Why local gauge invariance is necessary

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Outline

- Preliminaries
- Gauge invariance basics
- Implementation of local gauge invariance
- A practical example
- Summary
Setup: Currents

Photon may be real or virtual, incoming or outgoing

All hadrons on-shell:

\[ k_\mu J^\mu = 0 \] (necessary condition) \implies \text{gauge invariance}
Setup: Currents

\[ J^\mu = J^\mu_{\text{outgoing hadronic state}} - J^\mu_{\text{incoming hadronic state}} \]

Photon may be real or virtual, incoming or outgoing

All hadrons on-shell:

\[ k_\mu J^\mu = 0 \quad \text{(necessary condition)} \quad \Rightarrow \quad \text{gauge invariance} \]

Transition currents:

\[ B: \text{Baryon} \quad R: \text{Resonance} \]
Setup: Currents

\[ J^\mu = J^\mu \]

Photons may be real or virtual, incoming or outgoing.

All hadrons on-shell:

\[ k^\mu J^\mu = 0 \quad \text{(necessary condition)} \implies \text{gauge invariance} \]

Transition currents:

Current operators for transition currents must necessarily be manifestly transverse. Hence, gauge-invariance condition is trivially satisfied for on- and off-shell baryons.
Setup: Currents

\[ J^\mu = \begin{cases} \text{outgoing hadronic state} & \text{incoming hadronic state} \\ \text{outgoing hadronic state} & \text{incoming hadronic state} \end{cases} \]

Photon may be real or virtual, incoming or outgoing

All hadrons on-shell:

\[ k_\mu J^\mu = 0 \quad \text{(necessary condition)} \quad \Rightarrow \quad \text{gauge invariance} \]

Transition currents:

Current operators for transition currents must necessarily be manifestly transverse.

Hence, gauge-invariance condition is trivially satisfied for on- and off-shell baryons.

Why bother?
Production Current

\[ M^\mu = s\text{-channel} + u\text{-channel} + t\text{-channel} + \text{interaction current} \]

\[ = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

Generic expressions; summations over all possible intermediate states implied
Production Current

\[ M^\mu = M^\mu_{\text{int}} + M^\mu_s + M^\mu_u + M^\mu_t \]

\[ = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

Generic expressions; summations over all possible intermediate states implied

- **s-channel term contains transition-current contributions**
  
  Diagrams apply to spacelike process; for timelike process, read diagrams in time-reversed order

- **Entire production current must be gauge invariant**
  
  Without it, wrong background contribution for extraction of form factors
Production Current

\[ M^\mu = \left( \begin{array}{c}
\text{s-channel} \\
\text{u-channel} \\
\text{t-channel} \\
\text{interaction current}
\end{array} \right) 
\]

\[ = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta m F_t + M^\mu_{\text{int}} \]

Generic expressions; summations over all possible intermediate states implied

- **s-channel term contains transition-current contributions**
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- **Entire production current must be gauge invariant**
  - Without it, wrong background contribution for extraction of form factors

- **Any approximation will likely destroy gauge invariance**

- **For a microscopic theory, it is not sufficient to fix** \( k_\mu M^\mu = 0 \) **on shell**
Gauge invariance basics

Implementation of local gauge invariance

A practical example

Summary
Gauge Invariance

\[ M^\mu = s - \text{channel} + u - \text{channel} + t - \text{channel} + \text{interaction current} \]

\[ = F_s S_i J_i^\mu + J_f^\mu S_f F_u + J_m^\mu \Delta_m F_t + M_{\text{int}}^\mu \]

Global gauge invariance

\[ k_\mu M^\mu = 0 \]

all external hadrons on-shell

\[ \Phi \to \Phi e^{-i\Lambda} \]
Gauge Invariance

\[ M^\mu = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{int} \]

Global gauge invariance

\[ k_\mu M^\mu = 0 \]

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conserved current \( \Rightarrow \) implies charge conservation
Gauge Invariance

\[ M^\mu = s_{-}\text{channel} + u_{-}\text{channel} + t_{-}\text{channel} + \text{interaction current} \]

\[ = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

Global gauge invariance

\[ \Phi \rightarrow \Phi e^{-i\Lambda} \]

conserved current \( \Rightarrow \) implies charge conservation

Fixing global gauge invariance does not mean internal damage is fixed as well
Gauge Invariance

\[ M^\mu = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

Local gauge invariance

\[ \Phi \rightarrow \Phi e^{-i\lambda(x)} \]
Gauge Invariance

\[ M^\mu = s \rightarrow J^\mu_i S_i J^\mu_i + u \rightarrow J^\mu_f S_f F_u + t \rightarrow J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

Local gauge invariance

\[ \Phi \rightarrow \Phi e^{-i\lambda(x)} \]

Generalized Ward-Takahashi identities (gWTI)

\[
\begin{align*}
    k_\mu M^\mu &= (q^2 - M^2_m) Q_m F_t + S_f^{-1}(p') Q_f F_u - F_s Q_i S_i^{-1}(p) \\
    k_\mu J^\mu_m &= (q^2 - M^2_m) Q_m - Q_m (t - M^2_m) \\
    k_\mu M^\mu_{\text{int}} &= Q_m F_t + Q_f F_u - F_s Q_i
\end{align*}
\]

similarly for baryons

off-shell relations
Gauge Invariance

\[ M^\mu = \begin{cases} \text{s-channel} & F_s S_i J_i^\mu \\ \text{u-channel} & J_f^\mu S_f F_u \\ \text{t-channel} & J_m^\mu \Delta_m F_t \\ \text{interaction current} & M^\mu_{\text{int}} \end{cases} \]

Local gauge invariance

\[ \Phi \rightarrow \Phi e^{-i\lambda(x)} \]

Generalized Ward-Takahashi identities (gWTI)

\[ k_\mu M^\mu = (q^2 - M_m^2)Q_m F_t + S_f^{-1}(p')Q_f F_u - F_s Q_i S_i^{-1}(p) \]
\[ k_\mu J_m^\mu = (q^2 - M_m^2)Q_m - Q_m(t - M_m^2) \]
\[ k_\mu M^\mu_{\text{int}} = Q_m F_t + Q_f F_u - F_s Q_i \]

similarly for baryons

off-shell relations

local gauge invariance \[\Rightarrow\] implies existence of e.m. field
**Gauge Invariance**

\[ M^\mu = F_s S_i J^\mu_i + J^\mu_f S_f F_u + J^\mu_m \Delta_m F_t + M^\mu_{\text{int}} \]

**Local gauge invariance**

\[ \Phi \to \Phi e^{-i\lambda(x)} \]

**Generalized Ward-Takahashi identities (gWTI)**

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\begin{align*}
    k_\mu M^\mu &= (q^2 - M_m^2)Q_m F_t + S_f^{-1}(p')Q_f F_u - F_s Q_i S_i^{-1}(p) \\
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    k_\mu M^\mu_{\text{int}} &= Q_m F_t + Q_f F_u - F_s Q_i
\end{align*}
\]

**Local gauge invariance** \[\Rightarrow\] **implies existence of e.m. field**

**Without gWTI underlying e.m. field is damaged**
Implementation: Reorganize interaction current

\[ M_{\text{int}}^\mu = m_C^\mu + T + \cdots \]

- Interaction current contains all contributions from final-state interactions
Implementation: Reorganize interaction current

\[ M_{\text{int}}^\mu \] \[ m_c^\mu \] \[ T \] FSI loop

+ \cdots

Interaction current contains all contributions from final-state interactions

Decompose FSI loops into longitudinal and transverse contributions

For details, see HH, Nakayama, Krewald, PRC74, 045202 (2006); HH, Huang, Nakayama, PRC83, 065502 (2011); HH, Wang, He, PRC92, 055503 (2015)

Impose generalized Ward-Takahashi identities to longitudinal part
Generalized Ward-Takahashi Identities

(1) \( k_\mu M^\mu = (q^2 - M_m^2) Q_m F_t + S_f^{-1}(p') Q_f F_u - F_s Q_i S_i^{-1}(p) \)

(2) \( k_\mu J^\mu_m = (q^2 - M_m^2) Q_m - Q_m (t - M_m^2) \)

(3) \( k_\mu M^\mu_{int} = Q_m F_t + Q_f F_u - F_s Q_i \)
Generalized Ward-Takahashi Identities

\[(1) \quad k_\mu M^\mu = (q^2 - M_m^2) Q_m F_t + S_f^{-1}(p') Q_f F_u - F_s Q_i S_i^{-1}(p)\]

\[(2) \quad k_\mu J^\mu_m = (q^2 - M_m^2) Q_m - Q_m (t - M_m^2) \text{ trivial}\]

\[(3) \quad k_\mu M^\mu_{\text{int}} = Q_m F_t + Q_f F_u - F_s Q_i\]

Only two relations are independent \(\Rightarrow\) Use (2) & (3)
Generalized Ward-Takahashi Identities

\begin{align}
(1) \quad k_\mu M^\mu &= (q^2 - M_m^2) Q_m F_t + S_f^{-1}(p') Q_f F_u - F_s Q_i S_i^{-1}(p) \\
(2) \quad k_\mu J^\mu_m &= (q^2 - M_m^2) Q_m - Q_m (t - M_m^2) \quad \text{trivial} \\
(3) \quad k_\mu M_{int}^\mu &= Q_m F_t + Q_f F_u - F_s Q_i
\end{align}

Only two relations are independent \quad \Rightarrow \quad \text{Use (2) & (3)}

\textbf{Hadronic vertex}

\[ F(p_{b'}, p_b) = G(q_m) \tau f(q_m^2, p_{b'}^2, p_b^2) \]

\[ \begin{cases} 
  f_s(s) = f(M_m^2, M_{b'}^2, s) \\
  f_u(u) = f(M_m^2, u, M_b^2) \\
  f_t(t) = f(t, M_{b'}^2, M_b^2)
\end{cases} \]

This phenomenologically motivated ansatz can be improved in a systematic way
Generalized Ward-Takahashi Identities

\( k_\mu M^\mu = (q^2 - M_m^2) Q_m F_t + S_f^{-1}(p')Q_f F_u - F_s Q_i S_i^{-1}(p) \)  
\( k_\mu J_m^\mu = (q^2 - M_m^2) Q_m - Q_m(t - M_m^2) \quad \text{trivial} \)  
\( k_\mu M_{\text{int}}^\mu = Q_m F_t + Q_f F_u - F_s Q_i \)

Only two relations are independent \( \Rightarrow \) Use (2) \& (3)

**Hadronic vertex**

\[ F(p_{b'}, p_b) = G(q_m) \tau f(q_m^2, p_{b'}^2, p_b^2) \]

**Interaction-current Ansatz:**

\[ M_{\text{int}}^\mu = m^\mu_c f_t(t) + G(q) C^\mu + T^\mu_{\text{int}} \]

\[ k_\mu T^\mu_{\text{int}} \equiv 0 \]
Generalized Ward-Takahashi Identities

(1) \( k_\mu M^\mu = (q^2 - M_m^2) Q_m F_t + S_f^{-1}(p') Q_f F_u - F_s Q_i S_i^{-1}(p) \)

(2) \( k_\mu J^\mu_m = (q^2 - M_m^2) Q_m - Q_m(t - M_m^2) \) trivial

(3) \( k_\mu M^\mu_{int} = Q_m F_t + Q_f F_u - F_s Q_i \)

Only two relations are independent \( \Rightarrow \) Use (2) \& (3)

Hadronic vertex

\[ F(p_{b'}, p_b) = G(q_m) \tau f(q_m^2, p_{b'}^2, p_b^2) \]

Interaction-current Ansatz:

\[ M^\mu_{int} = m_c^\mu f_t(t) + G(q) C^\mu + T^\mu_{int} \]

\( k_\mu T^\mu_{int} \equiv 0 \)

\( \Rightarrow \) Determine \( C^\mu \) such that (3) is true
Auxiliary Contact Current $C^\mu$

\[
C^\mu = -e_m (2q - k)^\mu \frac{f_t - 1}{t - M_m^2} (\delta_s f_s + \delta_u f_u - \delta_s \delta_u f_s f_u)
\]

\[
- e_f (2p' - k)^\mu \frac{f_u - 1}{u - M_f^2} (\delta_t f_t + \delta_s f_s - \delta_t \delta_s f_t f_s)
\]

\[
- e_i (2p + k)^\mu \frac{f_s - 1}{s - M_i^2} (\delta_u f_u + \delta_t f_t - \delta_u \delta_t f_u f_t)
\]
Auxiliary Contact Current $C^\mu$

$$C^\mu = -e_m (2q - k)^\mu \frac{f_t - 1}{t - M_m^2} (\delta_s f_s + \delta_u f_u - \delta_s \delta_u f_s f_u)$$

$$- e_f (2p' - k)^\mu \frac{f_u - 1}{u - M_f^2} (\delta_t f_t + \delta_s f_s - \delta_t \delta_s f_t f_s)$$

$$- e_l (2p + k)^\mu \frac{f_s - 1}{s - M_i^2} (\delta_u f_u + \delta_t f_t - \delta_u \delta_t f_u f_t)$$

where

$$\delta_x = \begin{cases} 
1 & \text{channel contributes} \\
0 & \text{channel does not contribute}
\end{cases} \quad x = s, u, t$$
Auxiliary Contact Current $C^\mu$

$$C^\mu = -e_m (2q - k)^\mu \frac{f_t - 1}{t - M_m^2} (\delta_s f_s + \delta_u f_u - \delta_s \delta_u f_s f_u)$$

$$- e_f (2p' - k)^\mu \frac{f_u - 1}{u - M_f^2} (\delta_t f_t + \delta_s f_s - \delta_t \delta_s f_t f_s)$$

$$- e_i (2p + k)^\mu \frac{f_s - 1}{s - M_i^2} (\delta_u f_u + \delta_t f_t - \delta_u \delta_t f_u f_t)$$

where

$$\delta_x = \begin{cases} 
1 & \text{channel contributes} \\
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\end{cases} 
\quad x = s, u, t$$

Charge conservation:

$$Q_m \tau + Q_f \tau - \tau Q_i = e_m + e_f - e_i = 0$$
**Auxiliary Contact Current** $C^\mu$

$$C^\mu = -e_m(2q - k)^\mu \frac{f_t - 1}{t - M^2_m} (\delta_s f_s + \delta_u f_u - \delta_s \delta_u f_s f_u)$$

$$- e_f(2p' - k)^\mu \frac{f_u - 1}{u - M^2_f} (\delta_t f_t + \delta_s f_s - \delta_t \delta_s f_t f_s)$$

$$- e_i(2p + k)^\mu \frac{f_s - 1}{s - M^2_i} (\delta_u f_u + \delta_t f_t - \delta_u \delta_t f_u f_t)$$

where

$$\delta_x = \begin{cases} 1 & \text{channel contributes} \\ 0 & \text{channel does not contribute} \end{cases} \quad x = s, u, t$$

**Charge conservation:**

$$Q_m \tau + Q_f \tau - \tau Q_i = e_m + e_f - e_i = 0$$

**Four-divergence:**

$$k^\mu C^\mu = e_m f_t + e_f f_u - e_i f_s$$

ensures correct four-divergence for $M^\mu_{\text{int}}$
Example: Two-pion production at the no-loop level

\[ \text{NL} = \text{NL}_1 + \text{NL}_2 \]

\[ \text{NL}_1 = M + M - \]

\[ \text{NL}_2 = M + M - \]
Example: Two-pion production at the no-loop level:

\[ \text{NL} = \text{NL}_1 + \text{NL}_2 \]

\[ \text{NL}_1 = \text{M} + \text{M} - \text{M} \]

\[ \text{NL}_2 = \text{M} + \text{M} - \text{M} \]

Without gWTI, this amplitude will not be gauge invariant.
Summary

- Global gauge invariance is necessary but not sufficient to provide consistent implementation of the electromagnetic field throughout the interaction volume.
- Correct dynamical basis provided by generalized Ward-Takahashi identities as they follow from local gauge invariance.
- Results obtained within microscopic reaction models that do not obey local gauge invariance need to be viewed with caution.
- Applications show that contributions from gauge-fixing contact current can be substantial.
Summary

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- Correct dynamical basis provided by generalized Ward-Takahashi identities as they follow from local gauge invariance.

- Results obtained within microscopic reaction models that do not obey local gauge invariance need to be viewed with caution.

- Applications show that contributions from gauge-fixing contact current can be substantial.

Thank you!