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Soliton Form Factors from Lattice Simulations

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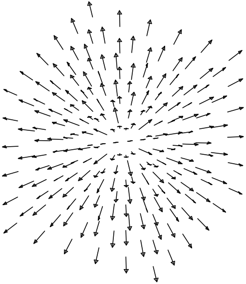
AR&D.J.Weir, arXiv:1006.2410

Topological Solitons

- Time-independent classical solutions, stabilised by topology:
    - Monopoles
        - Monopole condensation
    - Vortices
        - Dual to QCD string
    - Kinks
        - Simple



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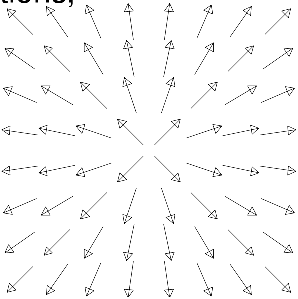

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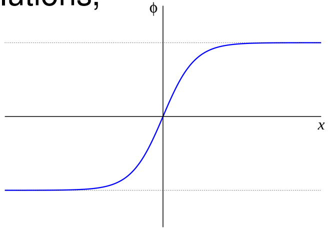

Topological Solitons

- Time-independent quantum states, stabilised by topology:
    - Monopoles
        - Monopole condensation
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        - Simple

$$|\psi\rangle ?$$

Quantum Observables

- Soliton mass (i.e. energy)
- Excitation spectrum (+wavefunctions)
- Kink form factor (Goldstone&Jackiw 1975)

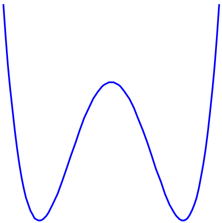
$$f(k,k') = \frac{1}{v}\langle k'|\hat{\phi}(0)|k\rangle$$

  - Momentum states $\langle k'|k\rangle = 2\pi\delta(k-k')E_k$
  - Lorentz invariance:
    Function of rapidity difference $\beta = \beta_k - \beta_k'$ only
    $(\beta_k = \mathrm{arcsinh}\; k/M)$ (Jackiw&Woo 1975)

1+1D Scalar Field Theory

$$\mathcal{L} = \int d^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4\right]$$

- Weak coupling $\lambda/m^2 \to 0$:
  - Perturbation theory
- Strong coupling $\lambda/m^2 \to \infty$:
  - Critical point:
    $\mathbb{Z}_2$ symmetry breaking at $m^2 = 0$
  - Ising universality class



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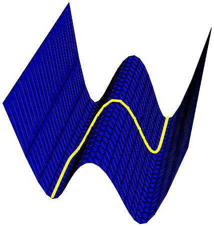
- Weak coupling:
  - Vev $\langle\phi\rangle = \pm v = \pm\sqrt{\frac{6m^2}{\lambda}}$
  - Kink solution
    $\phi_{\text{kink}} = v \tanh \frac{mx}{\sqrt{2}}$
  - Kink mass $M = 4\sqrt{2}\frac{m^3}{\lambda}$



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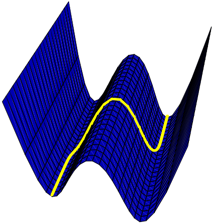
- Form factor:
  Fourier transform of classical kink

$$f_{\rm cl} = \frac{4}{3}\frac{i\pi v^2}{\sinh \frac{2}{3}\pi v^2\beta}$$



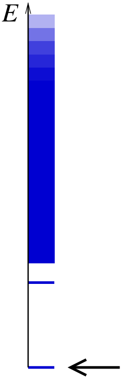
- Quantum definition of kink shape?

Perturbation Theory

- Perturb classical kink $\hat{\phi}(x) = \phi_{\mathrm{kink}}(x) + \delta\phi(x)$
- Energy eigenstates:

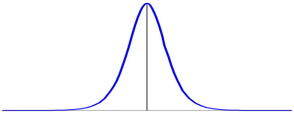
$$\left[-\frac{\partial^2}{\partial x^2} - m^2 + \frac{1}{2}\lambda\phi_{\mathrm{kink}}(x)^2\right]\psi_n(x) = E_n^2\psi_n(x)$$

Semiclassical Spectrum

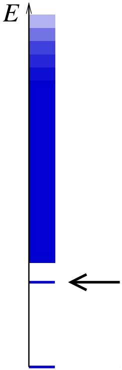


Goldstone mode

$$E_0 = 0$$

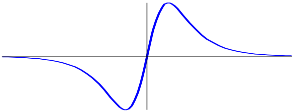


Semiclassical Spectrum

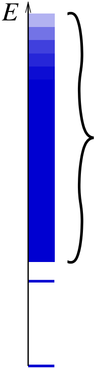


Bound state

$$E_1 = \sqrt{\frac{3}{2}} m$$

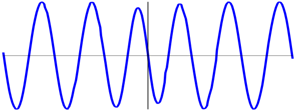


Semiclassical Spectrum



Scattering states

$$E_q = \sqrt{q^2 + 2m^2}$$



One-Loop Mass

- One-loop correction to kink mass: (Dashen et al 1974)
  Quantum zero-point energy in kink background

$$\begin{aligned} M &= M_{\rm cl} + \frac{1}{2}\sum E_n \\ &= M_{\rm cl} + m\left(\frac{1}{6}\sqrt{\frac{3}{2}} - \frac{3\sqrt{2}}{\pi} + O(\lambda/|m^2|)\right) \end{aligned}$$

1+1D Scalar Field Theory

$$\mathcal{L} = \int d^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \right]$$

- Strong coupling:
  - Scalar unstable → two kinks
  - Only one relevant scale: M
  - Critical exponents:
    $v \sim |m^2 - m_c^2|^{1/8}$
    $M \sim |m^2 - m_c^2|$
    $m_\phi = 2M$

1+1D Scalar Field Theory

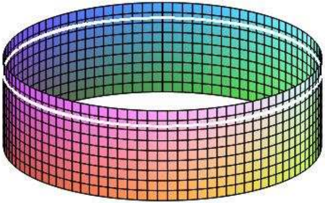
$$\mathcal{L} = \int d^2x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \right]$$

- Strong coupling:
  - Exact form factor: (Berg et al 1979)

$$f_{\mathrm{Ising}}(\beta) = i \coth\frac{\beta}{2}$$

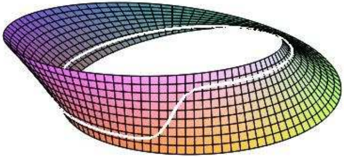
Lattice Formulation

- Define theory on $L \times T$ Euclidean lattice
- Boundary conditions: Preserve translation invariance
- Periodic: No kinks ($p_{\rm kink} \sim \exp(-2MT)$)



Lattice Formulation

- Define theory on $L \times T$ Euclidean lattice
- Boundary conditions: Preserve translation invariance
- Periodic: No kinks ($p_{\mathrm{kink}} \sim \exp(-2MT)$)
- Anti-periodic: One kink (Groeneveld et al 1980)
    - Kink ground state delocalised ($k = 0$)



Non-Perturbative Mass

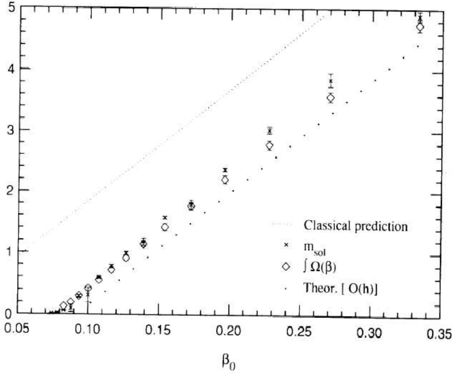
- Difference in ground state energies between the two sectors (Ciria&Tarancon 1994)

$$M = \frac{1}{T} \ln \frac{Z_{\rm per}}{Z_{\rm anti}}$$

- Partition function $Z = \int \mathcal{D}\phi \exp(-S[\phi])$ not calculable with lattice Monte Carlo
- However, derivatives are

$$\frac{\partial M}{\partial m^2} = \frac{L}{2}\left[\langle \phi^2 \rangle_{\rm anti} - \langle \phi^2 \rangle_{\rm per}\right]$$

Non-Perturbative Mass



(Ciria&Tarancon 1994)

Correlators in Kink Background

- Correlation function of operators $\mathcal{O}_i$:
  Local in time, but not necessarily in space

$$C_{ij}(t_2 - t_1) \equiv \langle \mathcal{O}_i(t_1)\mathcal{O}_j(t_2)\rangle$$

- Spectral expansion

$$C_{ij}(t) = \sum_\alpha \frac{\langle 0|\mathcal{O}_i|\alpha\rangle\langle\alpha|\mathcal{O}_j|0\rangle}{\langle 0|0\rangle} e^{-E_\alpha |t|}$$

where $|0\rangle$ is kink ground state

Excitations

- Choose operators labelled by separation Δx

$$\hat{\mathcal{O}}_{\Delta x} = \sum_x \hat{\phi}(x)\hat{\phi}(x+\Delta x)$$

- Zero overall momentum:
  Intermediate states $|\alpha\rangle$ = Kink excitations

Lüscher-Wolff Method

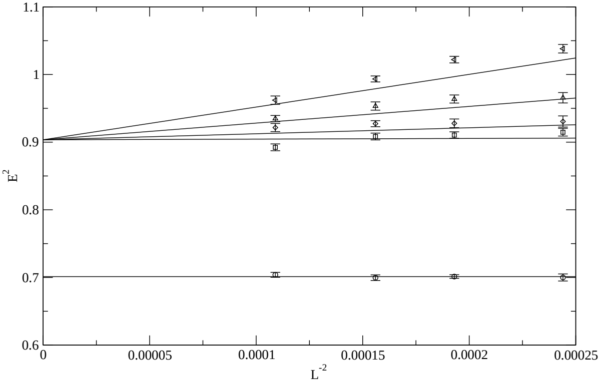
- $N_{\rm op}$ operators $\mathcal{O}_i$, $1 \le i \le N_{\rm op}$
- Generalised eigenvalue problem (Lüscher&Wolff 1990)

$$C_{ij}(t)\rho_j^n = \lambda_n(t,t_0)C_{ij}(t_0)\rho_j^n$$

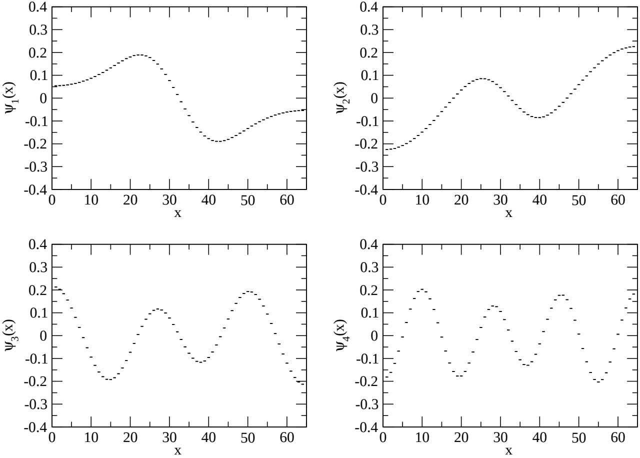
- Eigenvalues $\lambda_n = e^{-E_n t}$ for $n < N_{\rm op}$
- Eigenvectors $\rho_i^n \sim$ Energy eigenstates
- At weak coupling

$$\psi_n(x) = \sum_{\Delta x} \rho_{\Delta x}^n \left[\phi_{\rm kink}(x+\Delta x) + \phi_{\rm kink}(x-\Delta x)\right]$$

Excitation Spectrum



Excitation Wavefunctions



Non-Zero Momentum

- Choose operators $\hat{\mathcal{O}}_k = \hat{\phi}(k)$ with momentum k
- Interpolating states $|\alpha\rangle$ must have momentum k
- Lowest such state: Moving kink $|k\rangle$
  - Energy $E_k = \sqrt{k^2 + M^2}$
  - Gap before bound states or two-particle states

$$\langle \phi(0,k)\phi(t,q)\rangle \approx \int \frac{dk'}{2\pi E_{k'}} \frac{\langle 0|\hat{\phi}(k)|k'\rangle\langle k'|\hat{\phi}(q)|0\rangle}{\langle 0|0\rangle} e^{-t(E_k - M)}$$

- Form factors: $\langle k'|\hat{\phi}(q)|k\rangle = v f(k,k') 2\pi\delta(k - q - k')$

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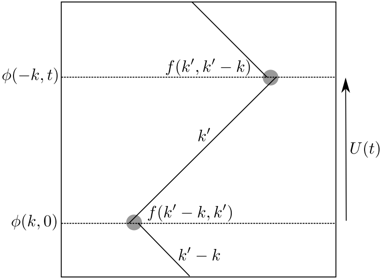
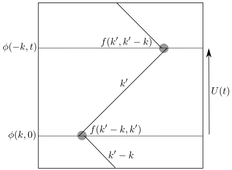
$$\langle \phi(0, k)\phi(t, q)\rangle \approx \frac{2\pi\delta(k+q)}{L} \frac{v^2 |f(k, 0)|^2}{E_k E_0} e^{-t(E_k - M)}$$

Finite Time Direction



$$\langle \phi(0,k)\phi(t,q)\rangle = \frac{\mathrm{Tr}\,\hat{U}(T-t)\hat{\phi}(q)\hat{U}(t)\hat{\phi}(k)}{\mathrm{Tr}\,\hat{U}(T)}$$

Finite Time Direction



- Including only moving kink states $|k\rangle$

$$\langle\phi(0,k)\phi(t,q)\rangle \approx \frac{2\pi\delta(q+k)}{L}\times \frac{\int\frac{dk'}{2\pi}\frac{v^2|f(k'-k,k')|^2}{E_{k'-k}E_{k'}}e^{-E_{k'}(T-t)-E_{k'-k}t}}{\int\frac{dk'}{2\pi}e^{-E_{k'}T}}$$

Finite Time Direction

- Using saddle point approximation

$$\langle\phi(0,k)\phi(t,q)\rangle \approx \frac{2\pi\delta(q+k)}{L}\times$$
$$\sqrt{\frac{T}{M}}\frac{f(k_0-k,k_0)|^2}{\sqrt{E_{k_0-k}E_{k_0}S''(k_0)}}e^{-S(k_0)}$$

where k_0 is the minimum of the classical action
$S(k') = E_{k'}(T-t) + E_{k'-k}t - MT$

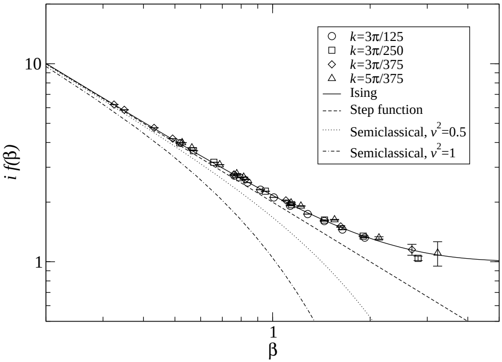
Form Factor from Correlator

- Inverting, we find

$$\begin{aligned} f(\beta) &= f(k_0, k_0 - k) \\ &= \pm i \frac{\sqrt{\langle \phi(0,k)\phi(t,-k)\rangle}}{v} \left(\frac{M E_{k_0-k} E_{k_0} S''(k_0)}{T} \right)^{1/4} e^{S(k_0)} \end{aligned}$$

where $\beta = \operatorname{arcsinh}\frac{k_0}{2M} - \operatorname{arcsinh}\frac{k_0-k}{2M}$

Form Factor at Strong Coupling



Conclusions

- Topological solitons: Twisted boundary conditions
- Quantum observables from correlator
    - Mass
    - Excitation spectrum
    - Form factor
- Kinks: Agreement with analytical results
    - Weak coupling: Excitations
    - Strong coupling: Form factor $f(\beta) = f_{\rm Ising}(\beta)$

QCD Strings?

- Form factor and spectrum of QCD string?
  - Extended object: No fundamental difference
  - Gauge theory: Plaquette correlator?
  - Non-topological: Real problem!
- Possible solutions:
  - Fix endpoints: Polyakov line correlator
    Lose translation invariance
  - Dual map to magnetic flux tube
    Only in toy models
  - Fourier tfm of magnetic flux sectors (de Forcrand&von Smekal 2001)
    Promising, technical issues