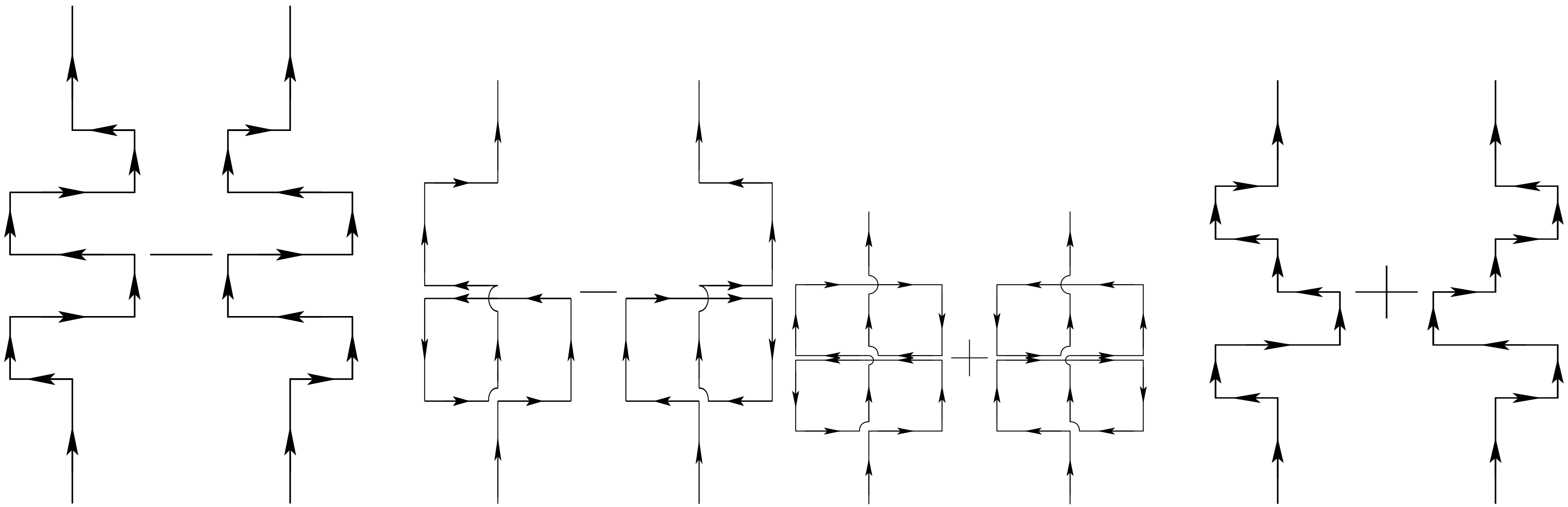
$2\pi q/l =$ total momentum along string;
$N_L, N_R =$ sum left and right oscillators ('phonons');
state $= \prod_k a_k^{n_k} \prod_{k'} \tilde{a}_{k'}^{n_{k'}} |0\rangle$

$$N_L = \sum_{k>0} n_L(k)\, k, \quad N_R = \sum_{k'>0} n_R(k')\, k', \quad N_L - N_R = q$$

J. Arvis, Phys. Lett. 127B(1983)106

Excited States

to have good overlaps onto excited string states, we need to include many more operators in our variational basis – in particular operators that 'look' excited and ones that have an intrinsic handedness so that we can construct $P = -$ as well as $P = +$, e.g.



typically we have 100-200 operators in our basis ...

• here we have in addition generalised to a variational calculation over a vector space V_ϕ spanned by some convenient blocked/smeared operators $\{\phi_i; i = 1, ..., n\}$ of the desired quantum numbers:

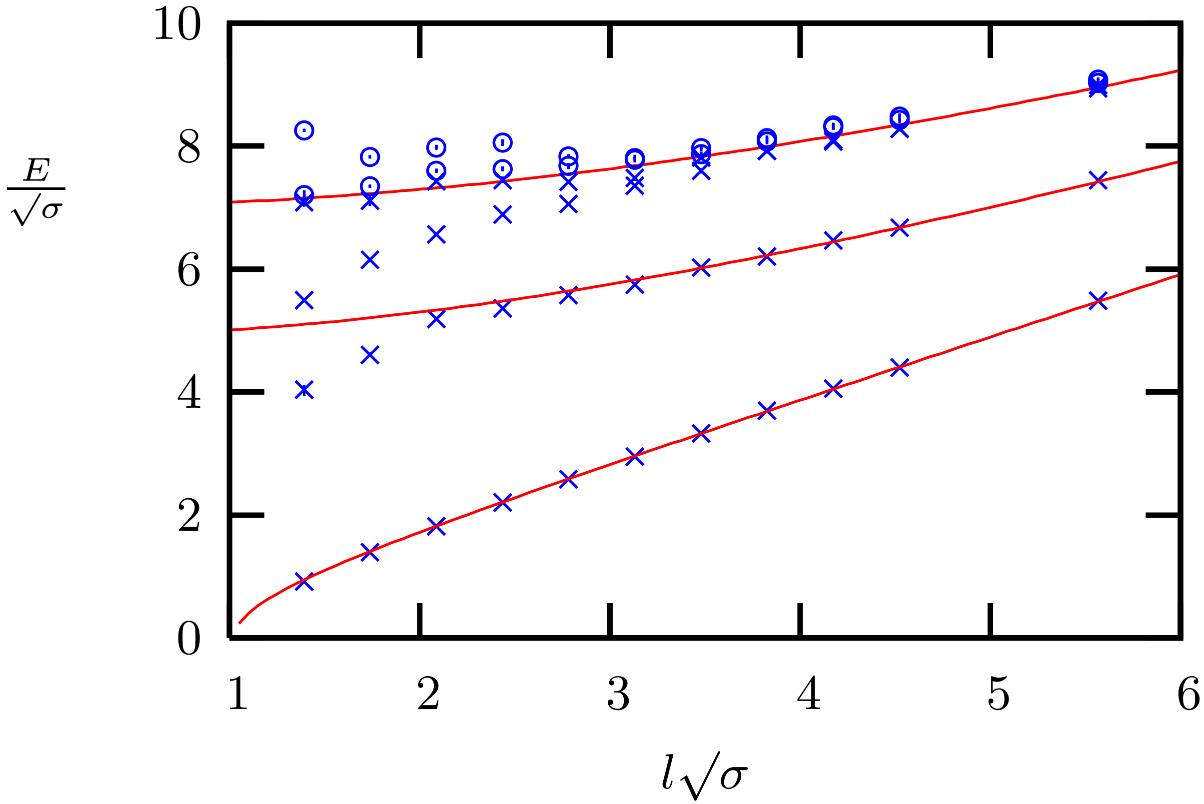
$$\langle {\psi_0}^\dagger(t_0)\psi_0(0)\rangle = \max_{\phi \in V_\phi} \langle \phi^\dagger(t_0)\phi(0)\rangle = \max_{\phi \in V_\phi} \langle \phi^\dagger e^{-Ht_0}\phi\rangle$$

where t_0 is some convenient value of t. Then ψ_0 is our best variational estimate for the true eigenfunctional of the ground state (with these quantum numbers). We can now use $\langle {\psi_0}^\dagger(t)\psi_0(0)\rangle$ to obtain our best estimate of the ground state mass.

• generalise this in an obvious way to calculating excited state energies

SU(3) : $q = 0$ closed string spectrum

$a\sqrt{\sigma} \simeq 0.174 \quad ; \quad l_c\sqrt{\sigma} \simeq 1.0$



— : Nambu-Goto : σ from ground state

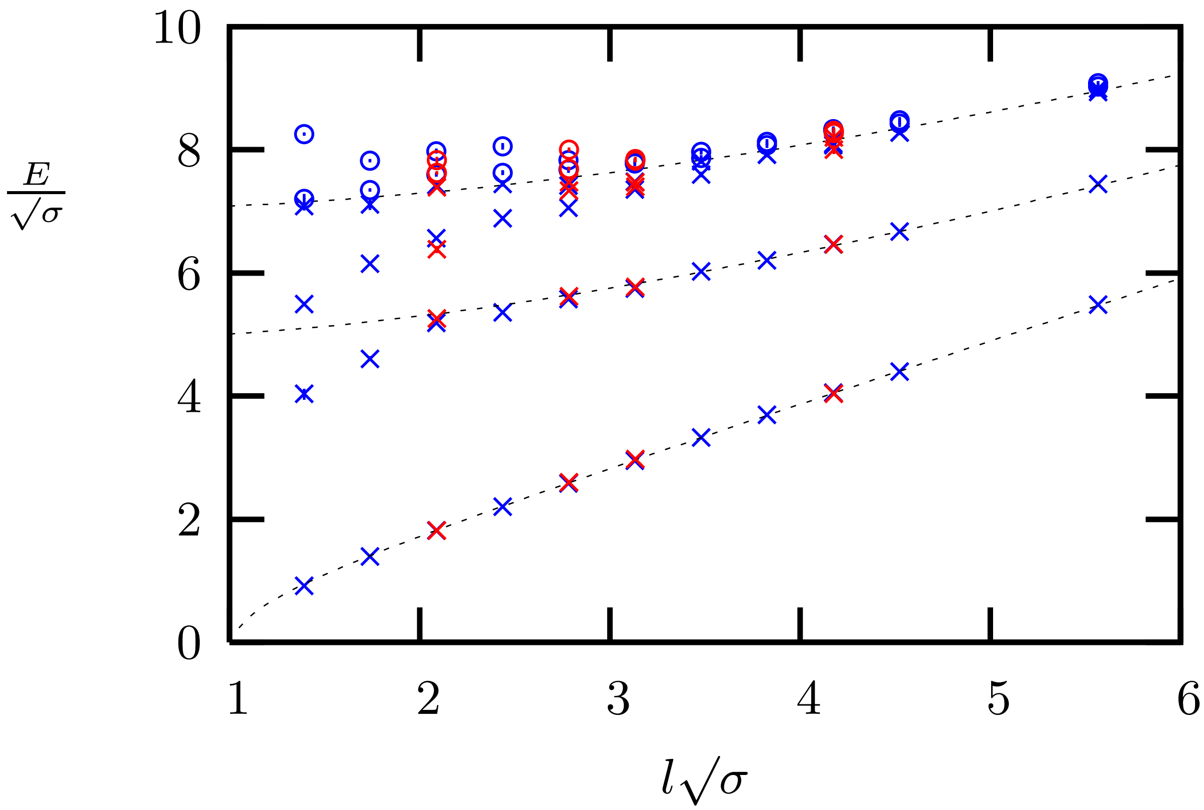
$\times$: +ve parity $\circ$: -ve parity content of lightest $q = 0$ NG states:

$\|0\rangle$	P=+, q=0
$a^R(k=1)a^L(k=1)\|0\rangle$	P=+, q=0
$a^R(k=2)a^L(k=2)\|0\rangle$	P=+, q=0
$a^R(k=1)a^R(k=1)a^L(k=1)a^L(k=1)\|0\rangle$	P=+, q=0
$a^R(k=2)a^L(k=1)a^L(k=1)\|0\rangle$	P=−, q=0
$a^R(k=1)a^R(k=1)a^L(k=2)\|0\rangle$	P=−, q=0

Since our lightest states have energies and degeneracies as in Nambu-Goto down to $l\sqrt{\sigma} \sim 2$, they are well-described by the above states.

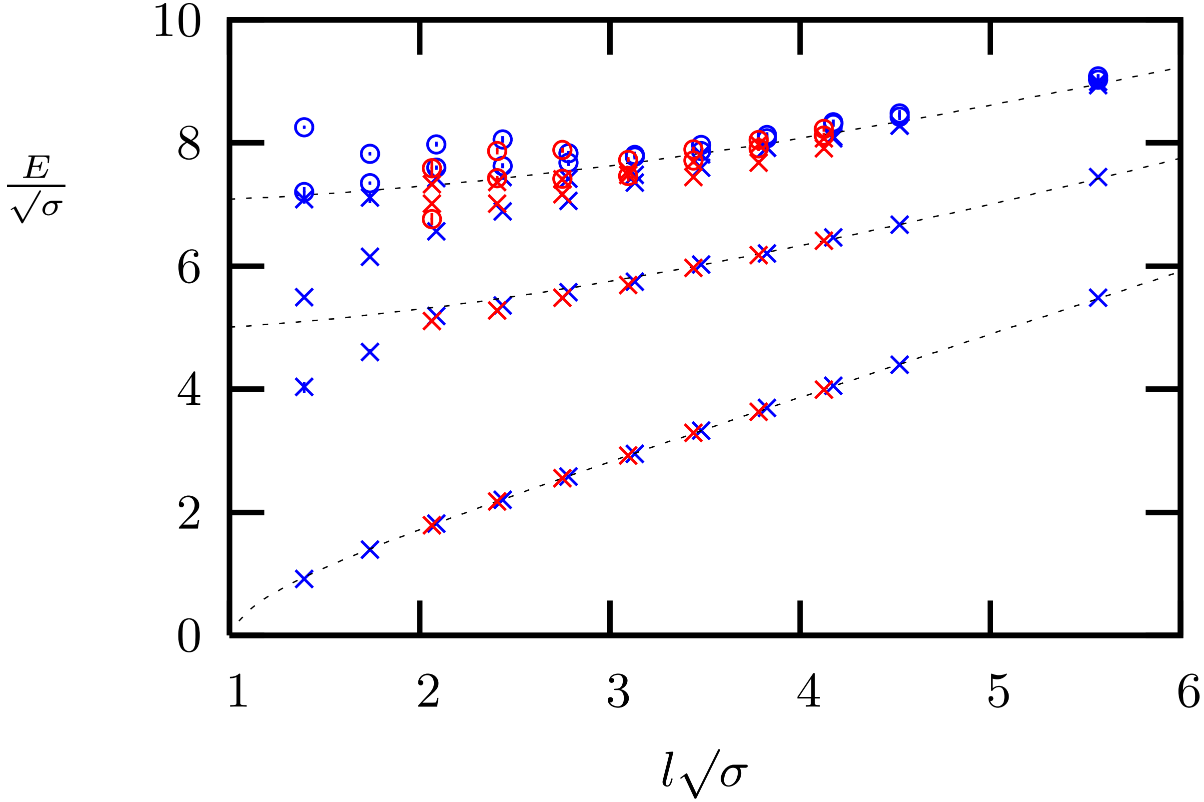
SU(3) : how close to continuum limit?

compare $a\sqrt{\sigma} \simeq 0.174 \quad vs \quad a\sqrt{\sigma} \simeq 0.087$



no significant difference as $a \to a/2 \Rightarrow$ we have ‘continuum’ physics

SU(3) vs SU(6) : same a



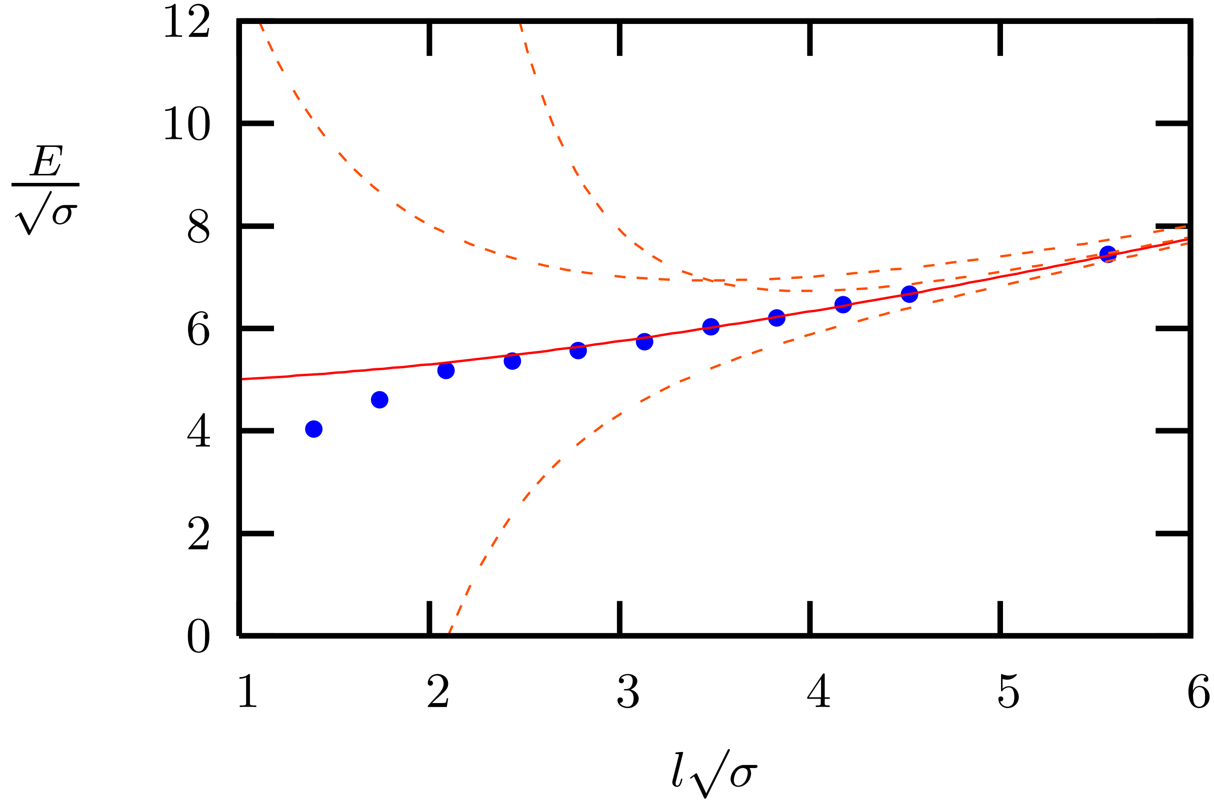
$\Rightarrow \quad SU(3) \simeq SU(\infty)$

$\diamond$ Striking agreement with free string model, down to $l\sqrt{\sigma} \simeq 2$.

$\diamond$ Remarkable since $l\sqrt{\sigma} \simeq 2 \Rightarrow$ the flux tube is maybe only twice as long as it is wide – hardly an ideal 'string'.

$\diamond$ Is this just a manifestation of the fact that we know (Luscher, Weisz, Drummond, Aharony, ...) that the first 3 or 4 terms in an expansion of $E_n(l)$ in powers of $1/\sigma l^2$ must be the same as Nambu-Goto?

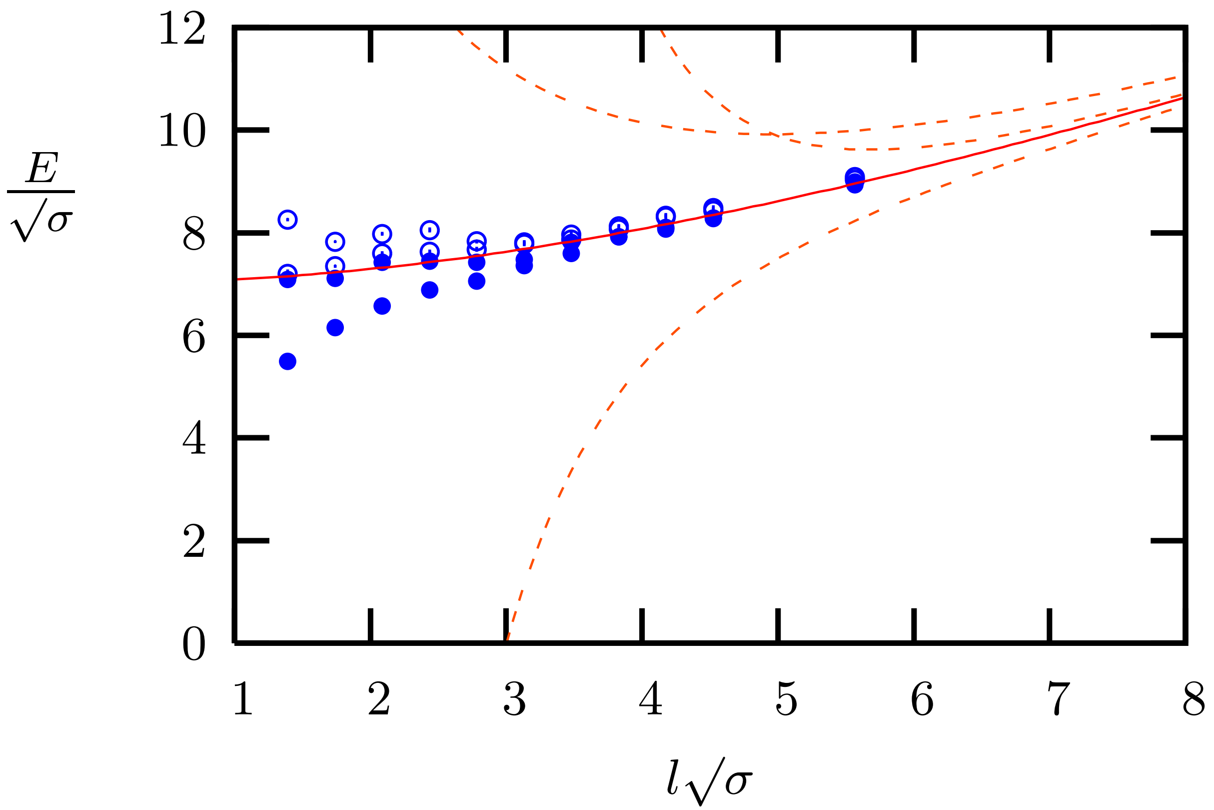
1st excited level



— Nambu-Goto : $E_n = \sigma l \sqrt{1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{1}{24}\right)}$

$\cdots$ Luscher 1980; Luscher-Drummond 2004; Aharony 2009

2nd excited level



— Nambu-Goto : $E_n = \sigma l \sqrt{1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{1}{24}\right)}$

$\cdots$ Luscher 1980; Luscher-Drummond 2004; Aharony 2009

Why ?

the covariant Nambu-Goto expression e.g. for $q = 0$,

$$E(l) = \sigma l \left(1 + \frac{8\pi}{\sigma l^2}\left(n - \frac{D-2}{24}\right)\right)^{\frac{1}{2}}$$

can only be expanded as a power series in $1/l\sqrt{\sigma}$ when

$$\frac{8\pi}{\sigma l^2}\left(n - \frac{1}{24}\right) \leq 1 \quad \leftrightarrow \quad l\sqrt{\sigma} \geq \sqrt{8\pi n} \sim 5\sqrt{n}$$

whereas in practice we have a very good fit by Nambu-Goto even down to

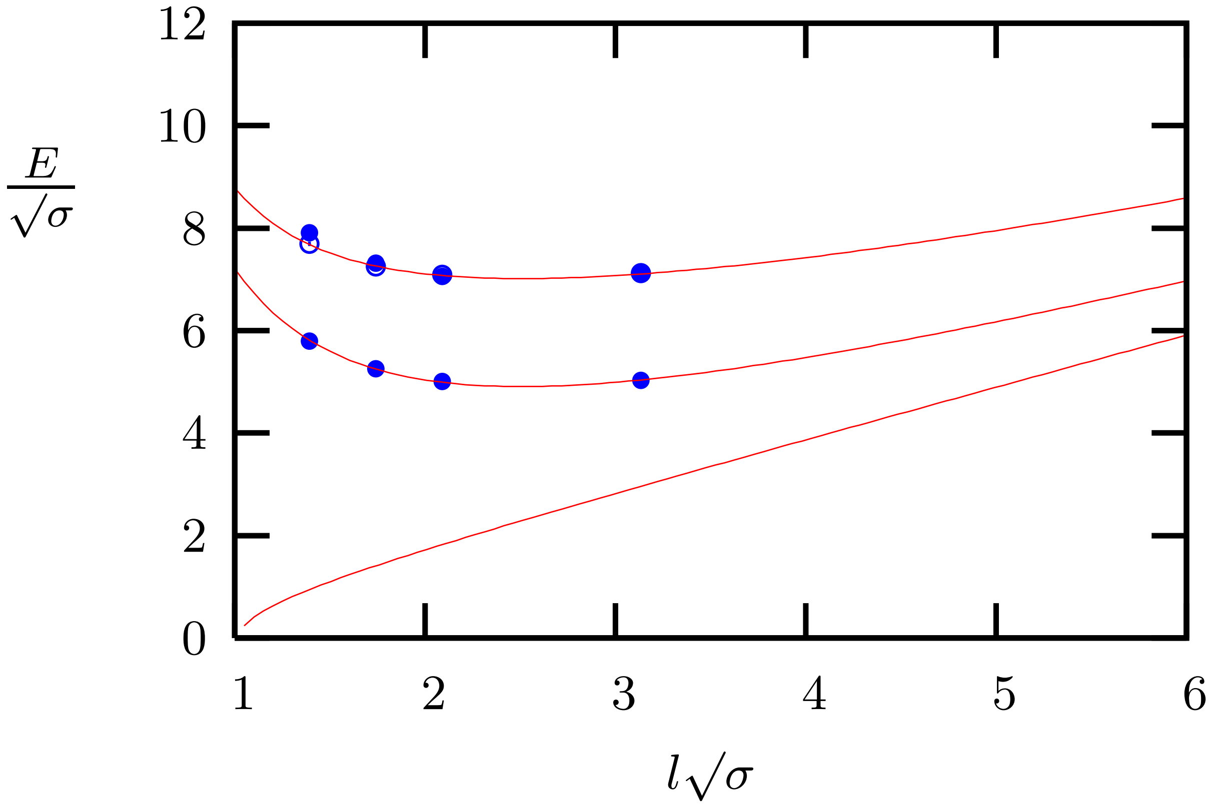
$$l\sqrt{\sigma} \sim 2\ ,\ n = 1, 2$$

which is well outside its radius of convergence

$\Rightarrow$

the agreement with NG that we see goes well beyond the range of valdity of an expansion of $\mathcal{L}_{eff}$ in powers of derivatives: it makes a statement about $\mathcal{L}_{eff}$ to *all* orders in $1/\sigma l^2$ – and its resummation

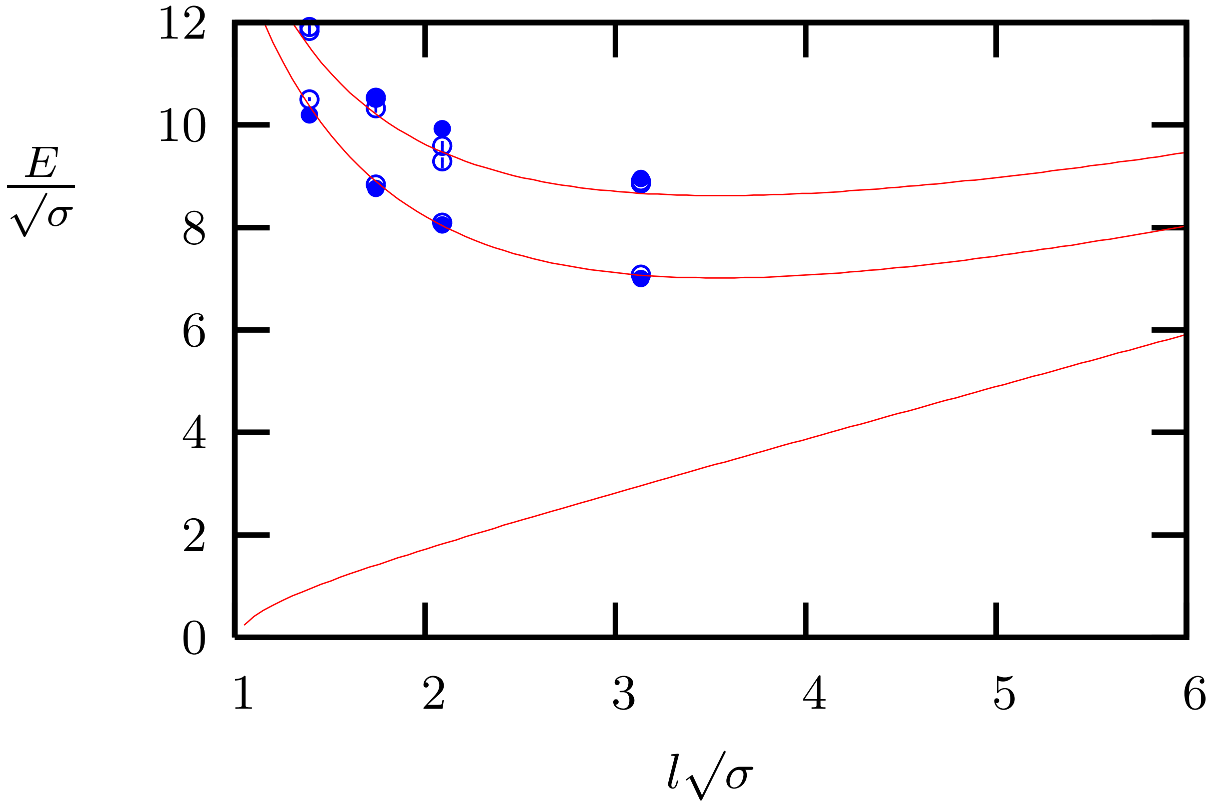
$p_{||} = \frac{2\pi q}{l} = \frac{2\pi}{l}$ spectrum : $a_1|0>;\ a_1 a_1 a_{-1}|0>;\ a_2 a_{-1}|0>$



$\bullet\ P = -$; $\circ\ P = +$

curves: NG predictions for $q = 1$

$p_{||} = \frac{2\pi q}{l} = \frac{4\pi}{l}$ spectrum : $a_1 a_1|0>; a_2|0>; \cdots$



$\bullet\ P = -$; $\circ\ P = +$

curves: NG predictions for $q = 2$

D=2+1 SU(N) gauge theories

• flux tubes are in the universality class of a simple bosonic string theory

• the Nambu-Goto free string spectrum

$$E^2(l) = (\sigma\, l)^2 + 8\pi\sigma \left(\frac{N_L+N_R}{2} - \frac{D-2}{24} \right) + \left(\frac{2\pi q}{l} \right)^2 .$$

very accurately describes the observed spectrum down to values of $l\sqrt{\sigma}$ where an effective string theory expansion in $x = l\sqrt{\sigma}$,

$$\frac{E_n}{\sqrt{\sigma}} = x\left(1 + \frac{c}{x^2}\right)^{\frac{1}{2}} = x + \frac{c}{2x} - \frac{c}{8x^3} + \cdots$$

no longer makes sense (i.e. is far past its range of convergence)

• $\Longrightarrow$ if we expand the effective string action around the Nambu-Goto action, then the correction terms should be small for almost all possible values of $l\sqrt{\sigma}$

SO :The lattice calculations overlap with but are also complementary to the $l \to \infty$ theoretical results.

BUT

• usually we think of the flux tube as

either

some non-Abelian dual Nielsen-Olesen vortex, with a finite intrinsic width $\sim 1/\sqrt{\sigma}$;

and/or

a string in some '5D' gravity dual, dangling near some 'horizon' where the metric will have a highly non-trivial curvature, so that it projects to a flux tube of non-zero width on our '4D' boundary

• in either scenario:

– where are the excited states due to excitations of the massive degrees of freedom generating the finite width?

– and the corrections to the stringy states from these massive degrees of freedom.

• very naively we might expect the mass scale of the lightest such extra states to be

$E(l) \sim E_0(l) + m_G \sim E_0(l) + 4\sqrt{\sigma}$

or maybe

$E(l) \sim E_0(l) + \Delta m_G \sim E_0(l) + 2\sqrt{\sigma}$

at low $l\sqrt{\sigma}$ this should be one of the lightest excitations – but we do not see it in our spectra ...

$\Rightarrow$ look at k-strings = bound states of k fundamental strings ....

$\longrightarrow$ D=3+1

ground state – universality class?

fit to each pair of neighbouring values of l

- effective Luscher correction

$$E_0(l) = \sigma l - c_{\rm eff}\,\frac{\pi(D-2)}{3l}$$

- effective Nambu-Goto

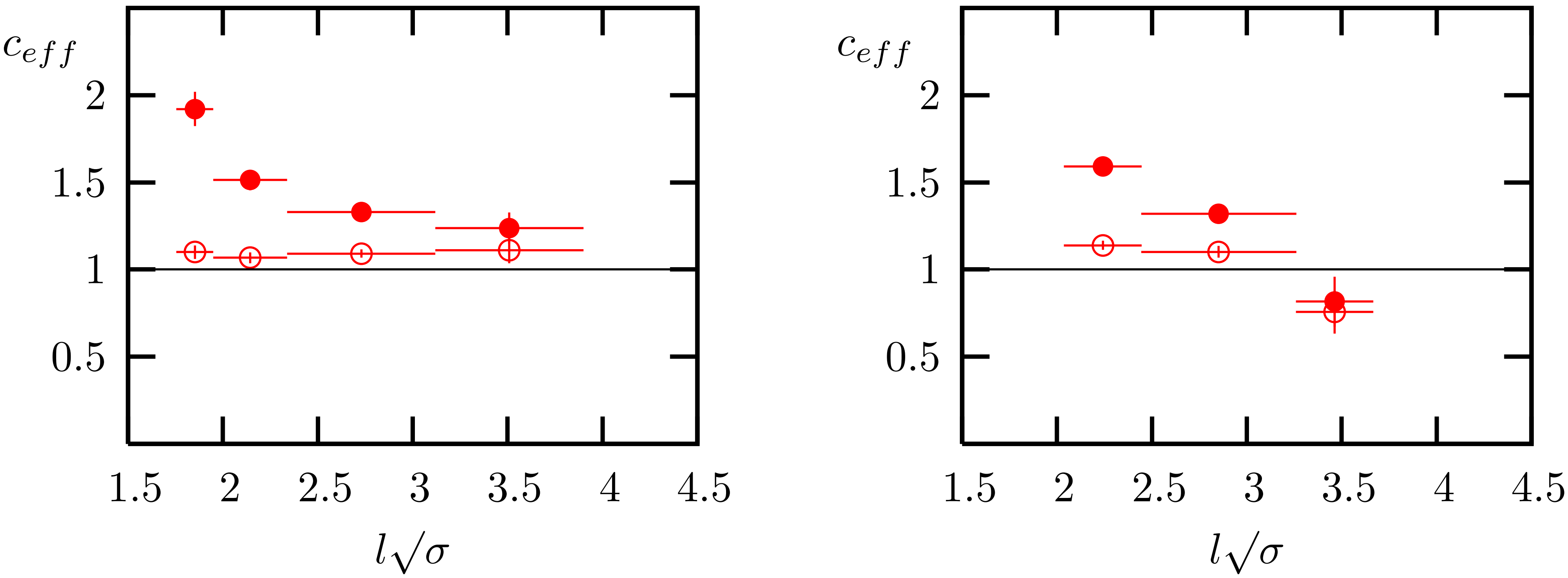
$$E_0(l) = \sigma l \left(1 - c_{\rm eff}\,\frac{2\pi}{3\sigma l^2}\right)^{\frac{1}{2}}$$

if

$$c_{eff}(l) \stackrel{l\to\infty}{\longrightarrow} c = 1$$

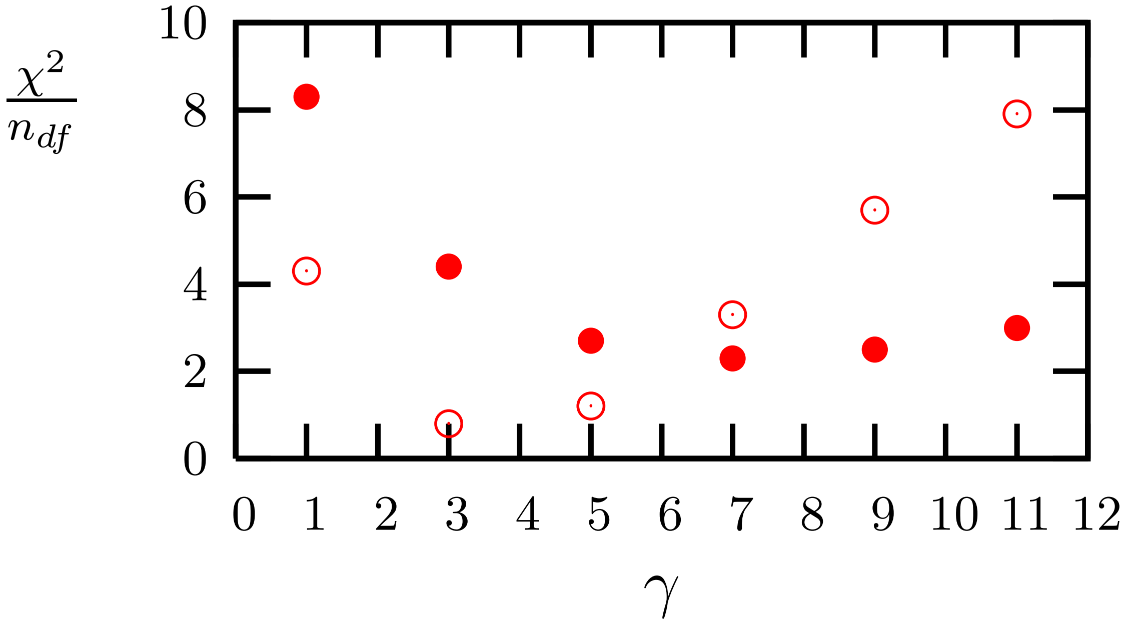
the effective string action belongs to the universality class of a simple bosonic where the only massless modes of the string are the transverse translations

SU(3) : $l_c\sqrt{\sigma} \simeq 1.56$ SU(6) : $l_c\sqrt{\sigma} \simeq 1.68$



c_{eff} : from Luscher ● , and from Nambu-Goto ○

aharony fit: $$E_0(l) = \sigma l - \sum_{n=0}^{2} \frac{c_n^{NG}}{\sigma^n l^{2n+1}} + \frac{\tilde{c}}{l^\gamma}$$



• SU(3) not great fit ; ∘ SU(6) consistent

– *but long way from any firm conclusion!*

relevant string quantum numbers in 3+1 dimensions:

∘ length, l.

∘ momentum along string, $p = 2\pi q/l$.

∘ angular momentum around string axis, $J = 0, 1, 2...$

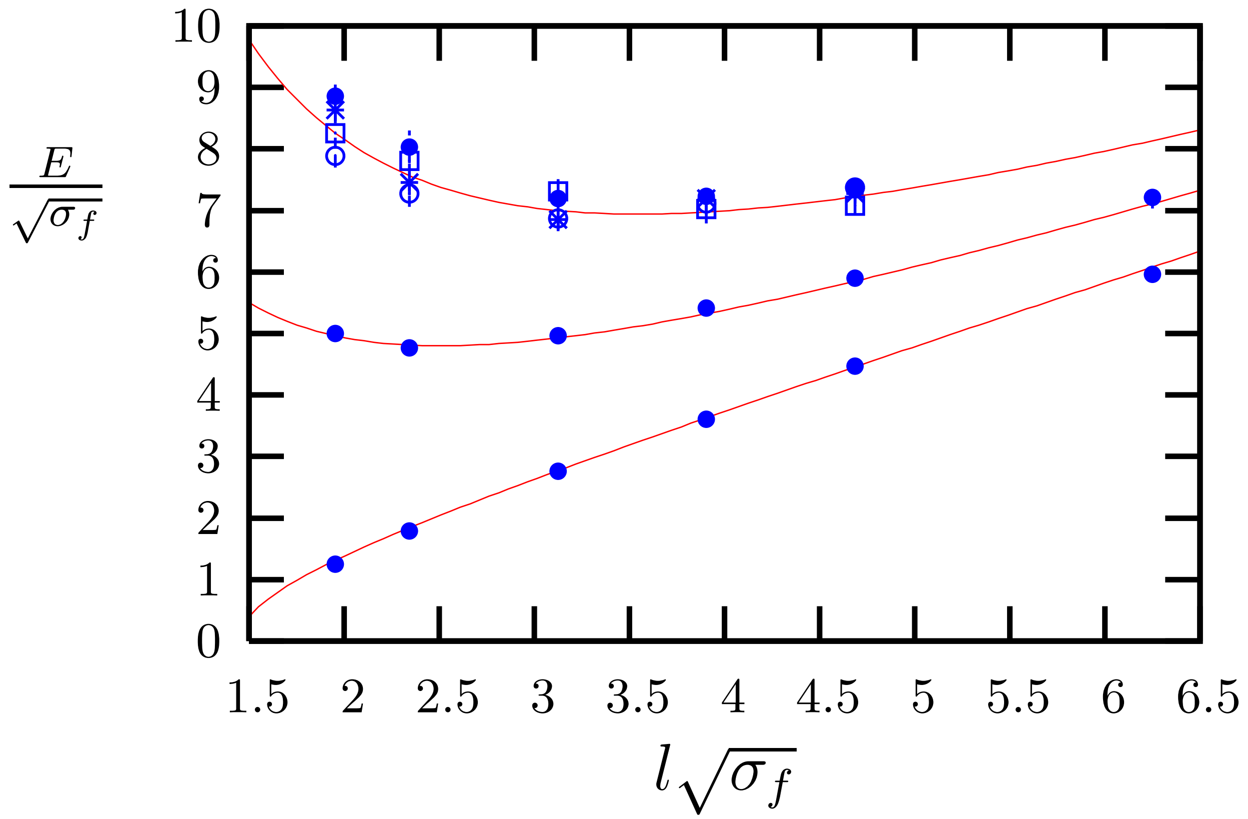
∘ $D = 2 + 1$ parity in plane orthogonal to string axis, P_ρ

∘ reflection 'parity' across this same plane, P_r

$\Rightarrow$

- phonons have momentum $\pm 2\pi k/l$ and helicity ± 1
- two transverse directions $\rightarrow$ need many more operators ...

SU(3) : lightest states with $p_{||} = 0,\ 2\pi/l,\ 4\pi/l$ vs NG (—)



$p = 0: \ J^P = 0^+$, $p = \frac{2\pi}{l}: \ |J|^P = 1^{\pm}$, $p = \frac{4\pi}{l}: \ |J| = 0^+, 1^{\pm}, 2^+, 2^-$

• States have the expected NG quantum numbers:

$|0> \sim J = 0 \qquad \text{for } p_{\shortparallel} = 0$

$a_1^{\pm}|0> \sim J = 1^{\pm} \qquad \text{for } p_{\shortparallel} = 2\pi/l$

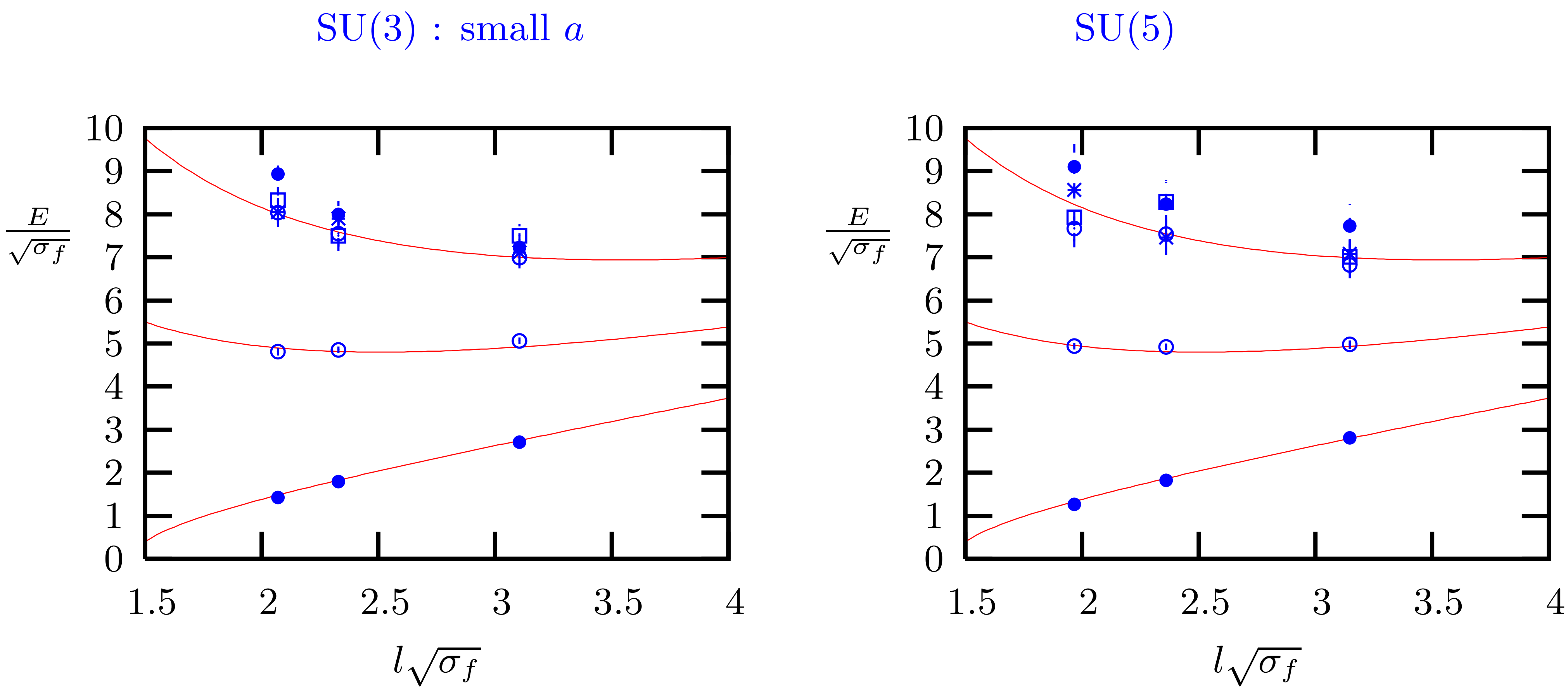
$a_1^{\pm}a_1^{\pm}|0>, \; a_2^{\pm}|0> \sim J = 0^{+}, 2^{\pm} \qquad \text{for } p_{\shortparallel} = 4\pi/l$

• σ obtained from NG fit to $E_0(p_{\shortparallel} = 0, l)$

$\Rightarrow$

the NG curves for $p_{\shortparallel} = 2\pi/l, \; 4\pi/l$ are parameter-free predictions

• agreement with free string prediction is excellent even for strings that are almost as short as they are wide, $l\sqrt{\sigma} \sim 2$



$\Longrightarrow$

- negligible $O(a^2)$ corrections
- negligible $O(1/N^2)$ corrections

how well is Nambu-Goto working ?

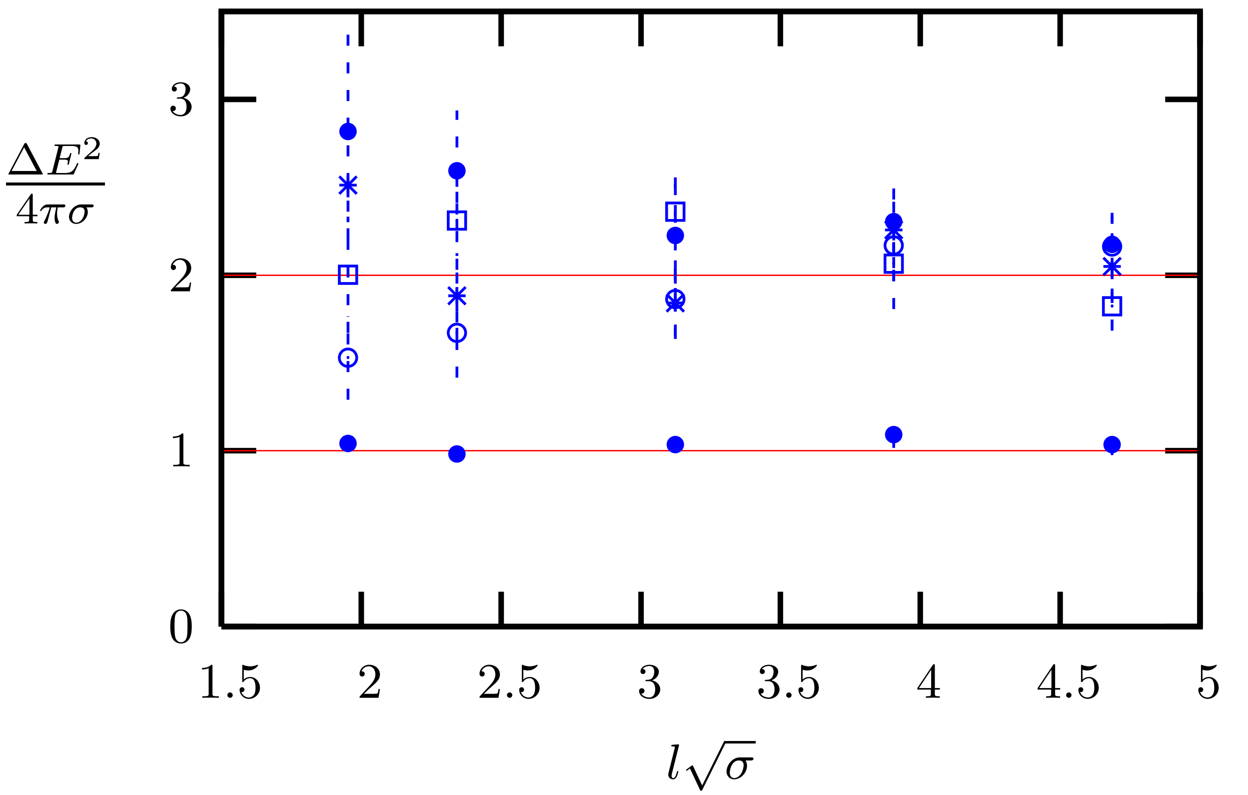
... since much of the extra energy may come just from the non-zero momentum

Recall

$$\Delta E^2(q,l) = E^2(q;l) - E_0^2(l) - \left(\frac{2\pi q}{l}\right)^2 \stackrel{NG}{=} 4\pi\sigma(N_L + N_R)$$

where $N_L - N_R = q$

SU(3) : excitation energies for $p_{||} = 0,\ 2\pi/l,\ 4\pi/l$



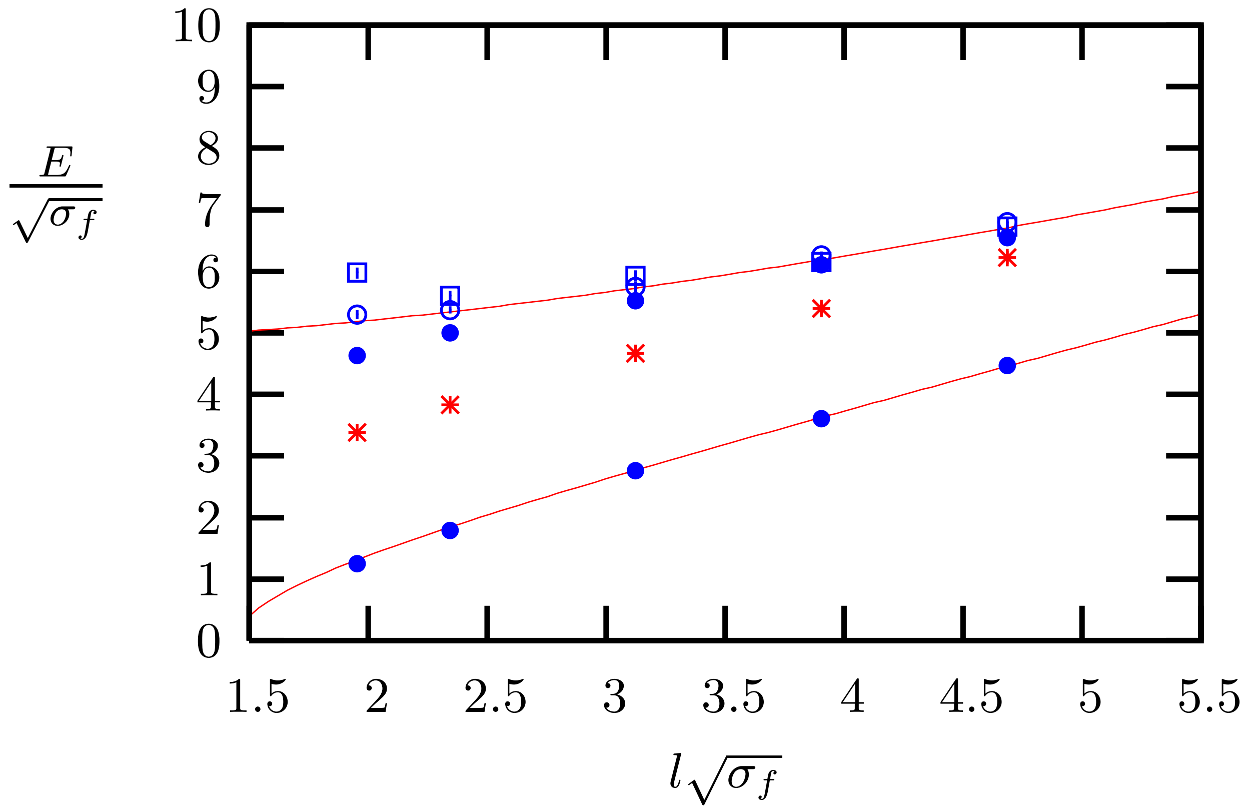
$p = 0 :\ \ J^P = 0^+ \ ,\qquad p = \frac{2\pi}{l} \ :\ \ |J|^P = 1^{\pm} \ ,\qquad p = \frac{4\pi}{l} \ :\ \ |J| = 0^+, 1^{\pm}, 2^+, 2^-$

So far just like $D = 2 + 1$: free string behaviour down to small l

BUT

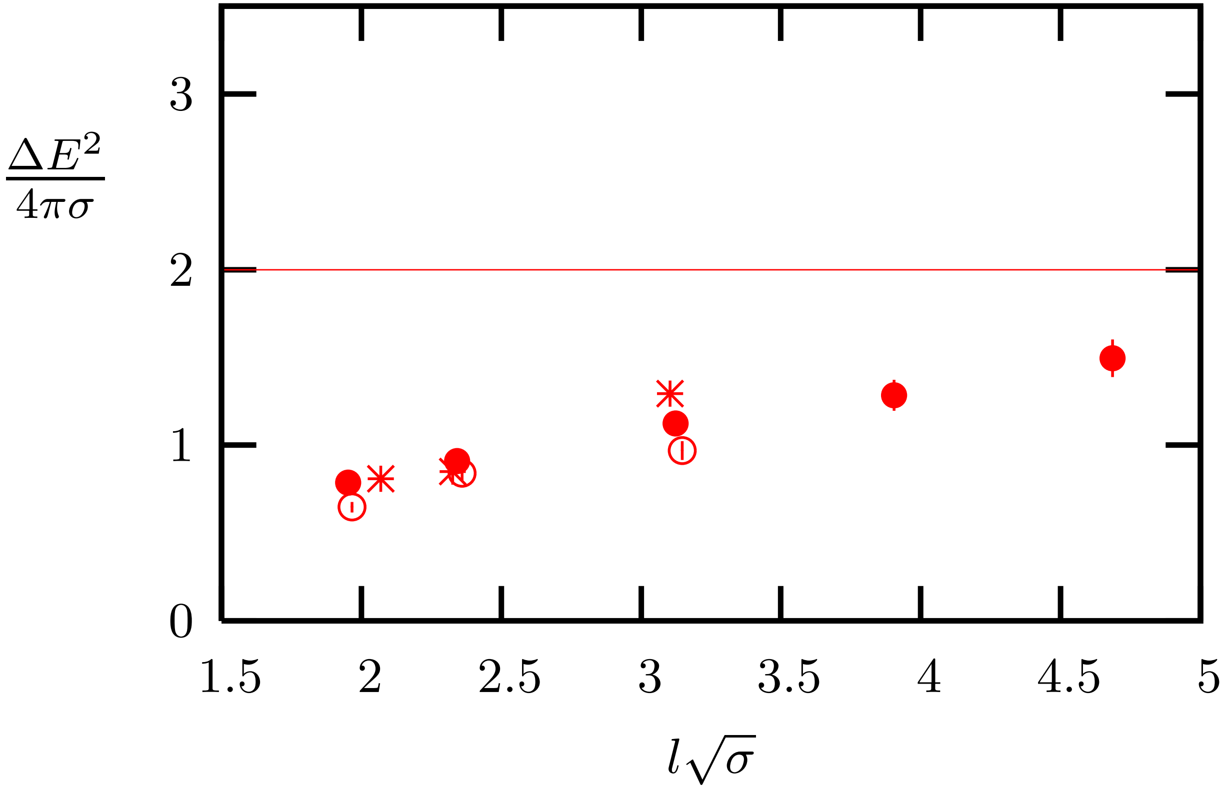
when we look at higher excited states, we find something new $\cdots$

lightest excited states with $p_{||} = 0$: SU(3)



$\bullet$, $J^P = 0^{++}$; $\star$, $J^P = 0^{--}$; $\circ$, $J^P = 2^{++}$; $\square$, $J^P = 2^{-+}$.

excitation energy of this anomalous 0^- state



$\bullet$, SU(3) coarse a ; $\circ$, SU(5) coarse a ; $\star$, SU(3) fine a.

So, for the first $p = 0$ excited energy level:

$a_1^{\pm} a_{-1}^{\pm} |0> \sim \ 0^{\pm}, \ 2^{\pm}$

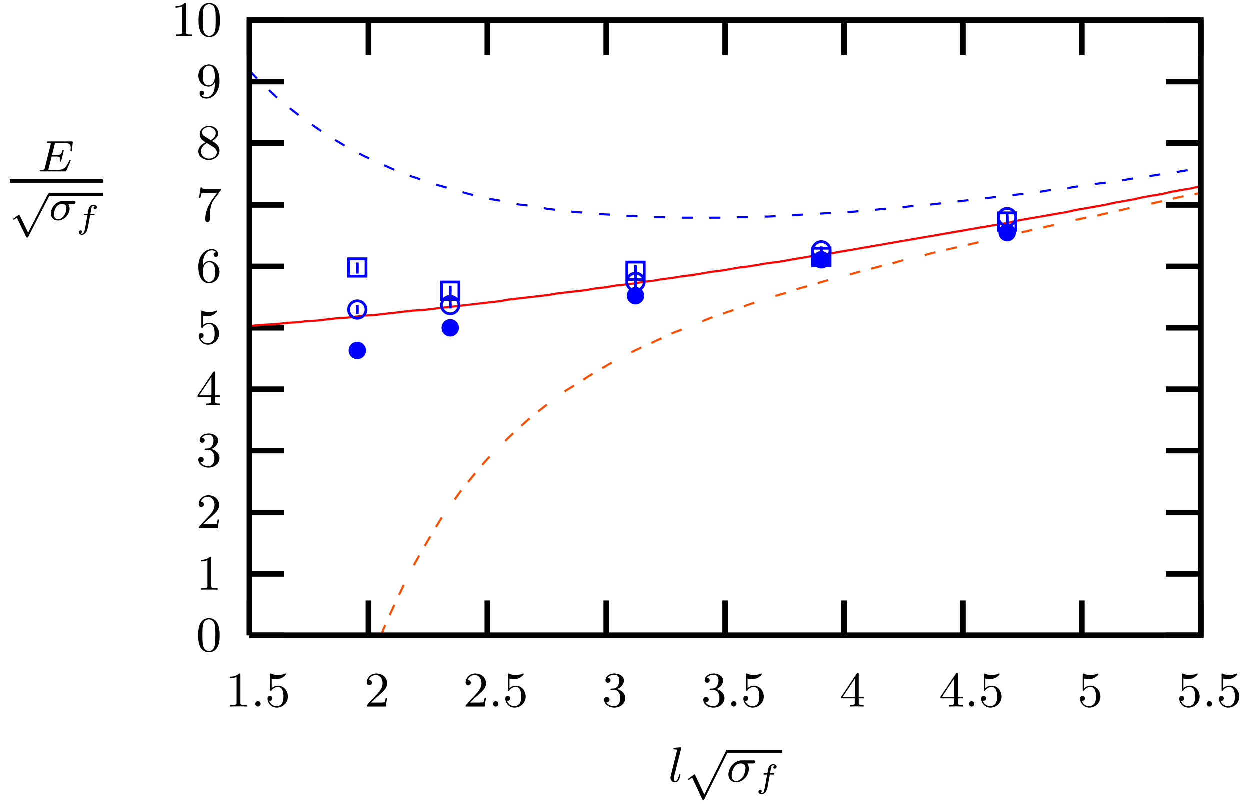
$0^{+}, \ 2^{\pm}$ close to NG

BUT

0^{-} very far from NG, with a very slow approach (?) to NG : $\propto \frac{1}{l^2}$ in E (?)
Is it a 'stringy' or 'massive' state – so that it crosses NG at larger l?

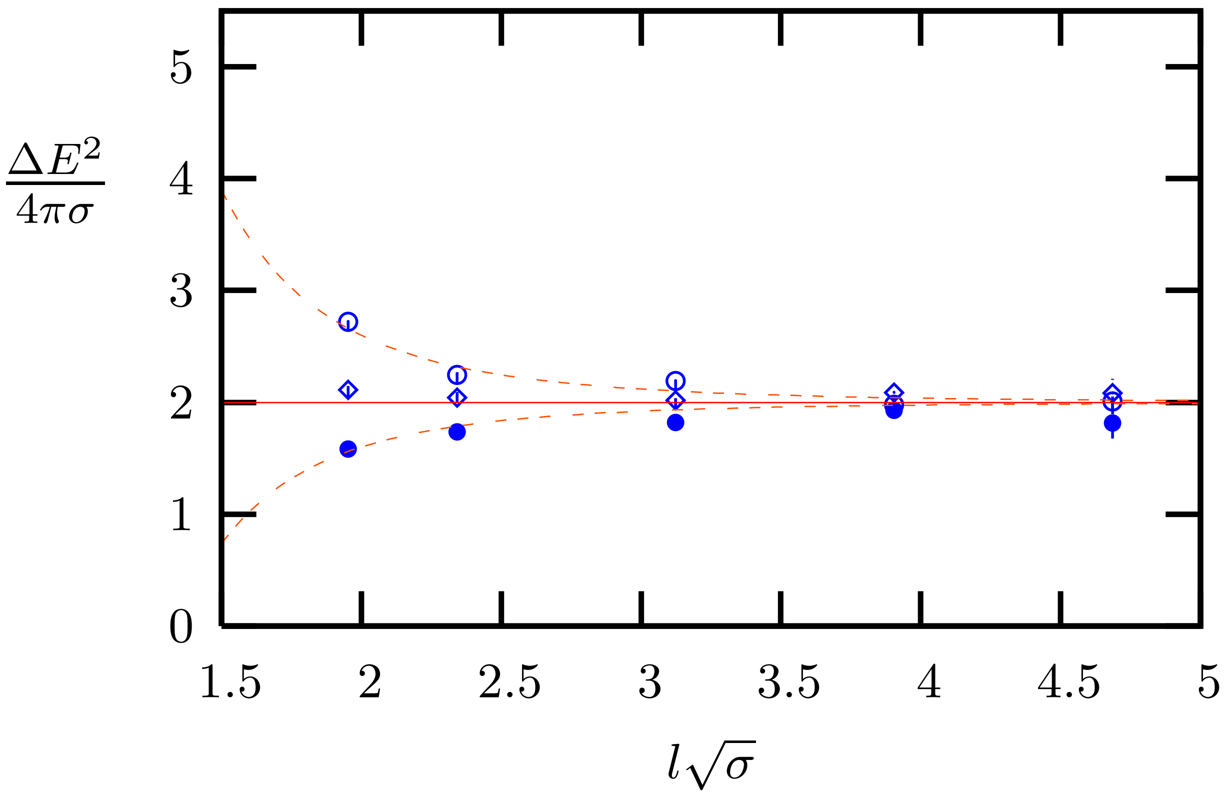
The 'good' states are much closer to NG than one would expect from the current universality results – just as for D=2+1:

$0^+, 2^\pm$ excited states with $p_{||} = 0$ – effective string fits



- - - Aharony $O(1/l^3)$; - - - Luscher $O(1/l)$; — Nambu-Goto.

$p_{||} = 0 \quad 0^+, 2^\pm$ excitation energies : Nambu-Goto fits



— Nambu-Goto ; $\cdots$ Nambu-Goto + $O(1/l^5)$ corrections

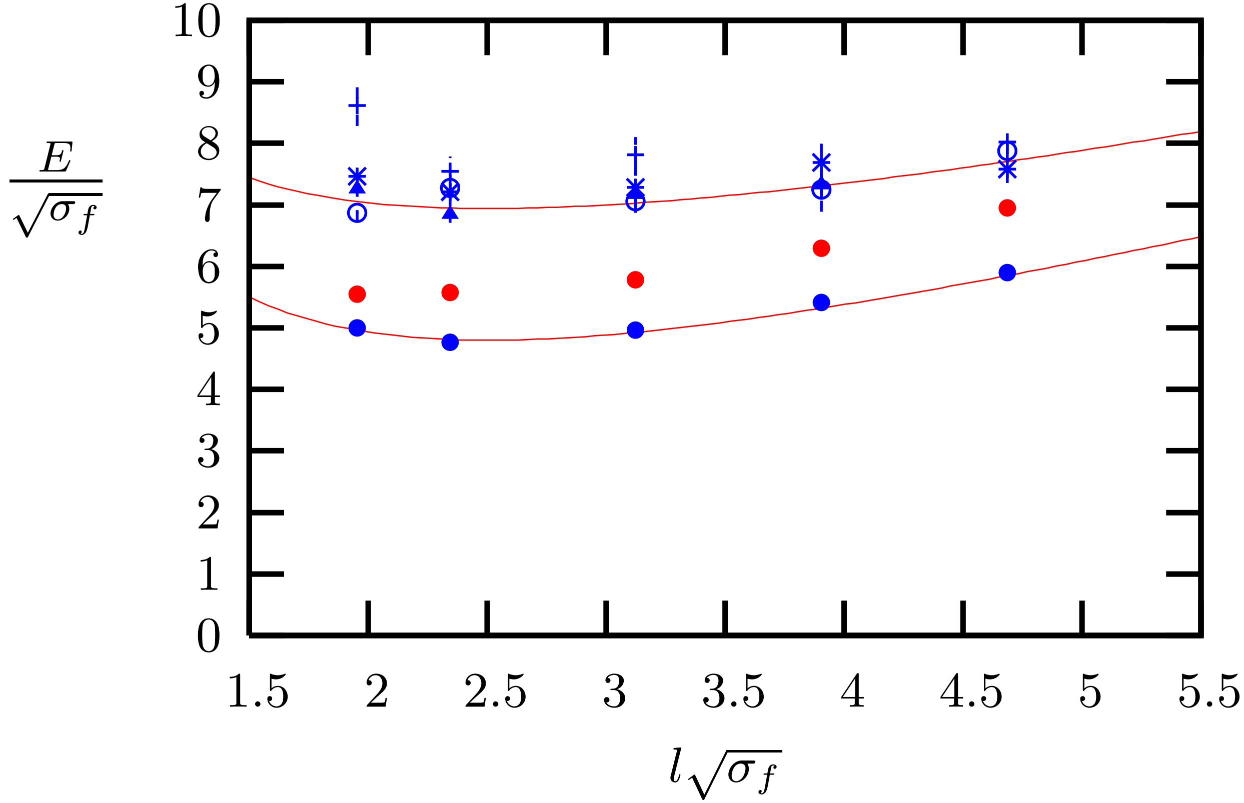
$\cdots$ return to the anomalous 0^- state:

are there other anomalous states?

∘ look to $p_{\shortparallel} \neq 0$

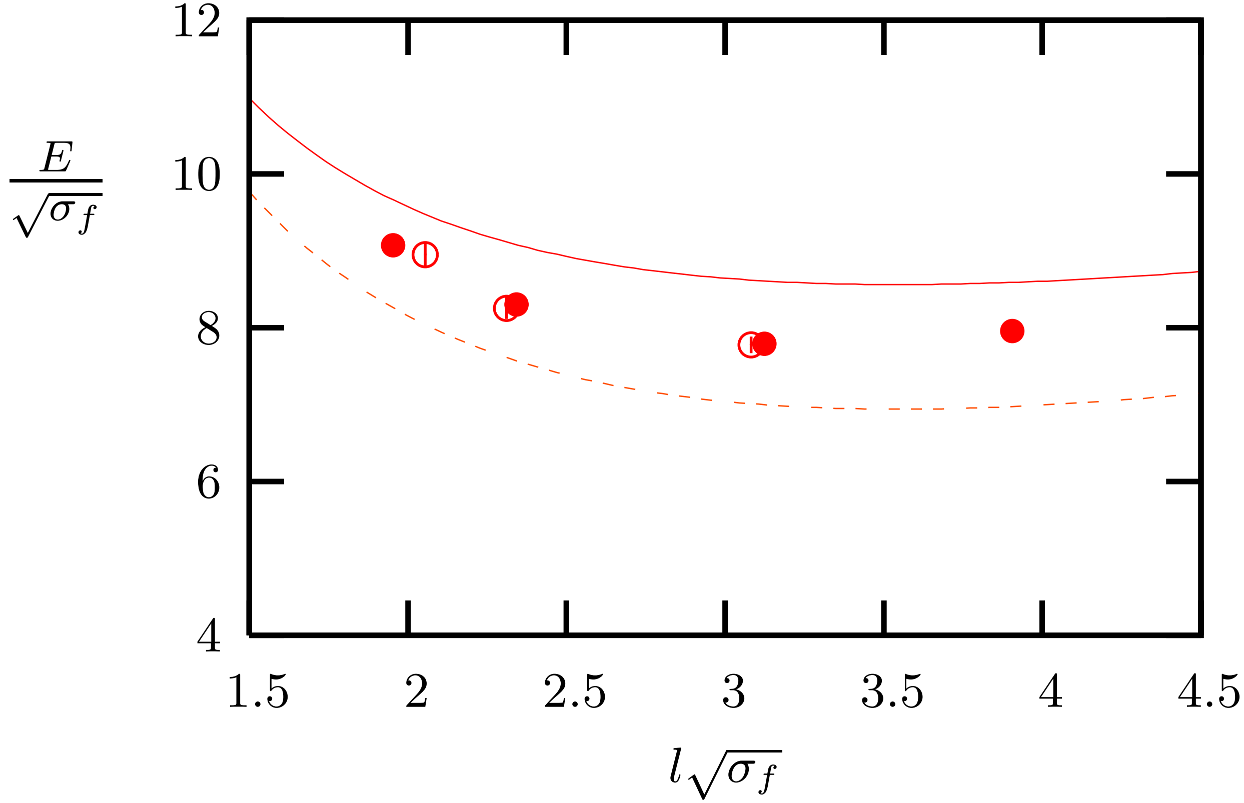
∘ look to higher excitations

lightest excited states with $p_{\shortparallel} = \frac{2\pi}{l}$: SU(3)



$\bullet J^P = 0^-$; $\bullet J^P = 1^{\pm} gs$;
$\circ J^P = 0^+$; $\star J^P = 2^+$; $+ J^P = 2^-$ NG — .

lightest excited 0^- state with $p_{\shortparallel} = \frac{4\pi}{l}$: SU(3)



● coarse a; ○ fine a

solid line: NG prediction; dashed line: ground state

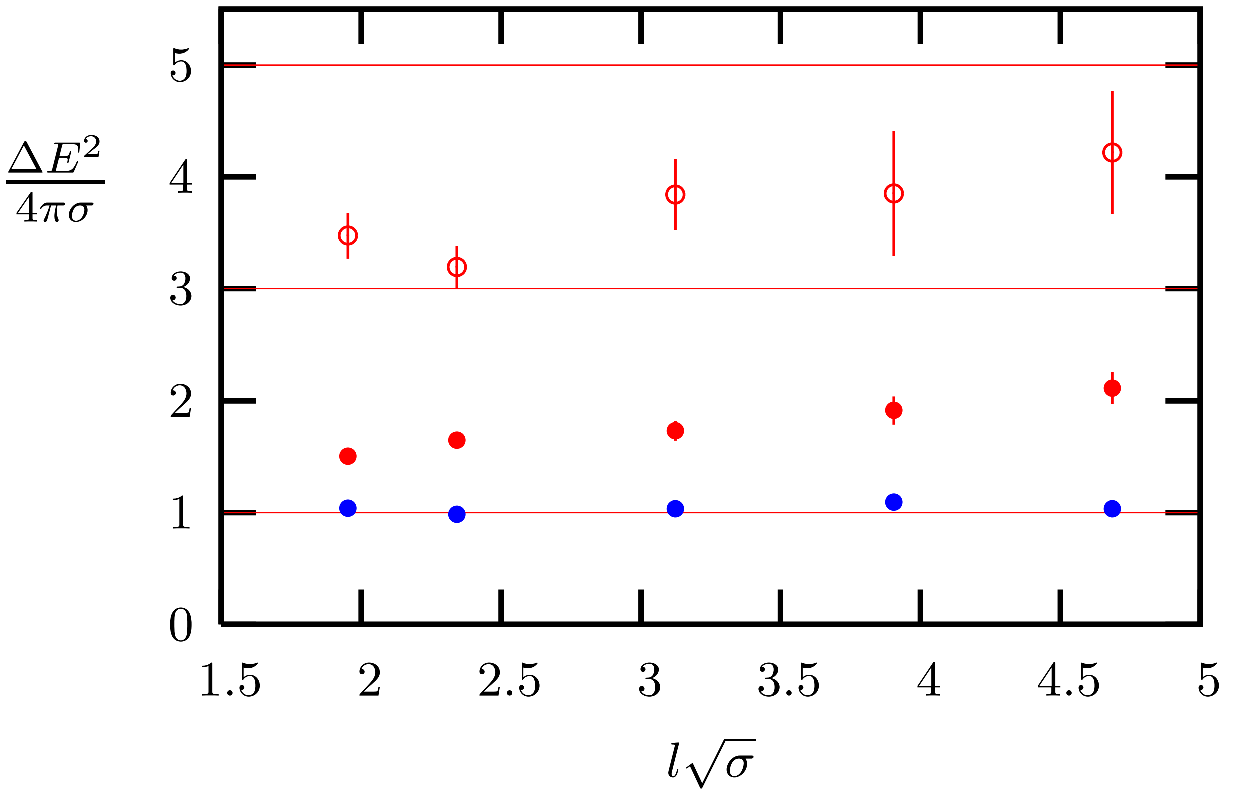
So we have a whole set of anomalous 0^- states

in terms of NG states these would be:

- $\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)|0\rangle$ for $p_{\shortparallel} = 0$
- $\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)|0\rangle$ for $p_{\shortparallel} \neq 2\pi/l$
- $\left(a_3^+ a_{-1}^- - a_3^- a_{-1}^+\right)|0\rangle$ for $p_{\shortparallel} \neq 4\pi/l$

but if these are massive mode states, then there may also be 0^- stringy states that are simply more massive than these $\cdots$

excitation energies of 0^- states with $p_{||} = \frac{2\pi}{l}$: SU(3)



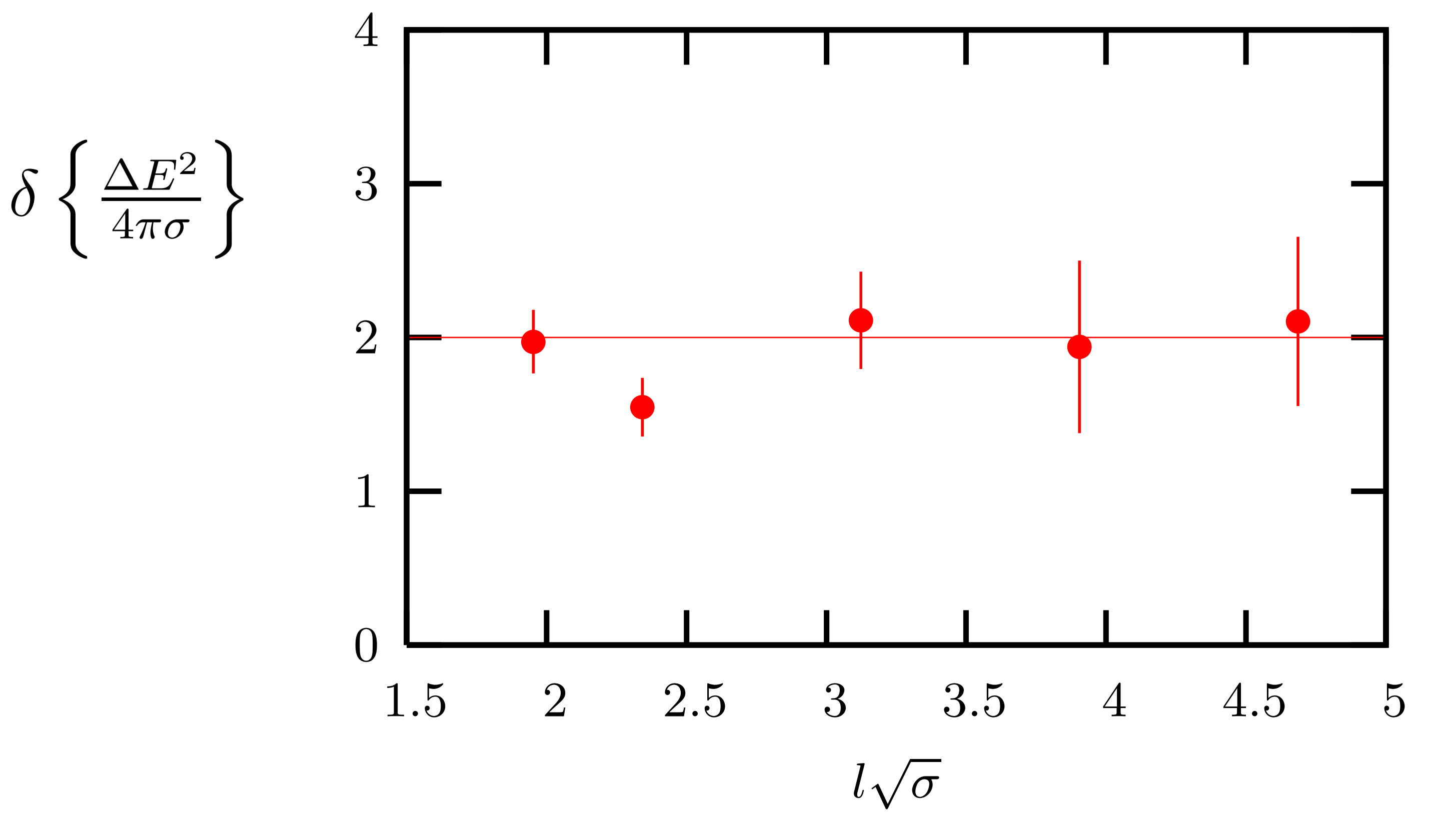
$\bullet J^P = 0^- 1st$; $\circ J^P = 0^- 2nd$; $\bullet J^P = 1^{\pm} gs$;

naively the next excited 0^- does not appear to approach the lowest NG state – but rather the next one up

caution : the errors are becoming very large

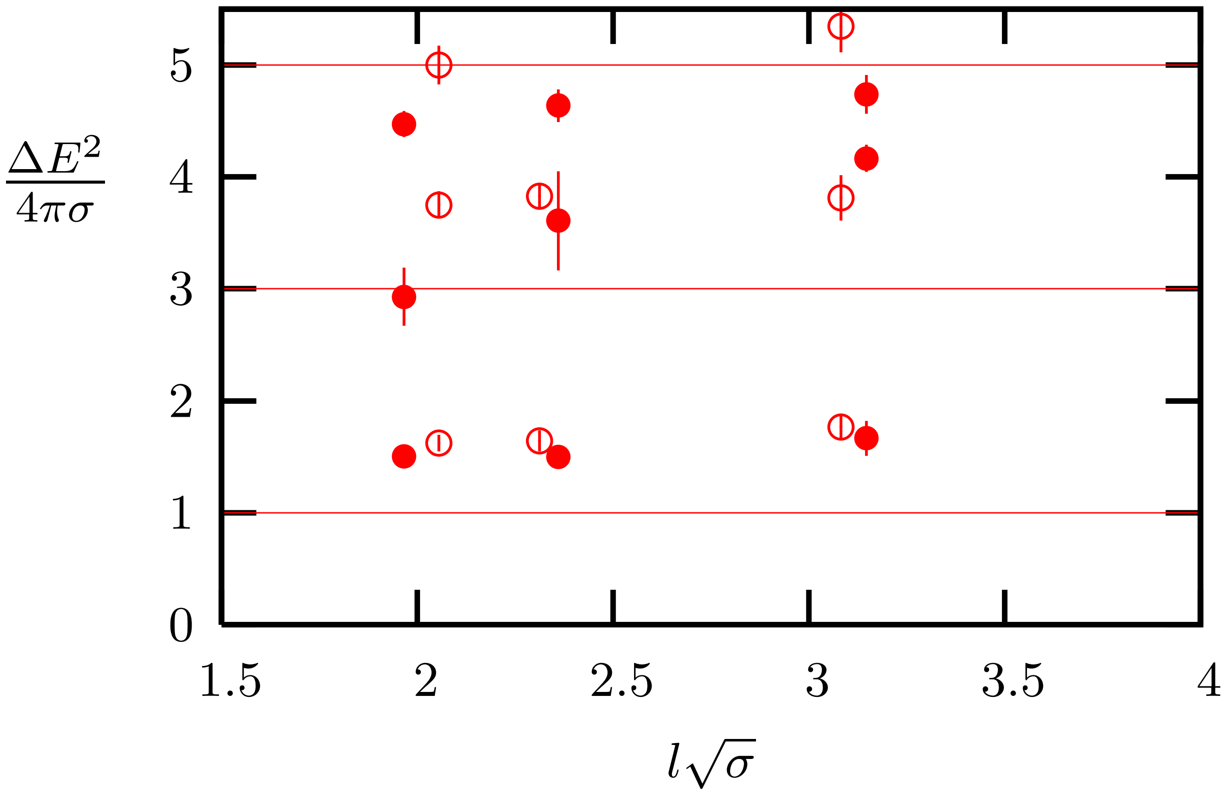
Nonetheless the difference in excitation energies is just $\sim$ NG : the anomaly is a common additive piece to all the ΔE^2 excitation energies

but some higher excited states of the 0^- – those with a more complex phonon description? – seem to be in agreement with free string theory excitation energy gap of these two lightest 0^- states with $p_{||} = \frac{2\pi}{l}$



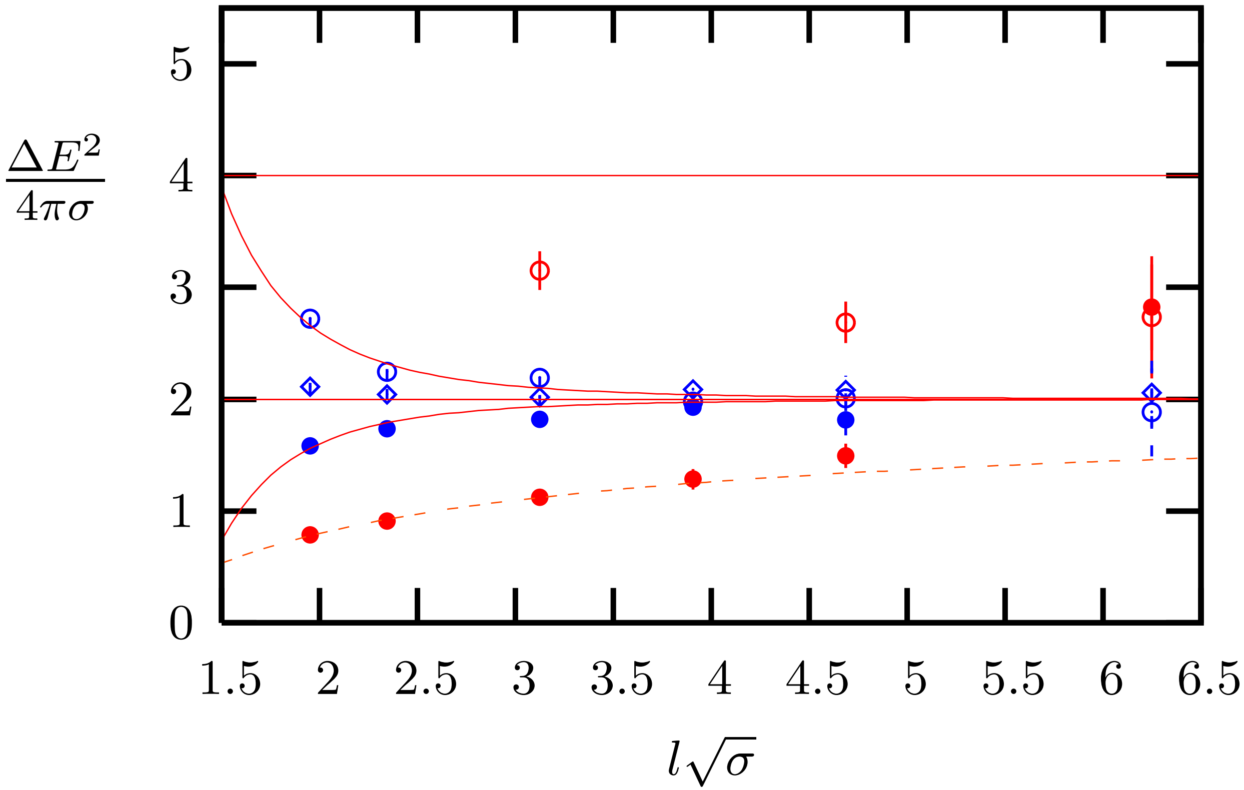
line : NG prediction

excitation energies of 0^- states with $p_{||} = \frac{2\pi}{l}$: SU(3) and SU(5)



○, SU(3) fine a ; ●, SU(5) coarse a ;

... and now back to $p = 0$ and some larger l ...



0^-; $0^+, 2^\pm$

Some Conclusions

• The flux tube spectrum in $D = 2 + 1$ is very close to the free string spectrum $\forall N$
— down to such small string lengths, $l\sqrt{\sigma} \sim 2$, that an expansion in $1/\sigma l^2$ no longer converges

$\Rightarrow$ we should perform our expansion in powers of $1/\sigma l^2$ around E_{NG} rather than around σl, when constructing our effective action

i.e. but where are the massive modes?

• The $D = 3 + 1$ flux tube spectrum in $D = 2 + 1$ is also very close to the free string spectrum for most states
— again down to such small string lengths that a $1/\sigma l^2$ expansion no longer converges

$\Rightarrow$ again we should perform our expansion in powers of $1/\sigma l^2$ around E_{NG}, with the theoretically expected $\propto 1/l^{\gamma \geq 5}$ corrctions

BUT

• Now there are a few states that are very far from Nambu-Goto ; so far all these *anomalous* states are $J^P = 0^-$.

$\Rightarrow$ are we seeing here the effects of massive modes?

• The lattice calculations have some overlap with the effective action analyses – but they are also nicely complementary in that their natural range is moderate to small l, rather than very large l:
this complementarity has been and should continue to be very useful for the physics of common interest.