Quantum Phases of Vortex Strings in N=1* theory

Roberto Auzzi

Based on work with S. P. Kumar

arXiv:0810.3201, 0908.4278

Introduction

This talk is about solitonic k-strings in the Higgs vacuum of mass-deformed $\mathcal{N} = 4$ theory

By S-duality, these objects are dual to the confining string in the confining vacuum of the theory

The tensions can be computed both from the field theory and from the string dual (Polchinski-Strassler), and they agree

These vortices are non-abelian because have orientational CP^1 zero modes; I'll discuss the quantum phases that occur in the worldsheet sigma model by changing the mass parameters and the θ angle.

Theoretical setting: $\mathcal{N} = 1^*$

Let us start with $\mathcal{N} = 4$ $SU(N_c)$ SYM (which contains an $\mathcal{N} = 1$ vector multiplet and 3 adjoint chiral multiplets, $\Phi_{1,2,3}$), with the superpotential

$$W = \frac{1}{g_{YM}^2} \mathrm{Tr}([\Phi_1, \Phi_2]\Phi_3)$$

The bosonic degrees of freedom are a gauge field and 3 complex scalars in the adjoint representation. There are also 4 Weyl fermions, also in the adjoint.

The $\mathcal{N} = 1^*$ is a mass deformation with the superpotential

$$\Delta W = \frac{1}{g_{YM}^2} \sum \frac{1}{2} m \mathrm{Tr}(\Phi_i^2)$$

The potential

$$V_F = \mathrm{Tr}\left(w_1 . w_1^\dagger + w_2 . w_2^\dagger + w_3 . w_3^\dagger\right) \,, \qquad w_i = \epsilon_{ijk}\Phi_j\Phi_k + m_i\Phi_i \,,$$

$$V_D = \frac{1}{4}\mathrm{Tr}\left([\Phi_1^\dagger, \Phi_1] + [\Phi_2^\dagger, \Phi_2] + [\Phi_3^\dagger, \Phi_3]\right)^2 \,.$$

In order for this to vanish we have to choose the VEVs as $SU(2)$ representations

$$\Phi_k = imJ_k$$

For large N_c really a lot of discrete vacua, totally or partially Higgsed/Confined/Obliqued confined, with and without mass gap

$SU(2)$ Colour Flavour Locking

$\mathcal{N} = 4$ has an $SU(4) \approx SO(6)$ global symmetry

The mass deformation, if we keep all the 3 masses $m_k = m$, breaks this to $SO(3)$

In the Higgs vacuum, for generic N_c, we can find a combination of the global and of some gauge generators which is unbroken

$$U_F = \exp(T_j a_j), \qquad W_C = \exp(iJ_l a_l)$$

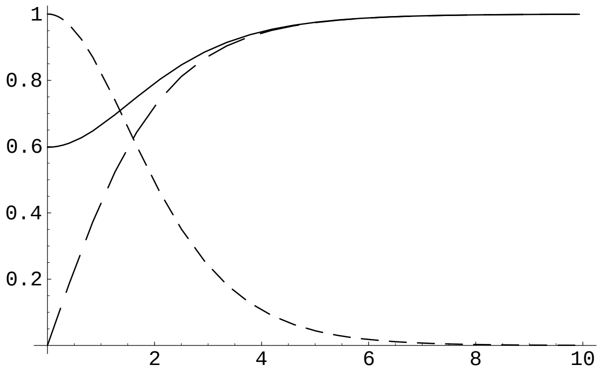
$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; T_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; T_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\vec{\Phi} \to U_F \, \vec{\Phi} \, , \qquad \Phi_i \to W_C \, \Phi_i \, W_C^\dagger \, .$$

Vortex $N_c = 2$

$$\Phi_1 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{im}{2}\psi_1(r)\begin{pmatrix} 0 & -ie^{i\varphi} \\ ie^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_3 = \frac{im}{2}\kappa_1(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}(1 - f(r))Y, \qquad A_y = \frac{x}{r^2}(1 - f(r))Y, \quad Y = \frac{1}{2}\sigma_3.$$



κ_1 (solid), ψ_1 (long dashes), f (short dashes).

Vortex $N_c = 4, k = 1$

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 & 0 \\ 0 & 2\psi_2 & 0 & \sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & \sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 & 0 \\ 0 & i2\psi_2 & 0 & -i\sqrt{3}\psi_3 e^{i\varphi} \\ 0 & 0 & i\sqrt{3}\psi_3 e^{-i\varphi} & 0 \end{pmatrix},$$

$$\Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \qquad A_y = \frac{x}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right),$$

$$Y_1 = \frac{1}{4}\mathrm{Diag}(1,1,1,-3), \qquad \lambda_1 = \frac{1}{\sqrt{12}}\mathrm{Diag}(1,1,-2,0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(1,-1,0,0).$$

$$\exp(2\pi i Y_1) = \mathrm{Diag}(e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2}, e^{\pi i/2})$$

$$f(0) = 1, \qquad g_\ell(0) = 0, \qquad f(\infty) = g_\ell(\infty) = 0\,.$$

Vortex $N_c = 4, k = 2$

$$\Phi_1 = \frac{mi}{2}\begin{pmatrix} 0 & \sqrt{3}\psi_1 & 0 & 0 \\ \sqrt{3}\psi_1 & 0 & 2\psi_2 e^{i\varphi} & 0 \\ 0 & 2\psi_2 e^{-i\varphi} & 0 & \sqrt{3}\psi_3 \\ 0 & 0 & \sqrt{3}\psi_3 & 0 \end{pmatrix}, \qquad \Phi_2 = \frac{mi}{2}\begin{pmatrix} 0 & -i\sqrt{3}\psi_1 & 0 & 0 \\ i\sqrt{3}\psi_1 & 0 & -i2\psi_2 e^{i\varphi} & 0 \\ 0 & i2\psi_2 e^{-i\varphi} & 0 & -i\sqrt{3}\psi_3 \\ 0 & 0 & i\sqrt{3}\psi_3 & 0 \end{pmatrix},$$

$$Phi_3 = \frac{mi}{2}\begin{pmatrix} 3\kappa_1 - 2\kappa_3 & 0 & 0 & 0 \\ 0 & \kappa_2 + 2\kappa_3 & 0 & 0 \\ 0 & 0 & -\kappa_2 + 2\kappa_3 & 0 \\ 0 & 0 & 0 & -3\kappa_1 - 2\kappa_3 \end{pmatrix}.$$

$$A_x = \frac{-y}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right), \qquad A_y = \frac{x}{r^2}\left((1-f)Y_1 + \sum_{\ell=1}^{2} g_\ell(r)\lambda_\ell\right),$$

$$Y_2 = \frac{1}{2}\mathrm{Diag}(1,1,-1,-1), \qquad \lambda_1 = \frac{1}{2}\mathrm{Diag}(1,-1,0,0), \qquad \lambda_2 = \frac{1}{2}\mathrm{Diag}(0,0,1,-1).$$

$$\exp(2\pi i Y_2) = \mathrm{Diag}(-1,-1,-1,-1)$$

$$f(0) = 1, \qquad g_\ell(0) = 0, \qquad f(\infty) = g_\ell(\infty) = 0\,.$$

Generic N_c, k

$$Y_k = \mathrm{Diag}\left(\underbrace{\frac{k}{N_c}, \cdots, \frac{k}{N_c}}_{N_c - k\,\mathrm{elements}},\ \ -\frac{N_c - k}{N_c}, \cdots, -\frac{N_c - k}{N_c}\right),$$

$$\Phi_{1,2}(r, \varphi) = e^{iY_k\varphi}\ \Phi_{1,2}(r, \varphi = 0)\ e^{-iY_k\varphi}$$

$3(N_c - 1)$ profile functions $(\psi_i, \kappa_i, f, g_i)$

Numerical solution found for $2 \leq N_c \leq 6$

Vortex Tensions

For $N_c = 4$ we find the following numerical result,

$$\frac{T_{N_c=4,\,k=2}}{T_{N_c=4,\,k=1}} = 1.334$$

while the prediction from Casimir scaling is $4/3$. For $N_c = 5$ we find

$$\frac{T_{N_c=5,\,k=2}}{T_{N_c=5,\,k=1}} = 1.501$$

while the Casimir scaling prediction is $3/2$. Finally, for $N_c = 6$:

$$\frac{T_{N_c,\,k=2}}{T_{N_c=6,\,k=1}} = 1.6008, \qquad \frac{T_{N_c=6,\,k=3}}{T_{N_c=6,\,k=1}} = 1.801,$$

while the Casimir scaling values are $8/5$ and $9/5$.

The string dual of the Higgs vacuum

Constructed by Polchinski and Strassler by considering an appropriate deformation of the $AdS_5 \times S^5$ background.
The $\mathcal{N} = 1^*$ deformation is achieved by switching on a three form

$$G_3 = F_3 - \tau H_3$$

where

$$\tau = \frac{i}{g_s} + \frac{C_0}{2\pi}$$

The N_c D3 branes on which the parent $\mathcal{N} = 4$ theory lives polarizes (by Myers dielectric effect) in a spherical D5 brane with N_c units of D3 charge on the top of it

String vs Gauge Theory quantities

$$4\pi g_s = g^2_{\rm YM}\,, \qquad \frac{R_{\rm AdS}}{\sqrt{\alpha'}} = (4\pi g_s N_c)^{1/4} \gg 1$$

In the Higgs vacuum, with $m_1 = m_2 = m_3 = m$, the metric in the interior matches onto the geometry generated by a D5-brane wrapped on an S^2 carrying N_c units of D3-charge.

The six transverse directions are denoted as $w^{1,2,3}$, $y^{1,2,3}$

The D3 branes spread out along the w^i directions with $y^i = 0$ and the resulting D5-brane wraps around sphere of radius:

$$r_0 = \pi\alpha' m N_c.$$

The $D5$ shell thickness is:

$$\rho_c = (\alpha' m)\sqrt{g_s N_c \pi}\,, \qquad \rho_c << r_0 \rightarrow \frac{N_c}{g_s} >> 1$$

Metric, Dilaton and B_2

Coordinates: $(x_\mu, \vec{y}, \vec{w})$

$$ds^2_{\rm string} = Z_x^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z_y^{1/2}(dy^2 + y^2 d\Omega_y^2 + dw^2) + Z_\Omega^{1/2} w^2 d\Omega_w^2,$$

$$Z_x = Z_y = \frac{R^4_{\rm AdS}}{\rho_+^2\rho_-^2}, \qquad Z_\Omega = \frac{R^4_{\rm AdS}\,\rho_-^2}{\rho_+^2(\rho_-^2+\rho_c^2)^2}, \qquad \rho_\pm = \sqrt{y^2 + (w \pm r_0)^2}.$$

$$e^{2\Phi} = g_s^2 \frac{\rho_-^2}{\rho_-^2+\rho_c^2}, \qquad C_0 = \theta_{3+1} = 0.$$

$$B_2 = -\frac{\alpha'\pi N_c}{1+\rho_-^2/\rho_c^2}\,\sin\theta_w\, d\theta_w \wedge d\phi_w\,.$$

Probes D1 branes: $k = 1$ vortex

$$S_{\mathrm{DBI}} = \frac{1}{2\pi\alpha'} \int d^2\xi \left\{ e^{-\Phi} \sqrt{(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}))} \right\} .$$

Let us consider a D1-brane oriented in the x_0, x_1 directions and the embedding $(\xi_0, \xi_1) = (x_0, x_1)$, so that the pullback of the metric onto the world-sheet is:

$$G_{00} = -Z_x^{-1/2} + \ldots, \qquad G_{11} = Z_x^{-1/2} + \ldots$$

This DBI is minimized when :

$$w = \frac{r_0 + \sqrt{r_0^2 - 2\rho_c^2}}{2} \approx r_0 - \frac{\rho_c^2}{2r_0}, \qquad y = 0.$$

Probes D1 branes: $k = 1$ vortex

Tension:

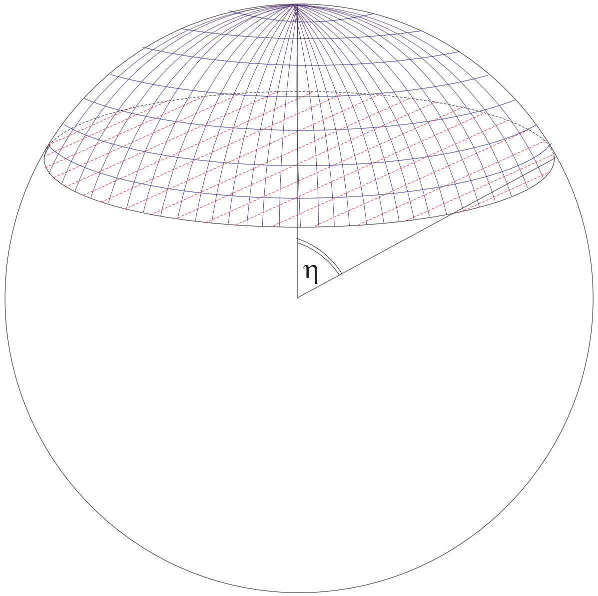
$$T_{\mathrm{D1}} = \frac{N_c m^2}{2 g_s} = \frac{2\pi N_c m^2}{g_{\mathrm{YM}}^2} \, .$$

Let us then consider an arbitrary dependence of $\vec{n}_w$ on the world-sheet coordinates (x_0, x_1) and introduce this into the DBI action.

$$\mathcal{L}_{\mathrm{kin}} = \frac{N_c}{4 g_s} (\partial_s \vec{n}_w)^2 = \frac{\pi N_c}{g_{\mathrm{YM}}^2} (\partial_s \vec{n}_w)^2 \, .$$

Probes D3 branes with flux: k strings

At large N, k (k/N fixed), a good description of the bound state of k D1 branes is given by a D3 with k units of flux on top of it



The minimal energy configuration for the probe D3 brane in the $\vec{w}$ space is given by the red disc and the blue “polar cap” shown in the figure.

The polar cap

We put all the flux on the top of the polar cap:

$$F_2 = \frac{k}{(\cos\bar{\eta}_k - 1)} \sin\theta_w (d\theta_w \wedge d\phi_w)$$

Then the DBI action reads:

$$S_{\text{cap}} = \frac{1}{(2\pi)^3\alpha'^2} \int d^4\xi \left\{ e^{-\Phi} \sqrt{(-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}))} \right\} .$$

At

$$|\vec{w}| = r_0 \,, \qquad (1 - \cos\bar{\eta}_k) = \frac{2k}{N_c}$$

the tension of the polar cap vanishes exactly.

The disc

The disc lies for most of its extension at $|\vec{w}| - r_0 \gg \rho_c$. In this limit the relevant metric (at $\vec{y} = 0$) is

$$ds^2\Big|_{w-r_0\gg\rho_c} = \frac{w^2 - r_0^2}{R_{\mathrm{AdS}}^2} dx_\mu dx_\nu \eta^{\mu\nu} + \frac{R_{\mathrm{AdS}}^2}{w^2 - r_0^2} (dw^2 + w^2(d\theta_w^2 + \sin^2\theta_w d\phi_w^2))\,;$$

the dilaton is simply $e^\Phi = g_s$. The B_2 field is small in this region. The warp factors from the two different subspaces cancel out and then the resulting DBI action is equivalent to the one for a membrane in flat space.

$$T_{\mathrm{D3}} = \frac{m^2}{2g_s} k(N_c - k)\,.$$

Effective world-sheet theory

For all N_c, k, the vortex moduli space is:

$$\frac{SO(3)_{C+F}}{U(1)_{C+F}} = \mathbb{CP}^1 = S^2$$

Switch on a (z,t) dependent C+F rotation on the vortex world-sheet:

$$\vec{\Phi} \to U_F(z,t) \cdot \Big(W_C(z,t)\, \vec{\Phi}\, W_C^\dagger(z,t) \Big) \,,$$

$$A_{x,y} \to W_C\, A_{x,y}\, W_C^\dagger \,, \qquad W_C\, J_3\, W_C^\dagger = \vec{n}(z,t) \cdot \vec{J} \,.$$

Ansatz for the gauge field along the vortex:

$$A_s = - \left(\vec{n} \times \partial_s \vec{n} \right)^a \; J^a \rho(r)$$

Effective world-sheet theory

Bosonic $\mathbb{CP}^1$ sigma model:

$$S_{1+1} = \int dz\, dt \left(B_{N_c,k}(\partial_s \vec{n})^2 - k(N_c - k)\frac{\theta_{3+1}}{8\pi}\epsilon^{sr}\,\epsilon^{abc}\,n^a\partial_s n^b \partial_r n^c \right) ,$$

where $B_{N_c,k}$ can be calculated numerically for each N_c, k

$$\theta_{1+1} = k(N_c - k)\theta_{3+1}\,.$$

For $\theta_{1+1} = 0, \pi$ this model is exactly solvable:
1) For $\theta_{1+1} = 0$ there is mass gap and the spectrum consists of an $SO(3)$ triplet (with a known S-matrix)
2) For $\theta_{1+1} = \pi$ the theory flows to an infrared conformal point, there is no mass gap and the spectrum consists of a deconfined $SO(3)$ doublet.

Different masses limit

In order to understand better the physics is useful to consider the limit

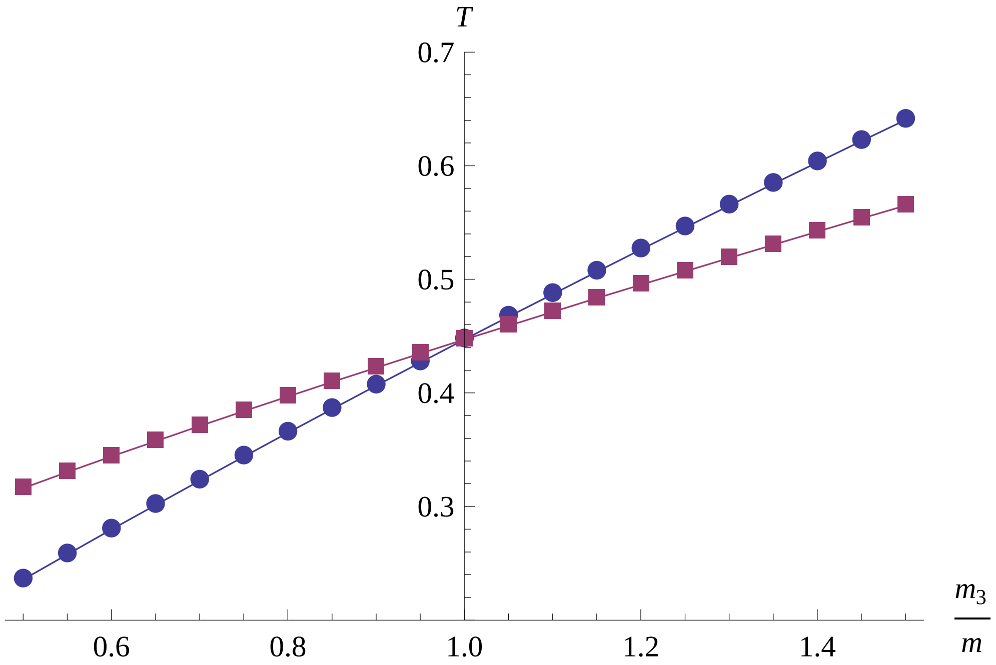
$$m_1 = m_2 = m\,, m_3 \neq m$$

In this way we reduce the global symmetry from $SO(3)$ to $SO(2)$.

The effect on the vortex effective theory is:

$$\delta S_{1+1} = A_{k,N}(m_3^2 - m^2)(n_3)^2$$

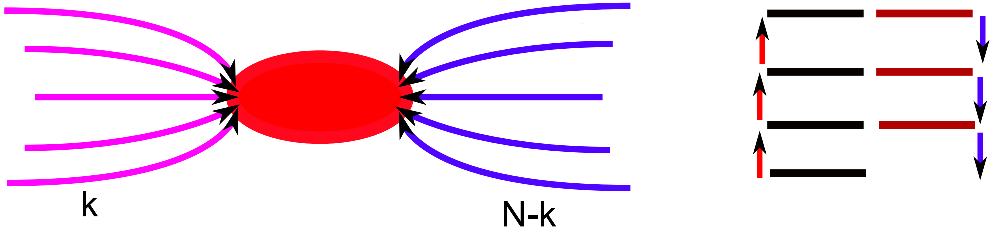
Tensions as a function of m_3/m



The round markers correspond to the “polar” vortex, while the square markers correspond to a vortex on the “equator”

$m_3 << m$

The sigma model has two distinct minima at $n_3 = \pm 1$; we can make kink out of them



Confined kink correspond to monopoles of the 4d gauge theory

Kinks have also their dyonic generalizations.
There is a level crossing between kink and dyonic kink for

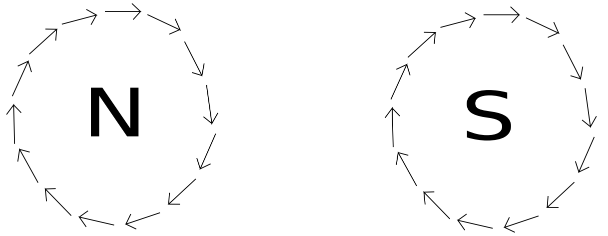
$$\theta_{1+1} = k(N_c - k)\theta_{3+1} = \pi \, .$$

$$\boxed{m_3 >> m}$$

The low energy effective description is an $O(2)$ sigma model. The $O(2)$ sigma model instantons look like “vortices inside the vortex”

This instantons correspond to merons of the original $O(3)$ sigma model

There are two different kinds of merons: they can have the north or the south pole at the core



Merons condensation can generate mass gap if the $O(2)$ sigma model coupling is strong enough (Kosterlitz-Thouless transition)

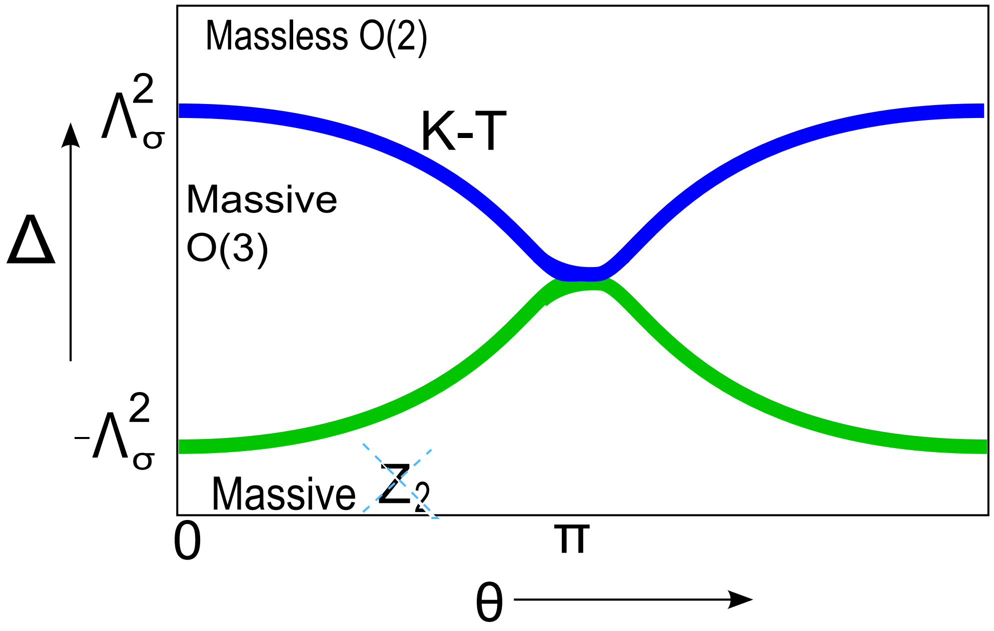
Kosterlitz-Thouless transition

The running of the coupling of the $\mathbb{CP}^1$ sigma model is "frozen" at the energy scale $\sqrt{\Delta} = \sqrt{m_3^2 - m^2}$. At that scale we have to do match with the coupling of the $O(2)$ sigma model.

As $\Delta = m_3^2 - m^2$ is decreased under a critical value, the effective $O(2)$ sigma model will be strong enough in order to have a Kosterlitz-Thouless transition. Mass is generated by merons condensation, at high enough m_3 the mass gap will disappear

At $\theta = \pi$, the mechanism which produced the mass gap fails due to the relative cancellation between the two different kinds of merons (I. Affleck, 1986)

Quantum phases of the vortex string



The KT phase transition manifest itself on the bulk as a jump in a the Lüscher term associated with the magnetic confining string

Confining vacuum

S-duality: maps the Higgs vacuum to the confining one, maps the D5 background to a NS5 background

Tension of a probe F string:

$$T_{\mathrm{F1}} = \frac{m^2 g_s N_c}{2},$$

Effective world-sheet theory for F string:

$$S_{\mathrm{F1}} = \int d^2x \left(\frac{g_s N_c}{4} (\partial_s \vec{n}_w)^2 \right) .$$

From S-duality the tension of the k-string:

$$T^{\mathrm{confining}}_{N_c,k} = m^2 \, \frac{g^2_{\mathrm{YM}}}{8\pi} \, k(N_c - k).$$

Conclusions

The tension of the k-vortex in the Higgs vacuum is proportional to $k(N_c - k)$, Casimir scaling law.

S-duality at $\theta_{3+1} = 0$ maps the Higgs vacuum to the confining one, so these results can be applied also for the confining vacuum in the strong coupling limit

The effective vortex world-sheet theory has been derived, both from the field theory picture and from the string dual. There are quantum phase transitions as a function of $\Delta = m_3^2 - m^2$ and of $\theta_{1+1} = k(N_c - k)\theta_{3+1}$.