Dual variables at work:
the 1+1-dimensional O(3) model

Falk Bruckmann
(Regensburg University)

Trento, Oct. 2015

with Tin Sulejmanpasic [1408.2229]
and Thomas Kloiber, Christof Gattringer [1507.04253, in prep.]
Dual variables at work: the 1+1-dimensional O(3) model

Falk Bruckmann
(Regensburg University)

Workshop AiDMCMfQFTCiNPaCMP, Trento, Oct. 2015

with Tin Sulejmanpasic [1408.2229]
and Thomas Kloiber, Christof Gattringer [1507.04253, in prep.]
Physics motivation

the $O(3)$ model in 1+1 dimensions

$$S = \frac{1}{g^2} \int d^2 x \frac{1}{2} \left( \partial_\nu n_a \right)^2 , \quad n_a^2 = 1 , \quad a = 1, 2, 3 , \quad \nu = 1, 2$$

rich variety of effects, all similar to QuantumChromoDynamics:

- no dimensionful parameter, but dynamical mass generation
- asymptotic freedom: strongly coupled in the infrared
- topology & instantons
- lowest model of the $O(N)$ and $CP(N)$ series, large-$N$ possible
  $O(3)$ best compared to $SU(2)$ Yang-Mills
- fermions can be coupled
- lattice formulation = Heisenberg model: $J \sum_{x,\nu} n_a(x) n_a(x + \hat{\nu})$
- global symmetry $\Rightarrow$ conserved charges
Euclidean path integrals

nonzero temperature $\Rightarrow$ base space $R^1 \cdot S^1_{1/T}$ (corresponding lattice)

- here: low temperatures = large temporal extent

nonzero density/chemical potential $\Rightarrow$ action modified:

- one of the O(3) generators rotates $n_{1,2}$
  - conserved current
  - couple $\mu$ to that symmetry (other $\mu$ couplings equivalent)

new action contains $\mu Q$ (= charge), but in bosonic theories also $\mu^2$
continuum: \[ g^2 \mathcal{L} = \frac{1}{2} (\partial_\nu \vec{n})^2 + i\mu (n_1 \dot{n}_2 - n_2 \dot{n}_1) - \frac{\mu^2}{2} (n_1^2 + n_2^2) \]

\[ + \frac{\mu^2}{2} (n_3^2 - 1) \]

⇒ suppression of the perpendicular component, effectively \( O(2) \)
like with magnetic field

⇒ complex action/sign problem: importance sampling??

note that imaginary \( \mu \) is fine
Physics expectations

(i) charged particles induced only after $\mu$ has reached the mass gap

= partition function remains $\mu$-indep. up to $\mu_c = m^1$

‘Silver-Blaze’ region & transition

• measuring $\mu_c$ (having solved the sign problem) one gets information about the mass of the charged particles

for other interesting insights see the talk by T. Sulejmanpasic

(ii) for large $\mu$ the system becomes more and more O(2)-like

• vortices

• condensing or in vortex-antivortex pairs depending on the coupling = Berezinskii-Kosterlitz-Thouless transition

no order parameter singular, two point correlators change

\[^1\text{QCD: }\mu_{\text{up}} = \mu_{\text{down}} \Rightarrow 3\mu_c = m_{\text{lightest baryon}}, \mu_{\text{up}} = -\mu_{\text{down}} \Rightarrow 2\mu_c = m_{\text{pion}}\]
Dualization in a nutshell

- lattice action with \( n_1 + in_2 = n_{12} e^{i \phi} \): \((n_{12}^2 + n_3^2 = 1, \text{ spacing } = 1)\)

\[
S = -J \sum_{x, \nu} \left( n_3(x) n_3(x + \hat{\nu}) + \frac{1}{2} n_{12}(x) n_{12}(x + \hat{\nu}) \cdot \left\{ e^{i(\phi(x + \hat{\nu}) - \phi(x))} e^{\mu \delta_{\nu,0}} + e^{-i(\phi(x + \hat{\nu}) - \phi(x))} e^{-\mu \delta_{\nu,0}} \right\} \right)
\]

- idea: for all terms \( A \) in the action, expand \( e^{-S_A[n]} \) with variables \( k_A \)
  then integrate out the original fields \( n \)

- result: the system is now represented in ‘dual variables’ \( k \)
  typically constrained, simulated by worm algorithms

provided all weights are positive = sign problem solved
Dualization in a nutshell

- lattice action with \( n_1 + i n_2 = n_{12} e^{i \phi} \): \( (n_{12}^2 + n_3^2 = 1, \text{ spacing } = 1) \)

\[
S = -J \sum_{x, \nu} \left( n_3(x) n_3(x + \hat{\nu}) + \frac{1}{2} n_{12}(x) n_{12}(x + \hat{\nu}) \right) \cdot \left\{ e^{i(\phi(x+\hat{\nu}) - \phi(x))} e^{\mu \delta_{\nu,0}} + e^{-i(\phi(x+\hat{\nu}) - \phi(x))} e^{-\mu \delta_{\nu,0}} \right\}
\]

- bond variables for the three terms above
  - e.g. for the third component:

\[
e^{-S} \propto \prod_{x, \nu} e^{J n_3(x) n_3(x + \hat{\nu})} = \prod_{x, \nu} \sum_{k_{\nu}(x)=0}^{\infty} \frac{J^{k_{\nu}(x)}}{k_{\nu}(x)!} n_3(x)^{k_{\nu}(x)} n_3(x + \hat{\nu})^{k_{\nu}(x)}
\]

- \( J \) comes with a power of all dual variables
Dualization in a nutshell

- lattice action with $n_1 + in_2 = n_{12}e^{i\phi}$: \( (n_{12}^2 + n_3^2 = 1, \text{ spacing} = 1) \)

\[
S = -J \sum_{x, \nu} (n_3(x)n_3(x + \hat{\nu}) + \frac{1}{2} n_{12}(x)n_{12}(x + \hat{\nu}) \cdot \\
\cdot \left\{ e^{i(\phi(x+\hat{\nu})-\phi(x))} e^\mu \delta_{\nu,0} + e^{-i(\phi(x+\hat{\nu})-\phi(x))} e^{-\mu \delta_{\nu,0}} \right\})
\]

- integrations over \( \{ n_3(x), n_{12}(x), \phi(x) \} = \{ \cos \theta(x), \sin \theta(x), \phi(x) \} \)
  the latter gives constraints:

\[
\frac{1}{2\pi} \int_0^{2\pi} d\phi(x) e^{i\phi(x)} \sum_{\nu} [m_\nu(x+\hat{\nu})-m_\nu(x)] \equiv \delta_{\text{Kron}}(\nabla_\nu m_\nu) \quad \forall x
\]

namely that \( m_\nu \in \mathbb{Z} \) is divergence-free

\( (m_\nu \) is the difference of two nonnegative dual variables)
$m_{\nu} \in \mathbb{Z}$ is divergence-free

- symmetry manifest
- obviously the conserved charge reads $Q = \sum_{x_1} m_0(x_0, x_1)$

$\Rightarrow$ on each time slice the net flux of the dual variable $m_{\nu}$ is the charge $Q$

- the coupling of the chemical potential is done via

$$e^{-\mu \sum_{x_1} m_0(x_0,x_1)} > 0$$

if the total weight of the dual variables $m_{\nu}$ etc. is positive at $\mu = 0$ \checkmark
it remains so at $\mu \neq 0$

$\Rightarrow$ sign problem solved

note that imaginary $\mu$ is bad
Full result and interpretation, in $O(4)$

$$Z = \sum_{m \in \mathbb{Z}} \prod_{x, \nu} \frac{(J/2)|m_{\nu}(x)| + 2\bar{m}_{\nu}(x)J k_{\nu}^{(3)}(x) + k_{\nu}^{(4)}(x)}{(|m^{(j)}_{\nu}(x)| + \bar{m}_{\nu}(x))! \bar{m}_{\nu}(x)! k_{\nu}^{(3)}(x)! k_{\nu}^{(4)}(x)!} \quad \text{(expans.)}$$

$$\times \prod_{x} \frac{\Gamma(1 + \frac{a(x)}{2}) \Gamma(\frac{1}{2} + \frac{b^{(3)}(x)}{2}) \Gamma(\frac{1}{2} + \frac{b^{(4)}(x)}{2})}{\Gamma(1 + \frac{a(x)}{2} + \frac{1}{2} + \frac{b^{(3)}(x)}{2} + \frac{1}{2} + \frac{b^{(4)}(x)}{2})} \quad \text{(integrated fields)}$$

$$\times \prod_{x} \delta(\nabla_{\nu} m_{\nu} \cdot e^{-\mu \sum x_1 m_0}) \quad \text{(constraints and chem. potential)}$$

$$\times \prod_{x} \delta_{\text{even}}(b^{(3)}(x)) \delta_{\text{even}}(b^{(4)}(x)) \quad \text{(evenness constraints)}$$

where

$$a(x) = \sum_{\nu} \left[ |m_{\nu}(x)| + |m_{\nu}(x + \hat{\nu})| + 2\bar{m}_{\nu}(x) + 2\bar{m}_{\nu}(x + \hat{\nu}) \right]$$

$$b^{(j)}(x) = \sum_{\nu} \left[ k_{\nu}^{(j)}(x) + k_{\nu}^{(j)}(x + \hat{\nu}) \right] \quad j = 3, \ldots, N$$
Full result and interpretation, in $O(4)$

$$Z = \sum_{m \in \mathbb{Z}} \prod_{x, \nu} \frac{(J/2) |m_\nu(x)| + 2 \tilde{m}_\nu(x) J k^{(3)}_\nu(x) + k^{(4)}_\nu(x)}{(|m_\nu^{(j)}(x)| + \tilde{m}_\nu(x))! \tilde{m}_\nu(x)! k^{(3)}_\nu(x)! k^{(4)}_\nu(x)!} \quad \text{(expans.)}$$

$$\times \prod_x \frac{\Gamma\left(1 + \frac{a(x)}{2}\right) \Gamma\left(\frac{1}{2} + \frac{b^{(3)}(x)}{2}\right) \Gamma\left(\frac{1}{2} + \frac{b^{(4)}(x)}{2}\right)}{\Gamma\left(1 + \frac{a(x)}{2} + \frac{1}{2} + \frac{b^{(3)}(x)}{2} + \frac{1}{2} + \frac{b^{(4)}(x)}{2}\right)} \quad \text{(integrated fields)}$$

$$\times \prod_x \delta\left(\nabla_\nu m_\nu \cdot e^{-\mu \sum x_1 m_0}\right) \quad \text{(constraints and chem. potential)}$$

$$\times \prod_x \delta_{\text{even}}\left(b^{(3)}(x)\right) \delta_{\text{even}}\left(b^{(4)}(x)\right) \quad \text{(evenness constraints)}$$

- second line = ‘beta function’
- fix length of original field by Lagrange multiplier:
  - numerator $\Gamma$’s from integrating out unconstrained fields,
  - denominator $\Gamma$ from integrating out Lagrange multiplier
- mass gap derived in original theory at large-N with Lagrange mult.
Full result and interpretation, in $O(4)$

$$Z = \sum_{\bar{m}, k^{(3)}, k^{(4)} \in \mathbb{N}_0} \prod_{x, \nu} \frac{(J/2)|m_\nu(x)|+2\bar{m}_\nu(x)Jk^{(3)}_\nu(x)+k^{(4)}_\nu(x)}{(|m^{(j)}_\nu(x)| + \bar{m}_\nu(x))! \bar{m}_\nu(x)!k^{(3)}_\nu(x)!k^{(4)}_\nu(x)!}$$

( expansive. )

$$\times \prod_x \frac{\Gamma(1 + \frac{a(x)}{2})\Gamma(\frac{1}{2} + \frac{b^{(3)}(x)}{2})\Gamma(\frac{1}{2} + \frac{b^{(4)}(x)}{2})}{\Gamma(1 + \frac{a(x)}{2} + \frac{1}{2} + \frac{b^{(3)}(x)}{2} + \frac{1}{2} + \frac{b^{(4)}(x)}{2})}$$

(integrated fields)

$$\times \prod_x \delta(\nabla_\nu m_\nu \cdot e^{-\mu \sum_{x_1} m_0})$$

(constraints and chem. potential)

$$\times \prod_x \delta_{\text{even}}(b^{(3)}(x))\delta_{\text{even}}(b^{(4)}(x))$$

(evenness constraints)

- third line: $\mu$ in (1,2)-components couples to conserved current manifest since parametrization with polar angle, adapted
- there must be a conserved current in e.g. the (3,4)-components not explicit in this representation (polar angle in (3,4)-component!)
- fourth line: more constraints (!)
Intermezzo: Full result for CP(N-1) models

- \( N \) chemical potentials \( \mu^{(j)}, j = 1, \ldots, N \):

\[
Z = \sum_{m^{(j)}(x) \in \mathbb{Z}} \sum_{\tilde{m}^{(j)}(x) \in \mathbb{N}_0} \prod_{x, \nu, j} \frac{J|m^{(j)}(x)| + 2\tilde{m}^{(j)}(x)}{|m^{(j)}(x)| + \tilde{m}^{(j)}(x))! \tilde{m}^{(j)}(x)!} \\
\times \prod_{x} \frac{\Gamma(1 + \frac{a^{(1)}(x)}{2})\Gamma(1 + \frac{a^{(2)}(x)}{2}) \cdots \Gamma(1 + \frac{a^{(N)}(x)}{2})}{\Gamma(N + \frac{a^{(1)}(x)}{2} + \frac{a^{(2)}(x)}{2} + \cdots + \frac{a^{(N)}(x)}{2})} \\
\times \prod_{x, j} \left( \delta(\nabla_{\nu} m_{\nu}) \cdot e^{-\mu \sum_{x} m_0} \right)^{(j)} \\
\times \prod_{x, \nu} \delta\left( \sum_{j} m^{(j)}_{\nu}(x) \right)
\]

(2N ‘colored’ dual variables)

(exponent expansion)

(integrated fields)

(constraints and chem. potentials)

(local U(1) symmetry, on bonds)

- no sign problem here either
Numerical results: Silver blaze transition

- particle number density and its susceptibility as functions of $\mu$ for different couplings $J$:

\[ \langle n \rangle, \chi_n \text{ independent of } \mu \text{ up to some } \mu_c \checkmark \]

movie ($10 \times 1000$)
\( \mu_c = m \) and expansions for mass (bare, in lattice units):

\[
am = \begin{cases} 
   - \log \frac{J}{3} & \text{strong coupling} \\
   \# \exp(-2\pi J) & \text{weak coupling (large } N) 
\end{cases}
\]

\[\log(m) = \log(\mu_{\text{crit}})\]

Dual data: \( m = c_1 J \exp(-c_2 J) \) with \( c_1 = 1167 \pm 501 \), \( c_2 = 6.61 \pm 0.28 \)

Also checked against conventional spectroscopy
Order of the transition

- at fixed $J = \text{fixed lattice spacing}$, scaling of $\langle Q \rangle$ with volume

\[ \langle n \rangle = \frac{\mu}{\mu_{\text{crit}}} \]

- temperature

\[ \langle n \rangle = \frac{\mu}{\mu_{\text{crit}}} \]

\[ L = 18 \ldots 35 \frac{1}{m} \quad (T = 0.02 m) \]

- a crossover . . .

\[ T = 0.02 \ldots 0.005 m \quad (L = 22 \frac{1}{m}) \]

- . . . that sharpens as $T \rightarrow 0$

- at $T = 0$ non-analytic?! “quantum phase transition”
scaling of the maximum of the particle number susceptibility $\chi$ with $1/LT = N_t/N_s$ (lattice aspect ratio):

$$\chi_{\text{max}} \propto \frac{1}{(LT)^\gamma} \quad \text{with} \quad \gamma = 0.998(3)$$
Nature of the transition

At large $\mu$ the system tends to be planar due to expl. breaking $+\frac{\mu^2}{2} n_3^2$

- Out-of-plane hoppings become suppressed:

$$J \langle n_3(x) n_3(x + \hat{\nu}) \rangle_{\text{orig}} = \langle k_\nu(x) \rangle_{\text{dual}}$$

(value at small $\mu \iff$ hoppings isotropic $\checkmark$)
Two point correlator and vortices

in-plane:

\[ C(x - y) = \langle e^{i\phi(x)} e^{-i\phi(y)} \rangle \]

- expectation in the O(2) model

\[ \mathcal{L}_{O(2)} = -J \sum_{x,\nu} \cos(\phi(x + \hat{\nu}) - \phi(x)) \]

th correlator decays exponentially/algebraically

\[ C(x - y) \rightarrow \begin{cases} 
J|x-y|/a & \text{for small } J = \text{strong coupling} \\
\frac{1}{|x-y|^{1/2\pi J}} & \text{for large } J = \text{weak coupling}
\end{cases} \]

because vortices (also called merons)

\[ \begin{cases} 
\text{condense} & \text{for small } J \\
\text{come in vortex-antivortex pairs} & \text{for large } J
\end{cases} \]

both \( C(x - y) \xrightarrow{|x-y|\to\infty} 0 \): no ordered phase in 2d

Mermin-Wagner
where will we land at $\mu > \mu_c$? $J_{O(2)} = J_{O(3)}^{\text{eff}}(\text{scale } \mu) \propto \log(\mu)$

will we see BKT as a second transition?

time-integrated spatial correlator (‘zero-momentum’) for the sake of high statistics

logarithmically:

exp. decay lost above $\mu_c$!? time integration spoils interpretation
dualization of O(N) and CP(N-1) models with chemical potential(s) sign problem gone, actually in all dimensions (some) symmetries manifest
more to learn from dual variables realization of mass gap?
interplay of pert. expansion & nonpert. physics = resurgence?

quantum phase transition? $\chi_{\text{max}} \sim 1/(LT)^{0.998(3)}$
Berezinskii-Kosterlitz-Thouless as a second transition?
more data, on two point and vortex correlator
outlook: topological angle $\theta$, higher models & large-$N$