Dual Simulations of Lattice Gauge Theories

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Motivation: Complex action problem

- In general lattice field theories with finite chemical potential $\mu$ have actions $S$ with an imaginary part.

- The Boltzmann factor

$$e^{-S} \in \mathbb{C}$$

thus has a complex phase and cannot be used as a probability weight.

- Standard Monte Carlo simulation techniques are not available for a non-perturbative analysis.

"Complex action problem" or "Sign problem"

- Generic feature of finite density field theories both, on the lattice and in the continuum, for bosonic and fermionic theories.
The toolbox of the lattice QCD community

- Reweighting / phase quenching
- Analytic continuation from imaginary $\mu$
- Expansions around the $\mu = 0$ ensemble: Taylor series.
- Expansions around the $\mu = 0$ ensemble: Fugacity series.
- Simulations with stochastic methods (complex Langevin etc)
- Canonical simulations
- Density of state / histogram methods
- Exploring symmetries - subset method
- Rewriting a system to new degrees of freedom - dual variables

Our dream: Map out the QCD phase diagram in the $\mu$-$T$ plane.
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Getting the idea across ..... 

Dual variables for charged $\phi^4$ fields with chemical potential
Formal definition in the continuum:

• **Action:**

\[
S = \int d^4 x \left[ -\phi^* \Delta \phi + (m^2 - \mu^2) |\phi|^2 + \lambda |\phi|^4 + i \mu 2 \Im \phi^* \partial_4 \phi \right]
\]

• **VEVs of observables:**

\[
\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] e^{-S[\phi]} O[\phi] , \quad Z = \int \mathcal{D}[\phi] e^{-S[\phi]} 
\]

• **The path integral is well defined on the lattice:** Discretize the action on a 4-d lattice and for \( \mathcal{D}[\phi] \) use the product of the measures at all sites.
Lattice path integral for the charged scalar field

- Lattice action: \( (\phi_x \in \mathbb{C}, \ M^2 = 8 + m^2) \)

\[
S = \sum_x \left[ M^2 |\phi_x|^2 + \lambda |\phi_x|^4 \right] - \sum_{x,\nu} \left[ e^{-\mu \delta_{\nu 4}} \phi^*_x \phi_{x+\nu} + e^{\mu \delta_{\nu 4}} \phi_x \phi^*_x \right]
\]

- Discretized partition sum \( Z = \int D[\phi] e^{-S[\phi]} \):

\[
Z = \prod_x \int_{\mathbb{C}} d\phi_x \prod_x e^{-M^2 |\phi_x|^2 - \lambda |\phi_x|^4} \prod_{x,\nu} e^{-\mu \delta_{\nu 4}} \phi^*_x \phi_{x+\nu} \prod_{x,\nu} e^{\mu \delta_{\nu 4}} \phi_x \phi^*_x
\]
Dual representation for the charged scalar field

- Expand the nearest neighbor terms of $e^{-S}$:

$$\prod_{x,\nu} \exp \left( e^{-\mu \delta_{\nu 4}} \phi_x^* \phi_{x+\hat{\nu}} \right) = \prod_{x,\nu} \sum_{j_{x,\nu}=0}^{\infty} \frac{(e^{-\mu \delta_{\nu 4}})^{j_{x,\nu}}}{j_{x,\nu}!} (\phi_x^*)^{j_{x,\nu}} (\phi_{x+\hat{\nu}})^{j_{x,\nu}}$$

$$\prod_{x,\nu} \exp \left( e^{\mu \delta_{\nu 4}} \phi_x \phi_x^* \right) = \prod_{x,\nu} \sum_{\bar{j}_{x,\nu}=0}^{\infty} \frac{(e^{\mu \delta_{\nu 4}})^{\bar{j}_{x,\nu}}}{\bar{j}_{x,\nu}!} (\phi_x)^{\bar{j}_{x,\nu}} (\phi_x^*_{x+\hat{\nu}})^{\bar{j}_{x,\nu}}$$

- The $j_{x,\nu}$ and $\bar{j}_{x,\nu}$ are the new, "dual" degrees of freedom.
Dual representation - integrating out the fields

- Integral over $\phi_x$ at site $x$: ($\Sigma_j, \overline{\Sigma}_j$ are sums of $j_{y,\nu}, \overline{j}_{y,\nu}$ connected to $x$)

$$\int_{\mathbb{C}} d\phi_x \ e^{-M^2|\phi_x|^2 - \lambda|\phi_x|^4} (\phi_x) \Sigma_j \ (\phi^*_x) \overline{\Sigma}_j$$

- Polar coordinates $\phi_x = re^{i\theta}$ to separate radial and U(1) parts (symmetry):

$$\int_0^{\infty} dr \ r (\Sigma_j + \overline{\Sigma}_j + 1) e^{-M^2r^2 - \lambda r^4} \int_{-\pi}^{\pi} d\theta \ e^{i\theta (\Sigma_j - \overline{\Sigma}_j)} = \mathcal{I}(\Sigma_j + \overline{\Sigma}_j) \ \delta(\Sigma_j - \overline{\Sigma}_j)$$

- At every site there is a weight factor $\mathcal{I}(\Sigma_j + \overline{\Sigma}_j)$ and a constraint.

- The constraint $\delta(\Sigma_j - \overline{\Sigma}_j)$ enforces vanishing flux of $\overline{j}_{x,\nu} - j_{x,\nu}$ at each $x$. 
Dual representation – final form

- The original partition function is mapped exactly to a sum over configurations of the dual variables $k_{x,\nu} \in \mathbb{Z}$ and $l_{x,\nu} \in \mathbb{N}_0$. $k_{x,\nu}$ and $l_{x,\nu}$ are linear combinations of the original $j$ and $\bar{j}$:

$$Z = \sum_{\{k,l\}} \mathcal{W}(k, l) \mathcal{C}(k)$$

- Real and positive weight factor from radial d.o.f. and combinatorics:

$$\mathcal{W}(k, l) = \prod_{x,\nu} \frac{e^{-\mu k_{x,\nu} \delta_{\nu,4}}}{(|k_{x,\nu}| + l_{x,\nu})!} \prod_{x} \mathcal{I} \left( \sum_{\nu} \left[ |k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(l_{x,\nu} + l_{x-\hat{\nu},\nu}) \right] \right)$$

- Constraint from integrating over the symmetry group:

$$\mathcal{C}(k) = \prod_{x} \delta \left( \sum_{\nu} [k_{x,\nu} - k_{x-\hat{\nu},\nu}] \right)$$
Admissible configurations are loops:

- Constraint from integrating over the symmetry group:

  \[ \forall \ x : \sum_{\nu} [k_{x,\nu} - k_{x-\nu,\nu}] = 0 \quad (\nabla \vec{k} = 0) \]

- Admissible configurations of dual variables are oriented loops of flux:

- Finite \( \mu \): Different weight for forward and backward temporal flux.

- We identify: Particle number = winding number of flux.

C. Gattringer, T. Kloiber, PLB 2013 & NPB 2013
Adding gauge fields $\Rightarrow$ U(1) gauge Higgs model
U(1) gauge Higgs model: Continuum

Continuum action:

\[
S = \int d^4x \left\{ - \phi(x)^* \left[ \partial_\nu + iA_\nu(x) \right] \left[ \partial_\nu + iA_\nu(x) \right] \phi(x) \right. \\
+ \left[ m_\phi^2 - \mu_\phi^2 \right] |\phi(x)|^2 + \lambda_\phi |\phi(x)|^4 \right\} + i\mu_\phi N_\phi \\
+ \int d^4x \left\{ - \chi(x)^* \left[ \partial_\nu - iA_\nu(x) \right] \left[ \partial_\nu - iA_\nu(x) \right] \chi(x) \right. \\
+ \left[ m_\chi^2 - \mu_\chi^2 \right] |\chi(x)|^2 + \lambda_\chi |\chi(x)|^4 \right\} + i\mu_\chi N_\chi \\
+ \frac{1}{4e^2} \int d^4x \, F_\rho\sigma(x) F_\rho\sigma(x)
\]
U(1) gauge Higgs model: Lattice

- **Compact U(1)-valued gauge fields:** \( U_{x,\nu} = e^{iA_{\nu}(x)} \)

- **Lattice action for matter fields:**
  \[
  S_M = \sum_x \left[ M^2 |\phi_x|^2 + \lambda |\phi_x|^4 \right] - \sum_{x,\nu} \left[ e^{-\mu \delta_{\nu 4}} U_{x,\nu} \phi^*_x \phi_{x+\hat{\nu}} + e^{\mu \delta_{\nu 4}} U^*_{x,\nu} \phi_x \phi^*_x \right]
  \]

- **Lattice action for gauge fields:** (Wilson plaquette action)
  \[
  S_G = -\frac{\beta}{2} \sum_x \sum_{\rho<\sigma} \left[ U_{x,\rho} U_{x+\hat{\rho},\sigma} U^*_{x+\hat{\sigma},\rho} U^*_{x,\sigma} + U^*_{x,\rho} U^*_{x+\hat{\rho},\sigma} U_{x+\hat{\sigma},\rho} U_{x,\sigma} \right]
  \]

- **Gauge field measure:**
  \[
  \int D[U] = \prod_{x,\nu} \int_{U(1)} dU_{x,\nu}
  \]
Mapping to dual variables

- Again we expand nearest neighbor terms, now with link variables $U_{x,\nu}$:

$$\exp \left( \phi_x^* U_{x,\nu} \phi_{x+\hat{\nu}} \right) = \sum_{j_{x,\nu}=0}^{\infty} \frac{(U_{x,\nu})^j_{x,\nu}}{j_{x,\nu}!} \left( \phi_x \right)^j_{x,\nu} \left( \phi_x^* \right)^j_{x,\nu}$$

$\Rightarrow$ Matter loops are dressed with gauge transporters $U_{x,\nu}$

- Inserting the dual representations for the matter fields:

$$Z = \int D[U] e^{-S_G[U]} Z_\phi[U] Z_\chi[U] = \sum_{\{k,l,\bar{k},\bar{l}\}} \mathcal{W}_\phi(k,l) \mathcal{W}_\chi(\bar{k},\bar{l}) C(k) C(\bar{k})$$

$$\times \int D[U] \prod_{x,\rho<\sigma} e^{\frac{g}{2} [U_{x,\rho} U_{x,\rho}^* + U_{x+\hat{\rho},\sigma} + c.c.]} \prod_{x,\nu} (U_{x,\nu})^{k_{x,\nu}-\bar{k}_{x,\nu}}$$
Integrating out the gauge fields \(\Rightarrow\) new constraints

- The remaining gauge field integral ...

\[
\prod_{x,\nu} \int_{U(1)} dU_{x,\nu} \prod e^{\frac{\beta}{2} U_{x,\rho} U_{x+\hat{\rho},\sigma} U_{x,\sigma}^*} \prod e^{\frac{\beta}{2} U_{x,\rho}^* U_{x+\hat{\rho},\sigma}^* U_{x,\sigma}} \prod (U_{x,\nu})^{k_{x,\nu} - \bar{k}_{x,\nu}}
\]

- ... is again tackled by expanding the Boltzmann factor ....

\[
e^{\frac{\beta}{2} U_{x,\rho} U_{x+\hat{\rho},\sigma} U_{x+\hat{\rho},\sigma}^*} = \sum_{p_{x,\rho\sigma}} \frac{(\beta/2)^{p_{x,\rho\sigma}}}{p_{x,\rho\sigma}!} \left[ U_{x,\rho} U_{x+\hat{\rho},\sigma} U_{x+\hat{\rho},\sigma}^* U_{x,\sigma}^* \right]^{p_{x,\rho\sigma}}
\]

... leading to new integer valued dual variables \(p_{x,\rho\sigma}, \bar{p}_{x,\rho\sigma}\) living on plaquettes.

- The gauge field integrals give rise to constraints on each link \((x, \nu)\) of the lattice.

- The link constraints connect the matter flux \(k_{x,\nu} - \bar{k}_{x,\nu}\) and the plaquette occupation numbers \(p_{x,\rho\sigma}, \bar{p}_{x,\rho\sigma}\) touching that link.

- Admissible configurations are closed surfaces or surfaces bounded by matter flux.
Dual form of the partition function for the gauge Higgs model

The partition sum is mapped exactly to a sum over loops and surfaces:

\[ Z = \sum_{\{p,k,l\}} W(p, k, l) \cdot C(p, k) \]

- \( W \) positive weight factors.
- \( C \) constraints that turn the sum over configurations of dual variables into summing over surfaces and loops in 4 dimensions.


M. Endres, PRD 75, 2007
C. Gattringer, A. Schmidt, PRD 86, 2012
T. Korzec and U. Wolff, NPB 871, 2013
P.N. Meisinger, M. Ogilvie, arXiv:1306.1495
Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013
An admissible configuration for dual U(1) gauge Higgs theory:

Chemical potential favors flux forward in time. (2 flavors)
Generalized worm algorithm for gauge Higgs systems:

Worm starts by inserting a unit of matter flux. Adding segments transports the defect across the lattice until the defect is healed in a final step.

1 2 3

4 5

Some results ...
Bulk observables

- Bulk observables are obtained as derivatives of the free energy with respect to the parameters.

- Example: Observables related to the particle number:

\[ n = \frac{1}{N_s^3 N_t} \frac{\partial \ln Z}{\partial \mu} \]

- Dual form: Particle number = temporal winding number of \( k \)-flux.

\[ n = \frac{1}{N_s^3 N_t} \left\langle \sum_x k_{x,4} \right\rangle = \frac{1}{N_s^3} \left\langle W[k] \right\rangle \]

- Dual bulk observables are related to moments of the dual variables.

- 2-point functions via strings made gauge invariant with surfaces.
Bulk observables for several $\mu > 0$
2-flavor gauge Higgs phase diagram at zero density

\[ M^2 \]

**Higgs phase**

- \( \langle U \rangle \sim 1 \)
- \( \langle |\phi|^2 \rangle \) growing with \(-m^2\)
- \( \chi_n \) growing with \(-m^2\)

**Confining phase**

- \( \langle U \rangle \) small
- \( \langle |\phi|^2 \rangle \) small
- \( \chi_n \sim 0 \)

**Coulomb phase**

- \( \langle U \rangle \sim 0.6-0.8 \)
- \( \langle |\phi|^2 \rangle \) small, \( \chi_n \sim 0 \)

Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013
In the confining phase the dependence on the chemical potential $\mu$ sets in only when $\mu$ reaches the mass of the lowest excitation. "Silver Blaze behaviour"

Condensation of dual variables.
Condensation of dual variables

BELOW THE TRANSITION

ABOVE THE TRANSITION

\( \bar{J} \)  \( \bar{I} \)  \( j \)  \( l \)  \( p \)
In the Higgs phase there is no mass gap and the non-trivial $\mu$-dependence starts at $\mu = 0$.

No condensation of dual variables.

Y. Delgado Mercado, C. Gattringer, A. Schmidt, PRL 111, 2013
Asymmetric propagation for $\mu < \mu_c \approx 0.17$.
Condensation (= constant propagator) for $\mu$ above $\mu_c$. 
Challenges ....

... and some progress
Challenges for dual variables in lattice field theory

- Non-abelian gauge fields
- Relativistic fermions
- Complex action problem from a $\theta$-term
- Physical interpretation of dual variables

--------- progress in low-dimensional lattice field theories.

See also the talks by Falk Bruckmann and Tin Sulejmanpasic.
Massless Schwinger model on the lattice

- Partition sum and lattice action: \((\mu \neq 0 \Rightarrow 2 \text{ flavors})\)

\[
Z = \int \mathcal{D}[U] \mathcal{D}[\bar{\psi}, \psi] \ e^{-S_G[U] - i \theta Q[U] - S_F[U, \bar{\psi}, \psi]}
\]

\[
S_F = \frac{1}{2} \sum_{x, \nu} \gamma_\nu(x) \left[ e^{+\mu \delta_{\nu, 2}} U_\nu(x) \bar{\psi}(x) \psi(x + \hat{\nu}) - e^{-\mu \delta_{\nu, 2}} U_\nu(x)^* \bar{\psi}(x + \hat{\nu}) \psi(x) \right]
\]

staggered sign function: \(\gamma_1(x) = 1, \quad \gamma_2(x) = (-1)^x\)

\[
S_G[U] + i \theta Q[U] = -\frac{\beta}{2} \sum_n [U_p(x) + U_p(x)^*] + \frac{\theta}{4\pi} \sum_x [U_p(x) - U_p(x)^*]
\]

\[
= -\eta \sum_n U_p(x) - \bar{\eta} \sum_x U_p(x)^*
\]

\[
U_p(x) = U_1(x) U_2(x + \hat{1}) U_1(x + \hat{2})^* U_2(x)^*
\]

\[
\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}, \quad \bar{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}
\]
Expansion of the Boltzmann factor of the gauge fields:

- Factorization of the gauge action Boltzmann factor as product over plaquettes:

\[
e^{-S_G[U] - i\theta Q[U]} = \prod_x e^{\eta U_p(x)} e^{\overline{\eta} U_p(x)^{-1}}
\]

- Expansion of the individual exponentials:

\[
e^{-\eta U_p(x)} e^{-\overline{\eta} U_p(x)^{-1}} = \sum_{p(x) \in \mathbb{Z}} (-1)^{p(x)} I_{|p(x)|} \left( \sqrt{\eta \overline{\eta}} \right) \left( \sqrt{\frac{\eta}{\overline{\eta}}} \right)^{p(x)} U_p(x)^{p(x)}
\]

\[p(x) \in \mathbb{Z} \quad \text{.... plaquette occupation numbers}\]

\[I_{|p(x)|} \left( \sqrt{\eta \overline{\eta}} \right) \quad \text{.... modified Bessel functions}\]

\[\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}, \quad \overline{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}\]
Fermion loops

- Saturating the Grassmann integral gives loops.
- Signs come from Grassmann ordering and from $\gamma$-matrices.
- The loops are dressed with gauge links that have to be saturated by plaquettes from the gauge field Boltzmann factor.
Dual form of the partition sum:

- The partition function is a sum over all admissible configurations of loops $l$, dimers $d$ and plaquette occupation numbers $p$:

\[
Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} (-1)^{N_L + N_P + \frac{1}{2} \sum_l L(l)} \prod_x I_{|p(x)|} \left(\sqrt{\eta \bar{\eta}}\right) \left(\sqrt{\frac{\eta}{\bar{\eta}}}\right)^{p(x)}
\]

Here we have introduced:

- $N_P = \sum_x p(x)$
- $N_L$ number of loops
- $L(l)$ length of the loop $l$

- Constructive proof:

\[
(-1)^{N_L + N_P + \frac{1}{2} \sum_l L(l)} = 1 \quad \forall \text{ admissible configurations (wrong with mass or } d > 2)\]

Dual representation of the massless Schwinger model with $\theta$-term:

- Partition function is a sum over admissible configurations of loops, dimers and plaquette occupation numbers:

$$Z = \left(\frac{1}{2}\right)^V \sum_{\{l,d,p\}} \prod_x I_{|p(x)|} \left(\sqrt{\eta \bar{\eta}}\right) \left(\sqrt{\frac{\eta}{\bar{\eta}}}\right)^{p(x)}$$

- Fermion loops that wind forward (backward) in time are weighted with $e^{+\mu N_2}$ ($e^{-\mu N_2}$)

- The partition sum has only real and positive terms for positive $\eta = \frac{\beta}{2} - \frac{\theta}{4\pi}$ and $\bar{\eta} = \frac{\beta}{2} + \frac{\theta}{4\pi}$.

- $\theta \neq 0$ introduces an imbalance between positive and negative $p(x)$.

Emergence of $2\pi$ periodicity in $\theta$ in the continuum limit (scalar QED$_2$):

C. Gattringer, T. Kloiber, M. Müller-Preussker, arXiv:1508.00681
Summary

- Complex action problems for scalar field theories and abelian gauge-Higgs models are resolved by mapping them to positive dual representations.

- Dual degrees of freedom are loops for matter and surfaces for gauge fields.

- Bulk observables, phase diagram, 2-point functions ...

- Models with positive dual representation serve as test cases for other techniques.

- Main challenges: Relativistic fermions, non-abelian gauge fields, topology ...

- Progress in low-dimensional lattice field theories.

- Dual variables shed light on different aspects of the physics. We should try to understand these aspects better.