# QCD/string Wilson loops and scattering amplitudes

by

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Based on: Y. M., Poul Olesen

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- Phys. Rev. D **80**, 026002 (2009) [arXiv:0903.4114 [hep-th]]
- arXiv:1002.0055 [hep-th]
- arXiv:1006.0078 [hep-th]

P. Buividovich, Y.M. Nucl. Phys. B **834**, 453 (2010) [arXiv:0911.1083 [hep-th]]

# Introduction and motivations since 1979

Y.M., Migdal (1979)

QCD string is not Nambu–Goto but the asymptote at large Wilson loops is universal:

classical string

$$W(C) \overset{\text{large } C}{\propto} \mathrm{e}^{-K S_{\min}(C)} \qquad \Longrightarrow \quad \text{the area law} = \text{confinement}$$

semiclassical correction (bosonic string) Lüscher, Symanzik, Weisz (1980)

$$W(C) \overset{\text{plane } C}{\propto} \mathrm{e}^{-K S_{\min}(C) + \frac{\#}{24\pi}\int \mathrm{d}^2 w \left(\partial_a \ln\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right|\right)^2} \qquad w(z) : \text{UHP} \Rightarrow D$$

$$\overset{\text{rectangle}}{\propto} \mathrm{e}^{-KRT + \frac{\#}{24}\frac{T}{R}} \qquad \Longrightarrow \quad \text{the Lüscher term}$$

for rectangle with $T \gg R$

$\# = d - 2$ for (bosonic string)

Ambjorn, Olesen, Peterson (1984)

De Forcrand, Schierholz, Schneider, Teper (1985)

What are the consequences for correlators of composite operators?

Regge behavior of scattering amplitudes at high energy and fixed momentum transfer under a few controllable approximations.

# Minimal area as boundary functional

QCD/string correspondence (expectation of 1970's):

$$W(C) \stackrel{?}{=} \text{open string disk amplitude} \quad \text{with Dirichlet b.c.}$$

$S(C)$ is highly nonlinear for Nambu–Goto $\Longrightarrow$ hopeless to calculate.

Reparametrization of the boundary is needed for conformal invariance in the Polyakov formulation Polyakov (1981)

Douglas algorithm for solving the Plateau problem Douglas (1931) (finding the minimal surface) is to minimize the boundary functional

$$A[x(\theta)] = \frac{1}{8\pi} \int_0^{2\pi} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\phi' \, \frac{\left[x(\theta(\phi)) - x(\theta(\phi'))\right]^2}{1 - \cos(\phi - \phi')}$$

with respect to the reparametrizations $\theta(\phi)$ ($\dot{\theta}(\phi) \geq 0$). In general

$$A[x(\theta)] \geq A[x(\theta_*)] = S_{\text{min}}(C)$$

The minimum is reached at $\theta(\phi) = \theta_*(\phi)$ which is contour-dependent.

# Minimal area as boundary functional (cont.)

The Douglas functional can be equivalently rewritten as

$$A = -\frac{1}{4\pi}\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2\, \dot{x}(\theta_1)\cdot\dot{x}(\theta_2)\,\ln\left(1-\cos\left[\phi(\theta_1)-\phi(\theta_2)\right]\right)$$

when only $\phi(\theta)$ is sensitive to reparametrizations.

Simplest example: $\phi_*(\theta)=\theta$ for a circle.

The solution for an ellipse $x_\mu(\theta)=(a\cos\theta, b\sin\theta)$

$$\theta_*'(\phi)=\frac{\pi}{2K(\nu)}\frac{1}{\sqrt{(1-\nu)^2+4\nu\sin^2\phi}} \qquad \frac{\pi K\left(\sqrt{1-\nu^2}\right)}{2K(\nu)}=\log\frac{a+b}{a-b}$$

where $K(\nu)$ is the complete elliptic integral of the first kind.

Elliptic integrals also emerge for a rectangle (the Schwarz–Christoffel mapping).

# Relation to boundary action in string theory

O. Alvarez (1983)

Durhuus, Olesen, Petersen (1984)

$A[x]$ is well-known as the classical boundary action in string theory.

Tree-level disk amplitude with Dirichlet b.c. for Polyakov string:

- integrate over fluctuations inside the disk, $X(r, \phi)$ with $r < 1$
- fix the value $X(1, \phi) \equiv x(\theta(\phi))$ at the boundary.

Subtlety with fixing conformal gauge:

- Liouville field decouples only in the interior of the disk.
- Its boundary value determines $\theta_*(\phi)$ at the classical level.
- The path integral over the boundary metrics restores the invariance under reparametrizations of the boundary (specific to QCD string).
- The path integral over the boundary metrics decouples in $d = 26$ for on-shell tachyon or massless vector $\Longrightarrow$ usual Koba–Nielsen amplitudes (specific to fundamental string).

# Relation to boundary action in string theory (cont.)

The proof of

$$A[x(\theta_*)] = S_{\text{min}}(C)$$

- reconstruct the surface $X_\mu(r,\phi)$ from the boundary value $X_\mu(1,\phi) = x_\mu(\theta_*(\phi))$ by the Poisson formula
- thus constructed $X_\mu$ automatically obeys conformal gauge

$$\frac{\partial X}{\partial r}\cdot\frac{\partial X}{\partial \phi} = 0\,, \qquad r^2\frac{\partial X}{\partial r}\cdot\frac{\partial X}{\partial r} = \frac{\partial X}{\partial \phi}\cdot\frac{\partial X}{\partial \phi}\,.$$

$\Longrightarrow$ the Nambu–Goto action $=$ the quadratic action

$\Longrightarrow$ the boundary action for Polyakov string $=$ the minimal area

The Douglas integral also appeared in models of statistical mechanics:

- boundary conformal field theories
- circular brane model (integrable) — Lukyanov, Zamolodchikov (2003)
- area law in turbulence — Migdal (1994)

# Integration over reparametrizations

Polyakov (1997)

Wilson loop in large-$N$ QCD $\Longleftrightarrow$ the tree-level string disk amplitude integrated over reparametrizations of the boundary contour.

Conformal map of the disk into the upper half-plane:
the disk boundary $\Longrightarrow$ the real axis

$$t(\theta) = -\cot\frac{\theta}{2} \qquad -\infty < t < +\infty$$

Reparametrization-invariant ansatz

$$W(C) = \int \mathcal{D}s(t) \exp\left( \frac{K}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 \, \dot{x}(t_1)\cdot\dot{x}(t_2) \ln|s(t_1)-s(t_2)| \right)$$

where the path integral is over reparametrizations $s(t)$ (with $s'(t) \geq 0$).

This classical boundary action is derivable for:

- bosonic string in $d = 26$,
- superstring in $d = 10$.

Area law for asymptotically large $C$ (or very large $K$) $\Longrightarrow$ a saddle point in the integral over reparametrizations at $s(t) = s_*(t)$.

# Large loops and minimal area

Gaussian fluctuations around the saddle-point $\theta_*(\phi)$ result in a pre-exponential factor

$$W\left[x(\cdot)\right] \stackrel{\text{large loops}}{=} F\left[\sqrt{K}x(\cdot)\right] \mathrm{e}^{-KS_{\text{min}}[x(\cdot)]}\left[1+\mathcal{O}\left(\left(KS_{\text{min}}\right)^{-1}\right)\right],$$

which is contour dependent

$$F\left[\text{circle}\right] \propto \sqrt{KR^2} \qquad \text{for a circle}$$

Asymptotic area law is recovered modulo the pre-exponential which is not essential for large loops.

More subtle effects (such as the Lüscher term) are due to the pre-exponential factor. For a $R \times T$ rectangle

$$F\left[\text{rectangle}\right] \propto \mathrm{e}^{\pi T/R} \qquad \text{for } T \gg R$$

reproducing the Lüscher term for bosonic string in $d = 26$.

# Lüscher term from reparametrizations

Olesen, Y.M. (2010)

Semiclassical expansion of the Douglas integral around $s_*(t)$
$s(t) = s_*(t) + \beta(t)$ to quadratic order in $\beta$

$$\Psi[x(\cdot)] = \int \mathcal{D}\beta(t)\, \mathrm{e}^{-KS_{\min} - KS_2}$$

$$S_2\,[\beta(t)] = \frac{K}{4\pi} \int_{-\infty}^{+\infty} \mathrm{d}t_1 \mathrm{d}t_2 \frac{\dot{x}(t_1) \cdot \dot{x}(t_2)}{(s_*(t_1) - s_*(t_2))^2} \,[\beta(t_1) - \beta(t_2)]^2$$

Next orders are suppressed (semiclassical expansion in $\beta \sim 1/\sqrt{KS_{\min}}$)

Rychkov (2002)

From the Schwarz–Christoffel mapping (for an $R \times T$ rectangle)

$$x_1(t_*(s)) + \mathrm{i}x_2(t_*(s)) = \frac{R}{2K(\mu)} F\left(\frac{s}{\sqrt{\mu}}, \mu\right), \qquad \frac{2T}{R} = \frac{K\left(\sqrt{1-\mu^2}\right)}{K(\mu)}$$

$S_2 \sim 1/\mu$ for $\mu \to 0$ (mode expansion for 4 segments near $s = \pm\sqrt{\mu}$)

$$\Psi[\text{rectangle}] \propto \Big(\prod_{\text{modes}} \sqrt{\mu}\Big)^4 \propto \Big(\frac{1}{\sqrt[4]{\mu}}\Big)^4 \propto \exp\left(\frac{\pi T}{R}\right)$$

The Lüscher term was calculated also for an ellipse: $\exp(\pi^2 a/2b)$.

# Functional Fourier transformation

Migdal (1986)

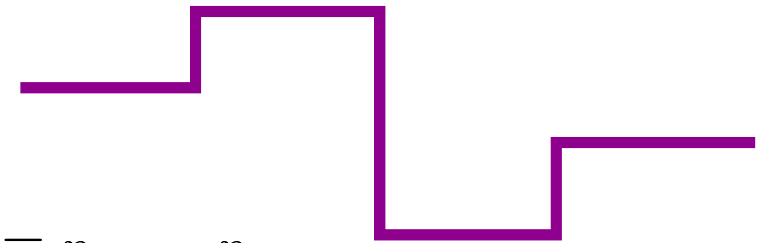
Reparametrization-invariant functional Fourier transformation

$$A[p(\cdot)] = \int \mathcal{D}x \; \mathrm{e}^{\mathrm{i} \int p \cdot \mathrm{d}x} \, W[x(\cdot)]$$

of disk amplitude for piecewise constant $p(t) \Longrightarrow$ scattering amplitude.

Piecewise constant momentum-space loop $p(t)$:

$$p(t) = p_i \qquad \text{for } t_i < t < t_{i+1}$$



$$\dot{p}(t) = -\sum_i \Delta p_i \delta(t - t_i) \qquad \text{with} \quad \Delta p_i \equiv p_{i-1} - p_i$$

representing $M$ momenta of (all incoming) particles.
Then momentum conservation is automatic while an (infinite) volume $V$ is produced, say, by integration over $x_0 = x_M$.

Relation to string vertex operators:

$$\int \mathrm{d}t \, p(t) \cdot \dot{x}(t) = - \int \mathrm{d}t \, \dot{p}(t) \cdot x(t) = \sum_i \Delta p_i \cdot x_i$$

reproducing the exponent of the Fourier transformation.

# Momentum-space disk amplitude

Performing the Gaussian path integration:

$$A[p(\cdot)] = \int \mathcal{D}s(t) \exp\left(\alpha' \int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2\, \dot{p}(t_1) \cdot \dot{p}(t_2) \ln|s(t_1) - s(t_2)|\right)$$

It looks like the disk amplitude with $x(t)$ substituted by $Kp(t)$ (and $1/K = 2\pi\alpha'$). The determinant is $s$-independent constant.

The principal-value prescription will be important for stepwise $p(t)$.

$p(t) = p_j$ at the $j$-th interval for the stepwise discretization $\Longrightarrow$ reparametrization changes $t_j$'s for $s_j$'s keeping their cyclic order — discrete reparametrization transformation.

Stepwise discretization of $x(t)$ itself would violate the continuity of the string end world line, but smeared steps will be OK!
They reproduce the principal-value prescription.

# Derivation of Koba–Nielsen amplitudes

First note that

$$\int_{-\infty}^{+\infty} \mathrm{d}t_1 \int_{-\infty}^{+\infty} \mathrm{d}t_2 \, \dot{p}(t_1) \cdot \dot{p}(t_2) \ln |s(t_1) - s(t_2)|$$
$$= -\frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1 \mathrm{d}s_2}{(s_1 - s_2)^2} \left[p\left(t(s_1)\right) - p\left(t(s_2)\right)\right]^2$$

Integration over $s_1$ or $s_2$ has divergences for adjacent sides $k = l \pm 1$

Principal-value regularization (achieved by smearing of steps in $p(s)$)
$\Longrightarrow$ omitting sides with $k = l \pm 1$:

$$-\frac{1}{2} \sum_{k \neq l \pm 1} \int_{s_{k-1}}^{s_k} \mathrm{d}s_1 \int_{s_{l-1}}^{s_l} \mathrm{d}s_2 \frac{(p_k - p_l)^2}{(s_1 - s_2)^2}$$
$$= \sum_{k \neq l} \Delta p_k \cdot \Delta p_l \log |s_k - s_l| + \sum_j \Delta p_j^2 \log \frac{(s_j - s_{j-1})(s_{j+1} - s_j)}{(s_{j+1} - s_{j-1})}$$

which is invariant under $PSL(2;\mathbb{R})$ projective transformations (subgroup of reparametrizations)

$$s \Rightarrow \frac{as + b}{cs + d} \qquad \text{with} \quad ad - bc = 1$$

# Derivation of Koba–Nielsen amplitudes (cont.)

Integrating over reparametrizations at intermediate points $(s_{i-1}, s_i)$ (where $\Delta p_i = 0$) results in the following measure

$$D^{(M)}s = \prod_{i=1}^{M} \frac{\mathrm{d}s_i}{|s_i - s_{i-1}|}$$

for the integration over $s_i$'s (where $\Delta p_i \neq 0$).
It is invariant under the projective transformation and gives

$$A(\Delta p_1, \ldots, \Delta p_M)$$
$$= \int_{s_{i-1}<s_i} \prod_i \frac{\mathrm{d}s_i}{|s_i - s_{i-1}|} \prod_{k\neq l} |s_k - s_l|^{\alpha' \Delta\vec{p}_k \cdot \Delta\vec{p}_l} \prod_j \left( \frac{|s_j - s_{j-1}|\,|s_{j+1} - s_j|}{|s_{j+1} - s_{j-1}|} \right)^{\alpha' \Delta p_j^2}$$

where the integration over $s_i$ emerges from the path integral over reparametrizations.

Fixing the $PSL(2;\mathbb{R})$ invariance in the standard way

$$s_1 = 0, \quad s_{M-1} = 1, \quad s_M = \infty$$

$\Longrightarrow$ Lovelace-type dual amplitudes in the Koba–Nielsen variables.

# Consistent off-shell amplitudes

For 4 scalars this reproduces projective-invariant off-shell amplitude

$$A(\Delta p_1, \Delta p_2, \Delta p_3, \Delta p_4) = \int_0^1 \mathrm{d}x\, x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1},$$

where $\alpha(t) = \alpha(0) + \alpha' t$ – linear Regge trajectory – and

$$s = -(\Delta p_1 + \Delta p_2)^2, \qquad t = -(\Delta p_2 + \Delta p_3)^2$$

are usual Mandelstam's variables (for Euclidean metric).

The tachyonic condition $\alpha' \Delta p_j^2 = 1$ has not to be imposed. The on-shell Veneziano tachyon amplitudes is obtained by setting $\alpha' \Delta p_j^2 = 1$

For critical bosonic string (in $d = 26$): Aoyama, Dhar, Namazie (1986)

$$A = \int \mathcal{D}\varphi(s) \int \prod_m \mathrm{d}s_m\, \mathrm{e}^{\varphi(s_m)/2 - \pi\alpha' \Delta p_m^2 G(s_m, s_m)} \prod_{j \neq m} |s_j - s_m|^{\alpha' \Delta p_j \cdot \Delta p_m}$$

the path integration over $\varphi(s)$ — boundary Liouville field decouples only for tachyonic scalar, massless vector, etc.
due to an invariant regularization Polyakov (1981)

$$G(s_m, s_m) \longrightarrow G_\varepsilon(s_m, s_m) = \frac{1}{\pi} \ln \frac{1}{\varepsilon} + \frac{1}{2\pi}\varphi(s_m)\,,$$

# Path integrals over reparametrizations

The measure on $Diff(\mathbb{R})$

$$\int_{\substack{s(\tau_0)=s_0\\ s(\tau_f)=s_f}} \mathcal{D}_{\text{diff}} s(\tau) \cdots = \lim_{N\to\infty} \int_{s_0}^{s_f} \prod_{j=1}^{N-1} \int_{s_0}^{s_{j+1}} \mathrm{d}s_j \, \frac{1}{(s_{j+1}-s_j)} \frac{1}{(s_1-s_0)} \cdots$$

is invariant under reparametrizations

$$s \to t(s)\,, \quad t(s_0)=s_0\,, \quad t(s_f)=s_f\,, \quad \frac{\mathrm{d}t}{\mathrm{d}s} \geq 0$$

Integration goes over $(N-1)$ subordinated values

$$s_0 \leq \cdots \leq s_{i-1} \leq s_i \leq \cdots \leq s_N = s_f$$

Discretizing $s' = \exp[-\varphi]$ that relates reparametrizations to the boundary value of the Liouville field $\varphi$ by $s_i - s_{i-1} = \exp[-\varphi_i]$ $\Longrightarrow$

$$\int_{s_0}^{s_f} \mathcal{D}_{\text{diff}} s \cdots = \lim_{N\to\infty} \prod_{i=1}^{N} \int_{-\infty}^{+\infty} \mathrm{d}\varphi_i \, \delta^{(1)}\Big(s_f - s_0 - \sum_{j=1}^{N} \mathrm{e}^{-\varphi_j}\Big) \cdots$$

with the only restriction on $\varphi_i$'s given by the delta-function.

# Path integrals over reparametrizations (cont.)

Regularization of (logarithmically) divergent integral

$$\frac{1}{(s_i - s_{i-1})} \longrightarrow \frac{1}{\Gamma(\delta_i)(s_i - s_{i-1})^{1-\delta_i}} \qquad \text{all } \delta_i = \delta$$

Main integral for the integration at the intermediate point $s_i$

$$\int_{s_{i-1}}^{s_{i+1}} \mathrm{d}s_i \frac{\Gamma^{-1}(\delta_i)\Gamma^{-1}(\delta_{i+1})}{(s_{i+1} - s_i)^{1-\delta_{i+1}}(s_i - s_{i-1})^{1-\delta_i}} = \frac{\Gamma^{-1}(\delta_i + \delta_{i+1})}{(s_{i+1} - s_{i-1})^{1-\delta_i-\delta_{i+1}}}$$

This is an analogue of the well-known formula

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}s_i}{\sqrt{2\pi}} \frac{\mathrm{e}^{-(s_f-s_i)^2/2\nu_1}}{\sqrt{\nu_1}} \frac{\mathrm{e}^{-(s_i-s_0)^2/2\nu_2}}{\sqrt{\nu_2}} = \frac{\mathrm{e}^{-(s_f-s_0)^2/2(\nu_1+\nu_2)}}{\sqrt{(\nu_1+\nu_2)}}$$

used for calculations with the usual Wiener measure.

The functional limit is when $N \to \infty$ with $N\delta \to 0$:

$$\int_{s_0}^{s_N = s_f} \mathcal{D}^{(N)}_{\text{diff}} s = \frac{1}{\Gamma(N\delta)} \frac{1}{(s_N - s_0)^{1-N\delta}} \stackrel{N\delta \to 0}{\longrightarrow} N\delta \frac{1}{\big(s_f - s_0\big)}$$

reproducing the projective-invariant result.

# Reparametrizations as Lévy stochastic process

Buividivich, Y.M. (2009)

What trajectories are typical in path integral over reparametrizations?

Subordinated stochastic process (gamma-subordinator) with PDF

$$P(\Delta s_i) = \frac{1}{\Gamma(\delta)\,(\Delta s_i)^{1-\delta}} \qquad \delta > 0 \text{ is a time step}$$

$$\mathrm{d}s_f \int_{s_0}^{s_f} \mathcal{D}^{(N)}_{\text{diff}}\, s \quad \text{— propagator from } s_0 \text{ to } \left[s_f, s_f + \mathrm{d}s_f\right]$$

during the time $\tau = N\delta$

Scaling variable

$$z = \tau \ln \frac{1}{(s_f - s_0)} \Longrightarrow \frac{\tau \mathrm{d}s_f}{\left(s_f - s_0\right)^{1-\tau}} = \mathrm{d}z\, \mathrm{e}^{-z}\,,$$

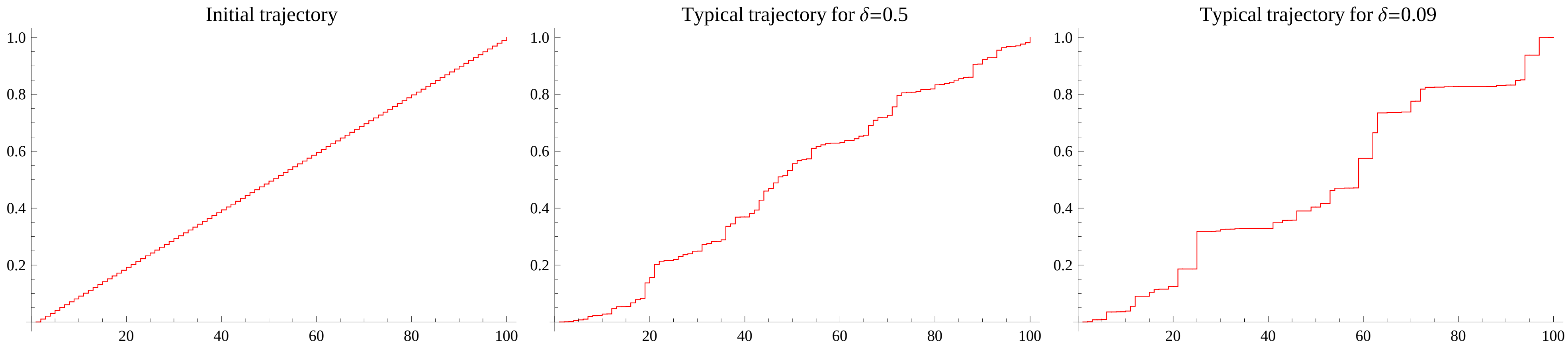
Scaling with

$$(s_f - s_0) \sim \mathrm{e}^{-1/\tau} \Longrightarrow \text{Hausdorff dimension}=0$$

supersedes $(s_f - s_0)^2 \sim \tau$ for the Brownian motion (whose $d_H = 2$).

# Sample trajectories in path integral

Typical trajectories for the gamma-subordinator (obtained by Metropolis–Hastings algorithm)




Lévy's flights are seen in the right figure.

Their origin is that $P(\Delta s_i)$ is very large at small $\Delta s_i$ $\Longrightarrow$ most of $\Delta s_i$'s are small.

Then some of $\Delta s_i$ has to be large to satisfy the boundary condition.

Hausdorff dimension decreases from 1 to 0 (left to right)

(Horowitz, 1968)

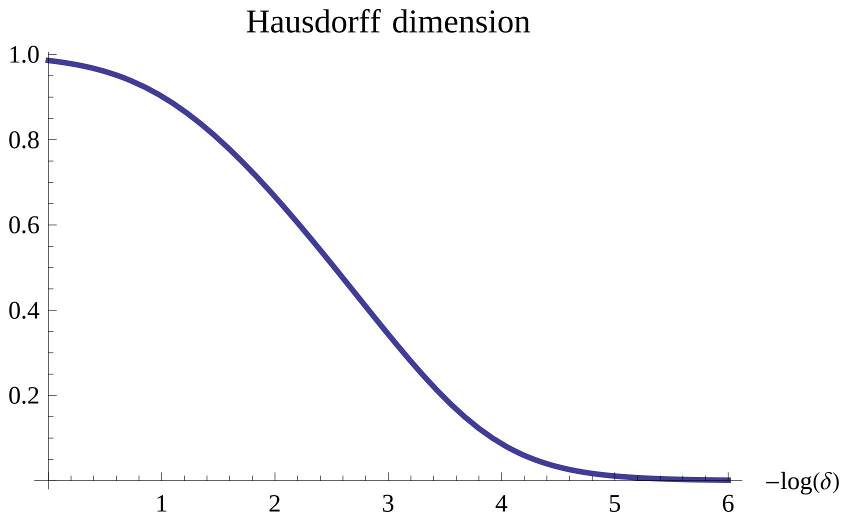
# Hausdorff dimension of sample trajectories

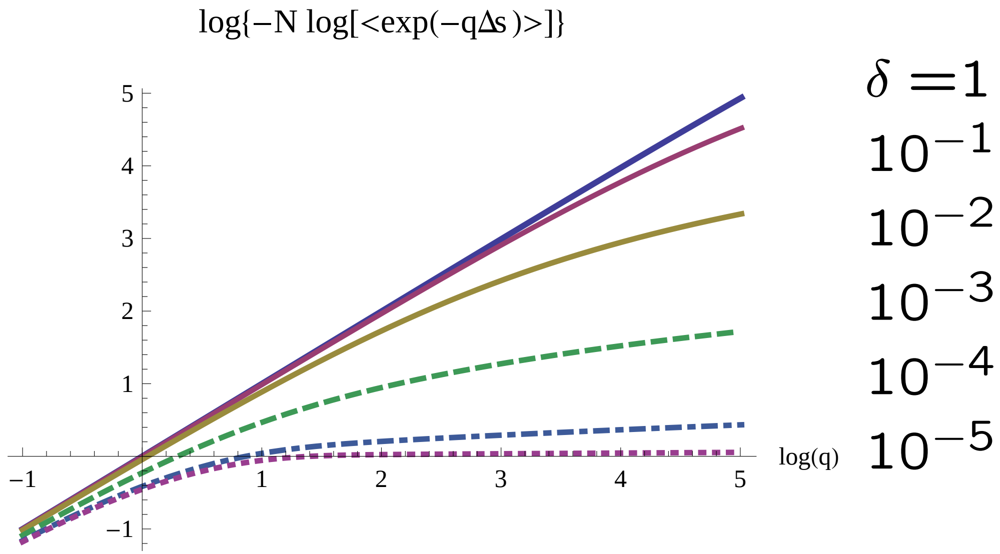
Hausdorff dimension of the discretized process is determined by its characteristic function (Lévy–Khintchine)

$$\left\langle \mathrm{e}^{-q\Delta s_i} \right\rangle = {}_1F_1(\delta, \delta N; -q)$$

as

$$d_H = \lim_{q\to\infty} \frac{\ln\left(-N \ln \left\langle \mathrm{e}^{-q\Delta s_i}\right\rangle\right)}{\ln q}$$




Hausdorff dimension versus $\ln(1/\delta)$ (left) from the slope of the lines (right).

It decreases from 1 for $\delta \gtrsim 1$ to 0 for $\delta N \to 0$

# QCD amplitudes through Wilson loops

Y.M., Migdal (1981)

Green's functions of $M$ colorless composite quark operators

$$\bar{q}(x_i)q(x_i) \qquad \bar{q}(x_i)\gamma_5 q(x_i) \qquad \bar{q}(x_i)\gamma_\mu q(x_i) \qquad \bar{q}(x_i)\gamma_\mu\gamma_5 q(x_i)$$

are given by the sum over Wilson loops passing via $x_i$ $(i = 1, \ldots, M)$

$$G \equiv \left\langle \prod_{i=1}^{M} \bar{q}(x_i)q(x_i) \right\rangle_{\text{conn}} = \sum_{\text{paths} \ni \{x_1,\ldots,x_M \equiv x_0\}} J[x(\tau)]\, W[x(\tau)]$$

The weight for the path integration is

$$J[x(\tau)] = \int \mathcal{D}k(\tau)\ \text{sp}\ \text{P}\, \text{e}^{\text{i}\int_0^{\mathcal{T}} \text{d}\tau\, [\dot{x}(\tau)\cdot k(\tau) - \gamma\cdot k(\tau)]}$$

for spinor quarks of mass $m$ and scalar operators or

$$J[x(\tau)] = \text{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \text{d}\tau\, \dot{x}^2(\tau)} = \int \mathcal{D}k(\tau)\ \text{e}^{\int_0^{\mathcal{T}} \text{d}\tau\, [\text{i}\dot{x}(\tau)\cdot k(\tau) - k^2(\tau)/2]}$$

for scalar quarks. $\tau$ is the proper time.

The Wilson loop $W(C)$ is in pure Yang–Mills at large $N$ (or quenched). For finite $N$, correlators of several Wilson loops are present.

# QCD amplitudes through Wilson loops (cont.)

QCD scattering amplitude = functional Fourier transform

$$A\,(\Delta p_1,\ldots,\Delta p_M) = \sum_{\text{paths}} \mathrm{e}^{\mathrm{i}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}(\tau)\cdot p(\tau)}\, J[x(\tau)]\, W[x(\tau)]$$

for piecewise constant momentum-space loop $p(\tau)$ as before.

Substituting the area-law and interchanging the integrals over $x(\tau)$ (Gaussian) and $s(\tau)$, we get

$$A\,(\{\Delta p_m\}) \propto \int_0^{\infty} \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_{M-1}}{1+s_{M-1}^2} \prod_{i=1}^{M-2} \int_{-\infty}^{s_{i+1}} \frac{\mathrm{d}s_i}{1+s_i^2}$$

$$\times \int \mathcal{D}k(t)\ \mathrm{sp}\ P\,\mathrm{e}^{-\mathrm{i}\mathcal{T}\int \mathrm{d}t\,\gamma\cdot k(t)/(1+t^2)}\, W\,[x(t) = \frac{1}{K}\,(p(t)+k(t))\,]$$

For small $m$ and/or large $M$, the integral over $\mathcal{T}$ is dominated by large $\mathcal{T} \sim (M-1)/m$ and the path integral over $k$ factorizes:

$$A\,(\{\Delta p_m\}) \propto W\,[x(t) = \frac{1}{K} p(t)]$$

It is just the same as the Lovelace amplitude for the (critical) string!

# Justification of large $\mathcal{T}$ as $m \to 0$

Path integral over $x(\tau)$ (for scalar quarks) can be calculated via

mode expansion $$x^\mu(\tau) = x_0^\mu + \sum_{n=1}^{\infty} \left( a_n^\mu \cos\frac{2\pi\tau}{\mathcal{T}} + b_n^\mu \sin\frac{2\pi\tau}{\mathcal{T}} \right) :$$

$$\int\limits_{x(0)=x(\mathcal{T})} \mathcal{D}x(\tau)\, \mathrm{e}^{-\frac{1}{2}\int_0^{\mathcal{T}} \mathrm{d}\tau\, \dot{x}^2(\tau) - \frac{K}{2}\dot{x}*G*\dot{x}} = \prod_{n=1}^{\infty} \left[ 2\pi \left( \frac{1}{\mathcal{T}} n^2 + K n \right) \right]^{-d}$$

$\zeta$-function regularization gives

$$\prod_{n=1}^{\infty} A = A^{\zeta(0)} = A^{-1/2} \qquad\qquad \prod_{n=1}^{\infty} n = \sqrt{2\pi}$$

$$\prod_{n=1}^{\infty} \left[ 2\pi \left( \frac{1}{\mathcal{T}} n^2 + K n \right) \right]^{-d} \quad \begin{array}{l} \xrightarrow{\mathcal{T}\to 0} \quad (2\pi\mathcal{T})^{-d/2} \\ \xrightarrow{\mathcal{T}\to \infty} \quad K^{d/2} \end{array}$$

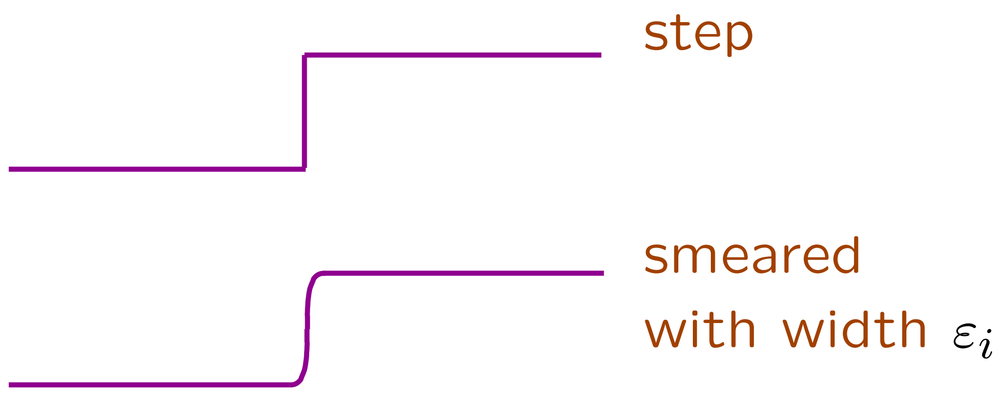
Miniconclusion: large $\mathcal{T}$ are essential in QCD perturbation theory only for $M > 4$, but are essential non-perturbatively: $\int \mathrm{d}\mathcal{T}\, \mathcal{T}^{M-1}\, \mathrm{e}^{-m\mathcal{T}} \ldots$

# Wilson-loop/scattering-amplitude duality in QCD

Wilson loop = scattering amplitude

provided that

$$C_* := \ x(t) = \frac{1}{K} p(t)$$




The above principal value prescription can be achieved by regularizing

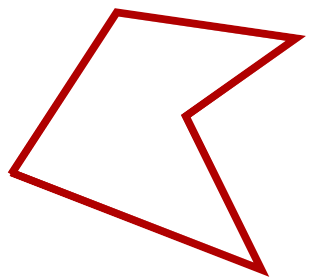
$$\dot{p}(t) = \sum_i \Delta p_i \frac{\varepsilon_i}{\pi[\varepsilon_i^2 + (t - t_i)^2]} \qquad \varepsilon_i = \varepsilon \, \mathrm{e}^{-\varphi(t_i)}$$

with the boundary Liouville field $\varphi(t_i)$ providing covariance

$$p(t) = \frac{1}{\pi} \sum_i \Delta p_i \arctan \frac{(t - t_i)}{\varepsilon_i}$$

$\Longrightarrow$ polygon with vertices

$$x_i = \frac{1}{K} p_i \qquad x_i - x_{i-1} = \frac{1}{K} \Delta p_i$$



where $K = 1/2\pi\alpha'$ is string tension.

# WL/SA duality in QCD (cont.1)

Similar to WL/SA duality in $\mathcal{N} = 4$ SYM — Alday, Maldacena (2007); Drummond, Korchemsky, Sokatchev (2008)

for Regge kinematic regime with

$$s \gg t \gg \Delta p_i^2$$

$2 \to 2$ kinematics (center-mass frame)

$$\begin{aligned}
\Delta p_1 &= (E, p, 0, 0) \\
\Delta p_2 &= (E, -p, 0, 0) \\
\Delta p_3 &= (-E, p\cos\alpha, p\sin\alpha, 0) \\
\Delta p_4 &= (-E, -p\cos\alpha, -p\sin\alpha, 0)
\end{aligned}$$

$C_*$ bounds the minimal surface of the area

$$K S_{\min}(C_*) = \alpha' t \ln \frac{s}{\max\{t, K\}}$$

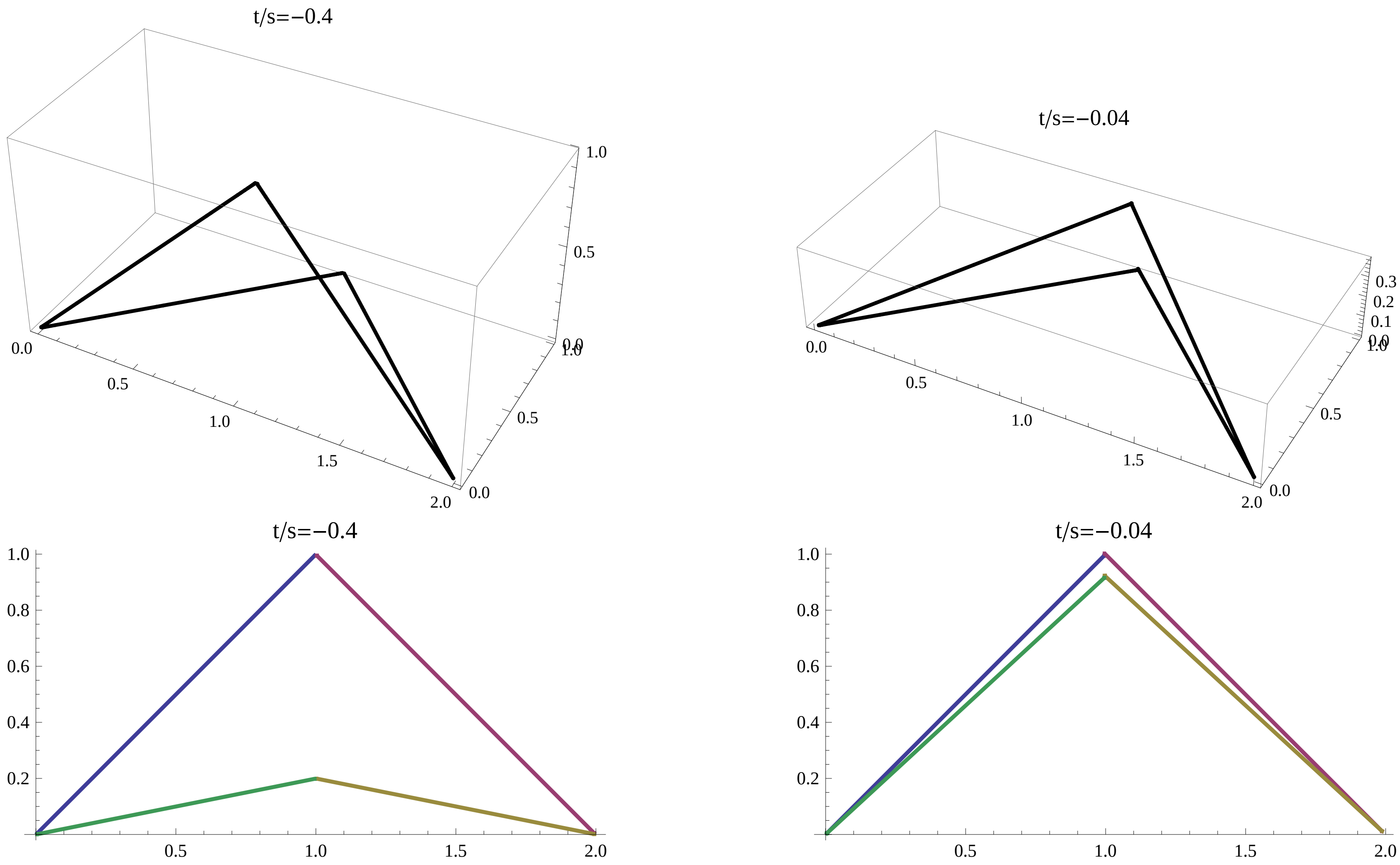
$S_{\min}(C_*)$ is large for very large $s$ = large loops dominate.

This justifies that Regge kinematic regime of scattering amplitudes corresponds to Wilson loop of large size, where the area law sets in.

Probably $t$ has to be large but $\ll s$ for the width of $C_*$ to be $\gg$ 1fm.

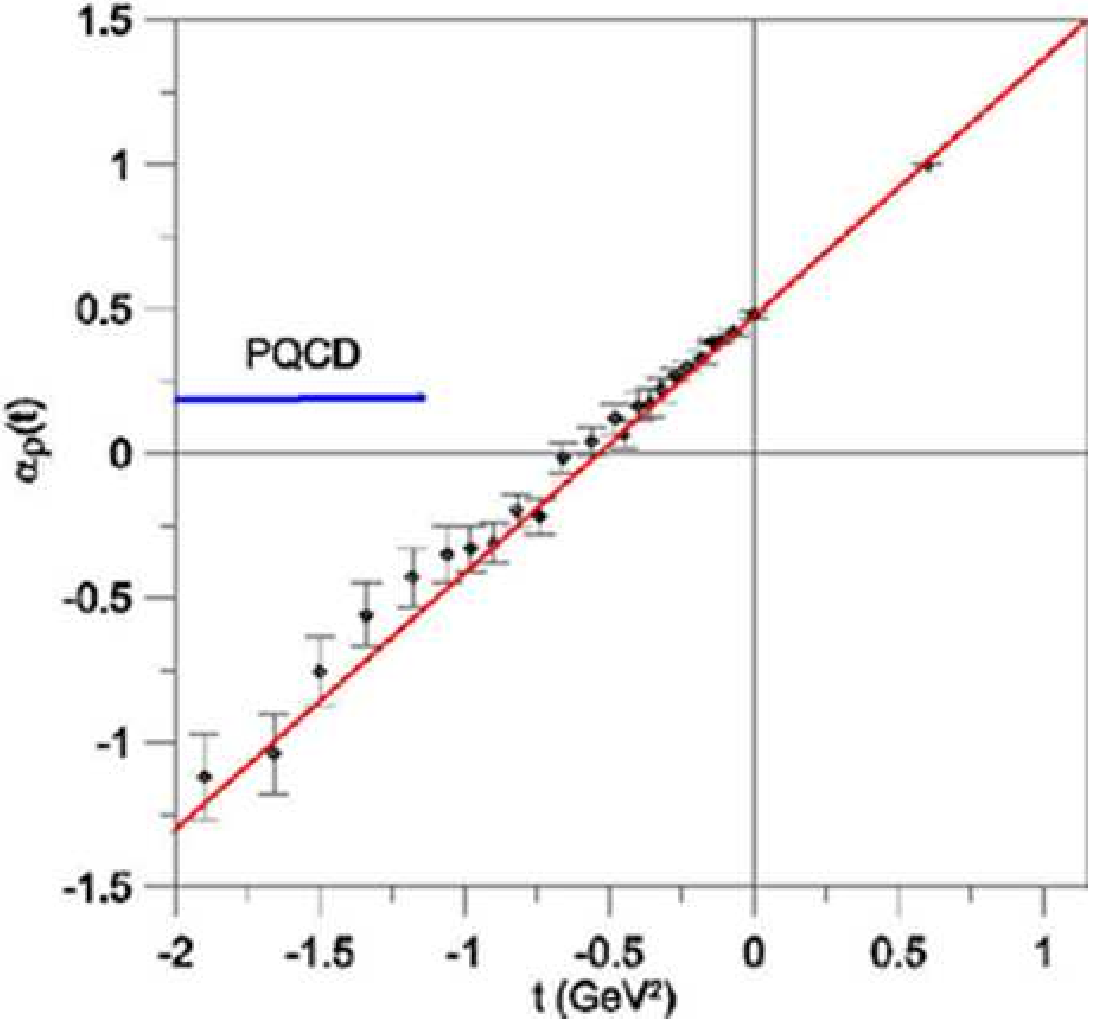
# WL/SA duality in QCD (cont.2)

Typical loop $C_*$ for $t/s = -0.4$ and $t/s = -0.04$



# Effective $\rho$-trajectory and pQCD prediction

The figure taken from A. B. Kaidalov, hep-ph/0612358




It is hard to believe that pQCD reggeization is relevant.

# Separation of pQCD and npQCD

Reggeization of $\bar{q}q$ in pQCD is due to double logarithms

Kirschner, Lipatov (1983)

$\mathcal{T}$ is restricted from above by $\tau_{\max} \sim 1/K$ to separate the contribution from small loops associated with pQCD. It plays the role of an infrared cutoff in pQCD, rather than a usual transverse mass $\mu$

With the double logarithmic accuracy:

$$\text{pQCD ladders} \;\propto\; I_1\left(\omega \ln |s| \tau_{\max}\right) \qquad \omega = \sqrt{\frac{g^2(t) C_F}{2\pi^2}} \approx .5$$

$g^2 \Rightarrow g^2(t)$ because of charge renormalization. Then asymptotically

$$\text{pQCD ladders} \;\propto\; (s\tau_{\max})^{\omega(t)}$$

standard pQCD $=$ $\tau_{\max} = \infty$ $\Longrightarrow$ IR regularization by $\mu$

The total amplitude = pQCD (this one) + npQCD (as before).
At finite $s$ the relative coefficient is of most importance.

# Semiclassical approach to Regge intercept

Olesen, Y.M. (2010)

The asymptotic ansatz can be modified to change the pre-exponential. A hint is to get the expected Lüscher term in $d < 26$.
The "improved" ansatz (not yet justified)

$$\Psi[x(\cdot)] = \int \mathcal{D}_{\rm diff} s(t)\, {\rm e}^{-KA} \left( \int \mathcal{D}\beta(t)\, {\rm e}^{-KS_2} \right)^{\gamma} \qquad \gamma = \frac{(d-26)}{24}$$

where $A$ is the Douglas integral and

$$S_2\,[\beta(t)] = \frac{K}{4\pi} \int {\rm d}t_1 \int {\rm d}t_2 \frac{\dot{x}(t_1)\cdot \dot{x}(t_2)}{(s(t_1)-s(t_2))^2}\,[\beta(t_1)-\beta(t_2)]^2\,.$$

The determinant is as from the integration over reparametrizations by expanding the Douglas integral to quadratic order in $\beta$:

$$s(t) = s_*(t) + \beta(t)$$

Only zero modes contribute since $\beta(t_i) = 0$ because of $s(t_i) = t_i$

This shifts the intercept by $\gamma$ to get $\alpha(0) = (d-2)/24$ in $d < 26$ for $\gamma = (d-26)/24$. Same result as Arvis (1983), Olesen (1985) found from the spectrum of the outstretched Nambu–Goto string (relation between tachyon and the Lüscher term).

# Conclusions

- Regge behavior of QCD scattering amplitudes follows from the area law. The only approximation is large $N$. Great simplification occurs for small $m$ and/or large $M$ (Lovelace-type amplitudes).

- It was crucial for the success of calculations that all integrals are Gaussian except for the one over reparametrizations which reduces to integration over the Koba–Nielsen variables.

- Derivation is legible for those momenta $\Delta p_i$ for which asymptotically large loops are essential in the sum over $C$:

$KS_{\text{min}}(C_*) = \alpha' t \ln \frac{s}{\max\{t,K\}}$ i.e. asymptotically large $s$ and $K \lesssim t \ll s$.

- The classical string has intercept of the $q\bar{q}$ Regge trajectory $\alpha(0) = 0$ ($\alpha(0) \approx 0.5$ from experiment) but is applicable only for $t \gg 1/\alpha'$. Semiclassical corrections result in $\alpha(0) = (d-2)/24$.

- 4-point scattering amplitude is valid only for asymptotically large $s$ and fixed $t$ associated with small angle or fixed momentum transfer.

- When $-t \ll s$ becomes large, there are no longer reasons to expect the contribution of large loops to dominate over perturbation theory, which comes from integration over small loops.