Isoscalar pairing and pairs in nuclei

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Definition of terms
Properties of isoscalar-pair condensates
$T=1$ pair vibrations
$T=0$ pair vibrations?
Aligned neutron-proton pairs?
Pairing and other interactions

Pairing refers to the interaction between nucleons in ‘time-reversed orbits’:

- **isovector (or singlet) pairing**: $J=0$ & $T=1$
- **isoscalar (or triplet) pairing**: $J=1$ & $T=0$

Aligned np interaction (**not** pairing): $J=2j$ & $T=0$
Nucleon-nucleon interaction
Nucleon-nucleon interaction

\[ (1g_{9/2})^2 \]

\[ u_{JT} \text{ (MeV)} \]

angular momentum \( J \)
Nucleon-nucleon interaction

\[ (1g_{9/2})^2 \]

\[ u_T \text{ (MeV)} \]

T=1

T=0

angular momentum J
Time-reversed orbits

A $jj$-coupled two-nucleon state with angular momentum $J=0$:

$$|j^2; J = 0\rangle \propto \sum_{m_j} \langle jm_j, j - m_j | 00 \rangle |jm_j\rangle |j - m_j\rangle \propto \sum_{m_j} |jm_j\rangle |jm_j\rangle$$

What about $J=1$? Use $LS$-coupled states:

$$\left| (l_{s})^2; L = 0, SM_S \right\rangle \propto \sum_{m_{\ell} m_{s} m'_{s}} \langle sm_{s}, sm'_{s} | SM_S \rangle \langle \ell m_{\ell}; sm_{s} | \ell m'_{\ell}; sm'_{s} \rangle$$

The Pauli principle allows ($L=0$, $S=0$, $T=1$) and ($L=0$, $S=1$, $T=0$) \( \Rightarrow \) two pairing interactions.
Pairing with neutrons and protons

For neutrons and protons two pairs and hence two pairing interactions are possible:

$^1S_0$ isovector or spin singlet ($S=0,T=1$): \[ \hat{S}_+ = \sum_{m>0} a_{m\uparrow}^+ a_{m\downarrow}^+ \]

$^3S_1$ isoscalar or spin triplet ($S=1,T=0$): \[ \hat{P}_+ = \sum_{m>0} a_{m\uparrow}^+ a_{m\uparrow}^+ \]
A matter of strength

Relation between $jj$-coupled and $LS$-coupled matrix elements:

$$V(j^2;JT) = \sum_{LS} \left[ \begin{array}{ccc} \ell & s & j \\ \ell & s & j \\ L & S & J \end{array} \right]^2 V[(\ell s)^2;LST]$$

For pairing matrix elements:

$$V(j^2;J = 0, T = 1) \approx \frac{1}{2} V[(\ell s)^2; L = 0, S = 0, T = 1]$$

$$V(j^2;J = 1, T = 0) \approx \frac{1}{6} V[(\ell s)^2; L = 0, S = 1, T = 0]$$
Neutron–proton pairing hamiltonian

Pairing hamiltonian in $LS$ coupling:
$$\hat{V}_{LS}(g,x) = -xg\hat{V}_{L=0,S=0,T=1} - (1-x)g\hat{V}_{L=0,S=1,T=0}$$

Properties (for a single-$l$ shell):

- $SO(8)$ algebraic structure;
- solvable for $x=0$, $x=1/2$ & $x=1$;
- reasonable ansatz in nuclei: $x=1/2$.

Neutron-proton pairing hamiltonian

Pairing hamiltonian in $jj$ coupling:

$$\hat{V}_{jj}(g,x) = -xg\hat{V}_{J=0,T=1} - (1-x)g\hat{V}_{J=1,T=0}$$

Properties (for a single-$j$ shell):

- solvable for $x=1$ based on $Sp(2j+1)$;
- $x=1$ solution based on quasi-spin algebra $SO(5)$;
- solution for $x=0$?

G. Racah, *L. Farkas Memorial Volume* (1952) 294
Augusto's question

Question from Augusto Macchiavelli at the Gordon conference in 2007: “An isoscalar pairing hamiltonian with spin-orbit splitting gives rise to an even-even $J \neq 0$ ground state. Why?”
1. \( H = g_1 \hat{S}_1 \cdot \hat{S}_2 + g_2 \hat{S}_2 \cdot \hat{S}_3 \), where \( g_1 \) and \( g_2 \) are g-factor pair. \( S_1, S_2, S_3 \) are angular momemtum.

Consider \( E_{g+sp} (E_{g+sp} > E_{g+df}) \). We know that \( E_g \) is a good approximation for \( g \) if this hamiltonian is small.

\[ |0> \approx \left( \sin \theta \left( |\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow| \right) + \cos \theta \left( |\uparrow\rangle \langle \downarrow| - |\downarrow\rangle \langle \uparrow| \right) \right) |0> \]

where coupling is in \( s^2 \), other states are.

\[ (|p> \otimes |0>) \quad (5, 0, 0) \quad \text{and} \quad (5, 1, 0) \quad (5, 0, 1) \]

In the limit of large \( J \) we may assume.

\[ \langle \uparrow | H | \uparrow \rangle = 2 J \sin \theta + 3 J \sin^3 \theta \]
\[ \langle \downarrow | H | \downarrow \rangle = 2 J \cos \theta \]
\[ \langle \uparrow | H | \downarrow \rangle = 2 J \sin \theta \]

Hence,

\[ \langle \uparrow | H | \uparrow \rangle + 2 \langle \uparrow | H | \downarrow \rangle + 2 \langle \downarrow | H | \downarrow \rangle = 2 J \]

We have the following behavior as a function of \( \theta \):

\[ \langle \uparrow | H | \uparrow \rangle \quad \langle \uparrow | H | \downarrow \rangle \quad \langle \downarrow | H | \downarrow \rangle \]

\[ \theta = 0 \quad \theta = \frac{\pi}{2} \quad \theta = \pi \]

2. With spin-orbit coupling.

Take for example the resonant limit \( g = \sqrt{2} \). The \( 2s^2 \) and \( 2p^4 1d^{10} \) are close at \( E \approx 2 g \). The spin-orbit splitting causes 0 to remain smaller than \( 2g \) and the \( 2p^4 1d^{10} \) is at an energy \( 2s^2 + 2 \), which is higher than \( 2g \), the energy of \( 2s^2 \). Hence, the \( 2s^2 \) becomes the ground state.
1. \( H = q, P_1^2 + P_2^2 + S_1^2 \), where \( P_1, P_2 \) = coordinate pair \((T_x, T_y)\) \\
\( S_1, S_2 \) = angular pair \((\theta_1, \phi_1)\).

Consider \( E_1 = E_2 \), \( \theta_2 > \theta_1 \). We know that \( E_1 \) is a good approximation. The \( \gamma \) of this Hamiltonian is \\
\[ |0> \propto \sin \theta (P_1^2) + \cos \theta (S_2^2) \]

We have coupling \( n \) in \((\gamma)\). Other states are \((0n)\), \((P_0) 1\rangle \propto (0, 0, \pm \gamma\rangle \), and \((S_1) 1\rangle \propto (0, 0, \pm \gamma\rangle \).

In the limit of large \( L \), we may assume:
\[ <P_0|e|P_0, P_0|P_0^2> = \gamma \], \[ <S_1|e|S_1, S_1|P_0^2> = 0 \], \[ <S_1|e|S_1, S_1|S_1^2> = 0 \], \[ <P_1^2|e|P_1^2, P_1^2> = \gamma \], \[ <P_1^2|e|P_1^2, P_1^2> = 0 \], \[ <P_1^2|e|P_1^2, P_1^2> = 2\gamma \].

Hence:
\[ <0|1|0> = 2\gamma \], \[ <3|2> = 2\gamma \].

We have the following behavior as a function of \( \gamma \):

\[ <0|1|0> \quad \gamma \]
\[ <3|2> \quad 0 \]
\[ <5|3> \quad 0 \]


Take, for example, the second order limit \( (\gamma > 0) \). The \( S^1 \) - \( I_x \) and \( S^1 \) - \( I_y \) are close to \( \pm \gamma \). The spin-orbit splitting makes \( E_x > \gamma \) be smaller than \( 2\gamma \) and \( E_y < \gamma \) - \( 2\gamma \) which is smaller than \( 2\gamma \), the energy of \( I_0^2 \). Hence the \( S^1 \) becomes the ground state.
Two $n+2p$ in $j=7/2$ shell

FIG. 1. Energies (in units of the pairing strength $v$) from the expected $0$ state for two particles interacting.

The shaded area in the top panel indicates the critical value of the spin-orbit splitting. In Fig. 2, the $0$ state changes to the spin-orbit splitting. In Fig. 3, in agreement with the estimate shown in Fig. 4, the ground state changes.

We can trace back the change in the properties of the ground state in terms of the spin-orbit splitting, as we saw also shown in Fig. 5, in agreement with the estimate shown in Fig. 6.

The critical value is obtained when the energy above equals the critical value of the condensate discussed above, we develop in the next section.

To obtain an estimate of the critical value of the condensate discussed above, we develop in the next section.
3n+3p in $j=7/2$ shell

![Graph showing the energy levels for different states in the isoscalar and isovector sectors.](image)
An approximate solution

Construct a shell-model subspace in terms of $P$ pairs with $J=1$ and $T=0$.  
Map $V_{jj}(g,x=0)$ onto $p$-boson hamiltonian

$$\hat{H}_b = \varepsilon_p \hat{n}_p + \frac{1}{2} \sum_{\lambda=0,2} v^b_{\lambda} (p^+ \times p^+)^{(\lambda)} \cdot (\vec{p} \times \vec{p})^{(\lambda)}$$

The boson-boson interaction strengths are

$$v^b_0 = \frac{6 \left( j^2 + j - 1 \right)}{j(j+1)(2j+1)} g \rightarrow \left[ \frac{3}{j} + O \left( \frac{1}{j^2} \right) \right] g$$

$$v^b_2 = \frac{3 \left( 4j^4 + 6j^3 + j^2 + 7j + 12 \right)}{j(j+1)(2j+1)(5j^2 + 7j + 3)} g \rightarrow \left[ \frac{6}{5j} + O \left( \frac{1}{j^2} \right) \right] g$$

An approximate solution

For an attractive isoscalar pairing interaction the boson-boson matrix elements are repulsive. The $\lambda = 2$ is less repulsive than $\lambda = 0$; therefore the aligned state is below the paired state.

\[
E(j^{2n}, J = 0) - E(j^{2n}, J = n) \approx \frac{3n(n+1)}{10j} g, \quad n \text{ even}
\]

\[
E(j^{2n}, J = 1) - E(j^{2n}, J = n) \approx \frac{3(n-1)(n+2)}{10j} g, \quad n \text{ odd}
\]
Summary 1

An approximate solution of the isoscalar pairing interaction in a single-\(j\) shell.
The aligned state is favoured over the paired state due to Pauli (finite-shell) effects.

Neutron-proton correlations

The question is not whether $T=0$ interactions between nucleons exist or whether they are important. They do and they are.

The question is whether

• $T=0$ pairing correlations exist?
• aligned $T=0$ pairs are dominantly important?
• quartet correlations exist?

⇒ “Study of np pairing through two-nucleon transfer reactions” by M. Assié et al.

Here: an analysis in terms of pair vibrations.
$T=1$ pair vibrations in Pb

$E = \mathcal{E} - \mathcal{E}^{208\text{Pb, gr. st.}} \lambda(A, 208) \text{ (MeV)}$

$\lambda = \frac{1}{2} (\mathcal{E}(2g_{9/2}) + \mathcal{E}(3p_{1/2})) = 5.66 \text{ MeV}$

- observed
- harmonic approximation
- with interactions between pairs of quanta

A. Bohr & B.R. Mottelson, Volume 2, page 646
$T=1$ pair vibrations in Ca

$^{40}_{20}$Ca$_{20}$

$0^+$
$T=1$ pair vibrations in Ca

$^{40}_{20}\text{Ca}_{20}^{42}_{20}\text{Ca}_{22}$

$|0^+\rangle = |S\rangle$

$0^+$
$T=1$ pair vibrations in Ca

$\frac{40}{20}\text{Ca}_{20}$  $\frac{42}{20}\text{Ca}_{22}$  $\frac{44}{20}\text{Ca}_{24}$

$|0^+\rangle \approx |S^2\rangle$

$|0^+\rangle = |S\rangle$
$T=1$ pair vibrations in Ca

\[ \frac{40}{20} \text{Ca}_{20} \quad \frac{42}{20} \text{Ca}_{22} \quad \frac{44}{20} \text{Ca}_{24} \]

$|0^+\rangle = |S\rangle$

$|0^+\rangle = |S^2\rangle$

$\sigma$

$2\sigma$

$0^+$
$T=1$ pair vibrations in Ca–Sc–Ti–V?

$^{40}_{20}\text{Ca}_{20}$

$0^+$
$T=1$ pair vibrations in Ca–Sc–Ti–V?

\[ \frac{40}{20} \text{Ca}_{20} \quad \frac{42}{21} \text{Sc}_{21} \]

\[ |0^+\rangle = |S\rangle \]
$T=1$ pair vibrations in Ca–Sc–Ti–V?
$T=1$ pair vibrations in Ca–Sc–Ti–V?

$^{40}_{20}\text{Ca}_{20}$  $^{42}_{21}\text{Sc}_{21}$  $^{44}_{22}\text{Ti}_{22}$

$\sigma |0^+_1\rangle \approx |S^2,0\rangle$

$2\sigma |0^+_2\rangle = |S\rangle$

$\sigma |0^+_3\rangle = |S\rangle$
$T=1$ pair vibrations in Ca–Sc–Ti–V?

- $^{40}_{20}\text{Ca}_{20}$
- $^{42}_{21}\text{Sc}_{21}$
- $^{44}_{22}\text{Ti}_{22}$
- $^{46}_{23}\text{V}_{23}$

- $0^+$
- $0^+ \approx |S^3;0\rangle$
- $0^+ \approx |S^2;0\rangle$
- $0^+ \approx |S;0\rangle$
$T=1$ pair vibrations in Ca–Sc–Ti–V?

<table>
<thead>
<tr>
<th>40[20]Ca[20]</th>
<th>42[21]Sc[21]</th>
<th>44[22]Ti[22]</th>
<th>46[23]V[23]</th>
</tr>
</thead>
</table>

- $|0^+\rangle = |S\rangle$
- $\sigma$
- $|0^+\rangle \approx |S^3;0\rangle$
- $|0^+\rangle = 0.98|S^2;0\rangle$
- $\sim 1.9\sigma$
- $5\sigma/3$
- $\sigma$
- $T=1$ pair vibrations in Ca–Sc–Ti–V?
$T=1$ pair vibrations in Ca–Sc–Ti–V?
$T=0$ pair vibrations in Ca-Sc-Ti-V?

$^{40}_{20}\text{Ca}_{20}$

$0^+$
$T=0$ pair vibrations in Ca-Sc-Ti-V?
$T=0$ pair vibrations in Ca-Sc-Ti-V?
$T=0$ pair vibrations in Ca-Sc-Ti-V?
$T=0$ pair vibrations in Ca-Sc-Ti-V?

\[
\begin{align*}
\frac{40}{20}\text{Ca}_{20} & \quad \frac{42}{21}\text{Sc}_{21} & \quad \frac{44}{22}\text{Ti}_{22} & \quad \frac{46}{23}\text{V}_{23} \\
1^+ & \quad 2^+ & \quad 3^+ & \quad 0^+ \\
5\sigma/3 & \quad |1^+\rangle \approx |P^3;1\rangle \\
\sim 1.07\sigma & \quad |0^+\rangle = 0.73|P^2;0\rangle \\
\sigma & \quad |1^+\rangle = |P\rangle \\
0^+ & \quad 0^+ 
\end{align*}
\]
$T=0$ pair vibrations in Ca–Sc–Ti–V?

\[ \begin{align*}
\begin{array}{c}
\frac{40}{20} \text{Ca}_{20} \\
\frac{42}{21} \text{Sc}_{21} \\
\frac{44}{22} \text{Ti}_{22} \\
\frac{46}{23} \text{V}_{23}
\end{array}
\end{align*} \]

\[ \begin{align*}
\sigma & \rightarrow 0^+ \\
\sim 0.65\sigma & \rightarrow 1^+ = 0.85|P^3 ; 1\rangle \\
\sim 1.07\sigma & \rightarrow 0^+ = 0.73|P^2 ; 0\rangle \\
1^+ & \rightarrow 0^+ \rightleftharpoons |P\rangle
\end{align*} \]
$T=0$ aligned pairs in Ca–Sc–Ti–V?

$^{40\text{Ca}}_{20}$
$T=0$ aligned pairs in Ca–Sc–Ti–V?
$T=0$ aligned pairs in Ca–Sc–Ti–V?

$^{40}_{20}\text{Ca}_{20}$, $^{42}_{21}\text{Sc}_{21}$, $^{44}_{22}\text{Ti}_{22}$

$\sigma |0^+\rangle \approx |B^2;0\rangle$

$\sigma |7^+\rangle = |B\rangle$
$T=0$ aligned pairs in Ca–Sc–Ti–V?
$T=0$ aligned pairs in Ca–Sc–Ti–V?
$T=0$ aligned pairs in Ca–Sc–Ti–V?
Summary 2

A systematic picture of $T=0$ and $T=1$ deuteron transfer can be obtained in terms of pair vibrations.

Is it valid in $1f_{7/2}$-shell nuclei? This also requires the careful consideration of the reaction mechanism.

Heavier $N=Z$ nuclei? E.g. $^{58}\text{Cu}$?
Supermultiplet primer

Two assumptions:

*The forces between nucleons are independent of spin and isospin.*

*The n-n interaction is short-range attractive.*

Consequences:

*Many-nucleon states can be classified according to their spatial or spin-isospin symmetry.*

*States with highest spatial symmetry are lowest in energy.*

→ Separate states according to their spatial symmetry: \( C_2[\text{SU}(4)]=\text{Majorana interaction} \).
Supermultiplet primer

Eigenvalue of $C_2[SU(4)]$:

$$3\lambda(\lambda + 4) + 4\mu(\mu + 4) + 3\nu(\nu + 4) + 4\mu(\lambda + \nu) + 2\lambda\nu$$

Favoured $SU(4)$ labels $(\lambda,\mu,\nu)$ for any nucleus:

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$[\tilde{h}']^a$</th>
<th>$(\lambda',\mu',\nu')$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even–even</td>
<td>$[k +</td>
<td>T_z</td>
</tr>
<tr>
<td>Odd-mass</td>
<td>$\left[ k +</td>
<td>T_z</td>
</tr>
<tr>
<td>Odd–odd ($N \neq Z$)</td>
<td>$[k' +</td>
<td>T_z</td>
</tr>
<tr>
<td>Odd–odd ($N = Z$)</td>
<td>$[k' + 1, k' + 1, k', k']$</td>
<td>$(0, 1, 0)$</td>
</tr>
</tbody>
</table>
Supermultiplet primer

Introduce favoured SU(4) labels in the eigenvalue expression:

\[(N - Z)^2 + 8|N - Z| + 8\delta_{N,Z}\pi_{np} + 6\delta_{\text{pairing}}(N,Z)\]

with \(\delta_{\text{pairing}}(N,Z)=0, 1, 2\) for even-even, odd-mass, odd-odd, and \(\pi_{np}=1\) for odd-odd.

Compare with Wigner binding energy:

\[B_W (N, Z) = -W(A)|N - Z| - d(A)\delta_{N,Z}\pi_{np}\]