Fluctuations and Nonlinearity in Hydrodynamics

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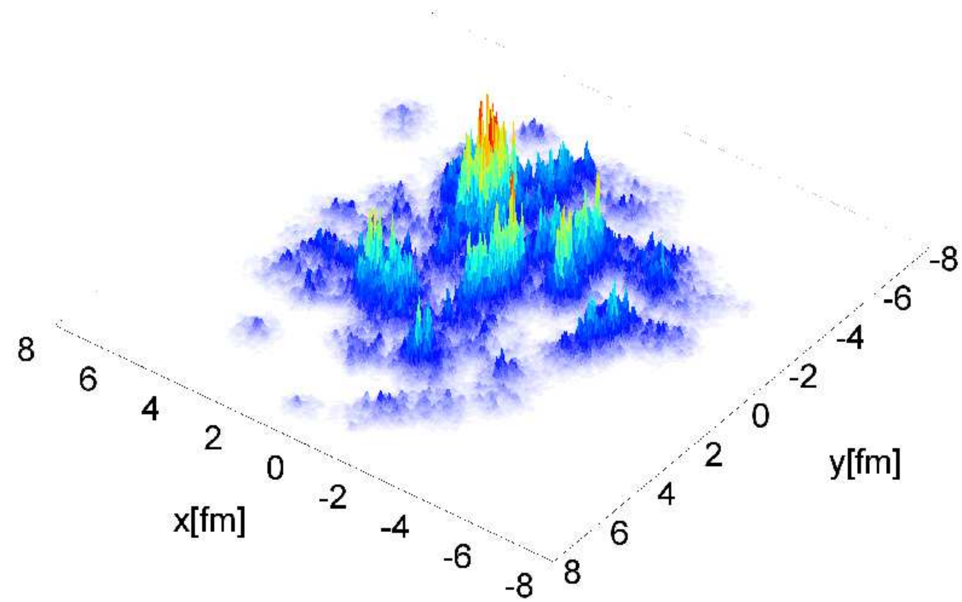
- Fluctuations and Fluctuations
- Review of second order hydrodynamics
- Equilibrium amplitude of hydro fluctuations
- Fluctuation-Dissipation and noise terms
- Impact on viscosity and 2'nd order hydro coefficients

Fluctuating initial conditions

We heard about importance of fluctuations in initial conditions



Arise from randomness in placement in ~ 200 participating nucleons.

Size $\sim 1/\sqrt{N}$ with $N \sim 200$.

Dissipative hydro will smooth out these fluctuations.

But there should be fluctuations it never smooths out.

Consider fluid *in equilibrium*. Split into local regions:

109 particles	97 particles	103 particles
86 particles	113 particles	104 particles
100 particles	92 particles	96 particles

Amount of stuff in each region has $1/\sqrt{N}$ fluctuations.

Fluctuations should always be there, also at strong coupling

Relative importance:

$\sqrt{N_{\mathrm{participants}}}/\sqrt{N_{\mathrm{final\ hadrons}}} \sim \sqrt{200}/\sqrt{5000} \sim 1/5$

How do I implement these fluctuations?

Let's start with the viscous hydro story.

Hydro is application of stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0$$

using some gradient-expansion model for $T^{\mu\nu}$.

Ideal hydro: assume equilibrium $T^{\mu\nu} \rightarrow$ strong assumption.
What are the corrections? Need viscous (1 order) hydro!

Viscous hydro, in relativistic setting, is an acausal and unstable theory. Practical applications: Need second order

Second Order Hydrodynamics

Hydro means we apply stress-energy conservation

$$\nabla_\mu T^{\mu\nu} = 0$$

with $T^{\mu\nu}$ gradient expanded about ideal form:

$$T^{\mu\nu}_{\text{ideal}}(x) \;\; = \;\; (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu}$$

$u^2 = -1$, $u_\mu T^{\mu\nu} = \epsilon u^\nu$ (Landau-Lifshitz) *define* u, ϵ.

$$\Pi^{\mu\nu} \equiv T^{\mu\nu} - T^{\mu\nu}_{\text{ideal}} \,, \qquad u_\mu \Pi^{\mu\nu} = 0 \text{ and } \Pi^\mu_\mu = 0$$

if u, ϵ vary slowly in spacetime, expect $\Pi^{\mu\nu}$ well described by *gradient expansion* about ideal form.

In conformal theory, first order term is unique:

$$T^{\mu\nu}(x) \;=\; (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} - \eta_b \sigma^{\mu\nu}$$

Order ∇^2, 5 terms Baier Romatschke Son Stephanov Starinets 0712.2451

$$\begin{aligned} T^{\mu\nu} \;=\; & \text{above} + \eta_b \tau_{\pi,b} \left(u \cdot \nabla \sigma^{\mu\nu} + \frac{1}{3} \sigma^{\mu\nu} \nabla \cdot u \right) \\ & + \kappa \left(R^{\langle\mu\nu\rangle} - 2 u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta} \right) \\ & + \lambda_{1b} \sigma^{\langle\mu}{}_\alpha \sigma^{\nu\rangle\alpha} + \lambda_{2b} \sigma^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} \end{aligned}$$

with shear tensor $\sigma^{\mu\nu} = 2\nabla^{\langle\mu} u^{\nu\rangle}$, vorticity tensor $\Omega^{\mu\nu} = (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$, and $\langle \ldots \rangle$ means spatial symmetrized traceless.

Fluctuation-Dissipation

The viscous term $-\eta_b \sigma^{\mu\nu}$ leads to dissipation of fluctuations.

Equilibrium: dissipation always accompanied by fluctuation. Local dissipative term matched by local noise term:

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \eta_b \sigma^{\mu\nu} + \xi^{\mu\nu}$$

Physically, $\xi^{\mu\nu}$ is local fluctuations in the number of particles (or stuff) flying in each direction in each local region. Should not affect every component (equation of state unchanged...).

Noise should only influence $T^{\mu\nu}$ components which feel η_b. So what should be form of $\xi^{\mu\nu}$?

A little more on the viscous tensor

Define spatial projector $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. Rewrite

$$
\begin{aligned}
-\eta_b \sigma^{\mu\nu} &= -2\eta_b P_T^{\mu\nu\alpha\beta} \partial_\alpha u_\beta \,, \\
2P_T^{\mu\nu\alpha\beta} &\equiv \Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}
\end{aligned}
$$

P_T is a transverse-traceless projection operator.

So shear tensor is $\partial_\mu u_\nu$ put through a projector.

Expect that

$$
\langle \xi^{\mu\nu}(x)\xi^{\alpha\beta}(y)\rangle = \sqrt{4\eta_b T}\, P_T^{\mu\nu\alpha\beta}\delta^4(x-y)\,.
$$

How to be sure about size of the noise?

Equilibrium, equal and small k, standard results give

$$\int d^3x e^{i\vec{k}\cdot\vec{x}} \langle T^{xy}(x) T^{xy}(0) \rangle = PT \,,$$

$$\int d^3x e^{i\vec{k}\cdot\vec{x}} \langle T^{0x}(x) T^{0x}(0) \rangle = (\epsilon + P) T \,.$$

Make sure that if you evolve hydro for a while, these will stay true. Dissipative hydro without noise destroys initial fluctuations and gives

$$\langle T^{xy} T^{xy} \rangle = 0 = \langle T^{0x} T^{0x} \rangle$$

Noise term amplitude chosen to keep fluct amplitudes right.

Meaning of coefficients $\eta_b, \tau_{\pi,b}, \lambda_{12b}$

Coefficients $\eta_b, \tau_{\pi,b}, \lambda_{1b,2b}$ in constitutive relation for $T^{\mu\nu}$ are

- Wilsonian coefficients for an effective theory
- Established by microphysics in a matching calculation between Hydro and the more microscopic theory
- Not the values which Kubo relations will give!

Ask for instance about equilibrium value of

$$
\begin{aligned}
G^{xyxy}_{\mathrm{ret}}(\omega) &= \int d^4x e^{-i\omega x^0} \Theta(x^0) \left\langle \left[T^{xy}(x) \, , \, T^{xy}(0) \right] \right\rangle \\
&= P - i\eta_f \omega - \eta \tau_{\pi,f} \omega^2 + \ldots
\end{aligned}
$$

Kubo for $\eta_f, \tau_{\pi,f}$

Really I want $\langle [T,T] \rangle$ + contact term $\partial T^{xy}/\partial g_{xy}$;

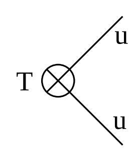
$$\partial T^{xy}/\partial g_{xy} = P - i\eta_b\omega - \eta_b\tau_{\pi,b}\omega^2$$

but there is also a long-distance contribution:

$$\left\langle \left[T^{xy}_{\rm Hydro}(x) \, , \, T^{xy}_{\rm Hydro}(0) \right] \right\rangle$$

where T^{xy} is the hydro expression:

$$T^{xy} = (\epsilon + P)u^x u^y - \eta P^{xy\mu\nu}\partial_\mu u_\nu =$$



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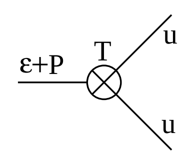


$\ldots$

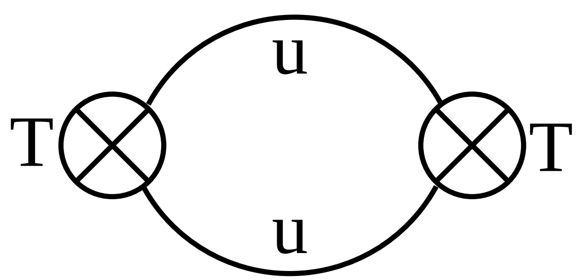
and then I need to know $u^x(x)u^x(0)$ correlators *etc*

Correlator calculation

$$\text{Feynman rules:} \quad T^{ij} = (\epsilon + P)u^i u^j + P g^{ij} \,,$$

$$\langle u^i u^j(k,\omega)\rangle = \frac{T}{\epsilon+P}\frac{(\delta^{ij} - \hat{k}^i\hat{k}^j)2\gamma_\eta k^2}{(\gamma_\eta k^2 - i\omega)(\gamma_\eta k^2 + i\omega)}\text{shearwave}$$

$$\left[\gamma_\eta = \frac{\eta_b}{\epsilon+P}, \gamma'_\eta = \frac{4}{3}\gamma_\eta\right] \qquad + \frac{T}{\epsilon+P}\frac{(\hat{k}^i\hat{k}^j)2\gamma'_\eta k^2\omega^2}{(\omega^2 - k^2/3)^2 + (\gamma'_\eta k^2\omega)^2}\text{soundwave}$$

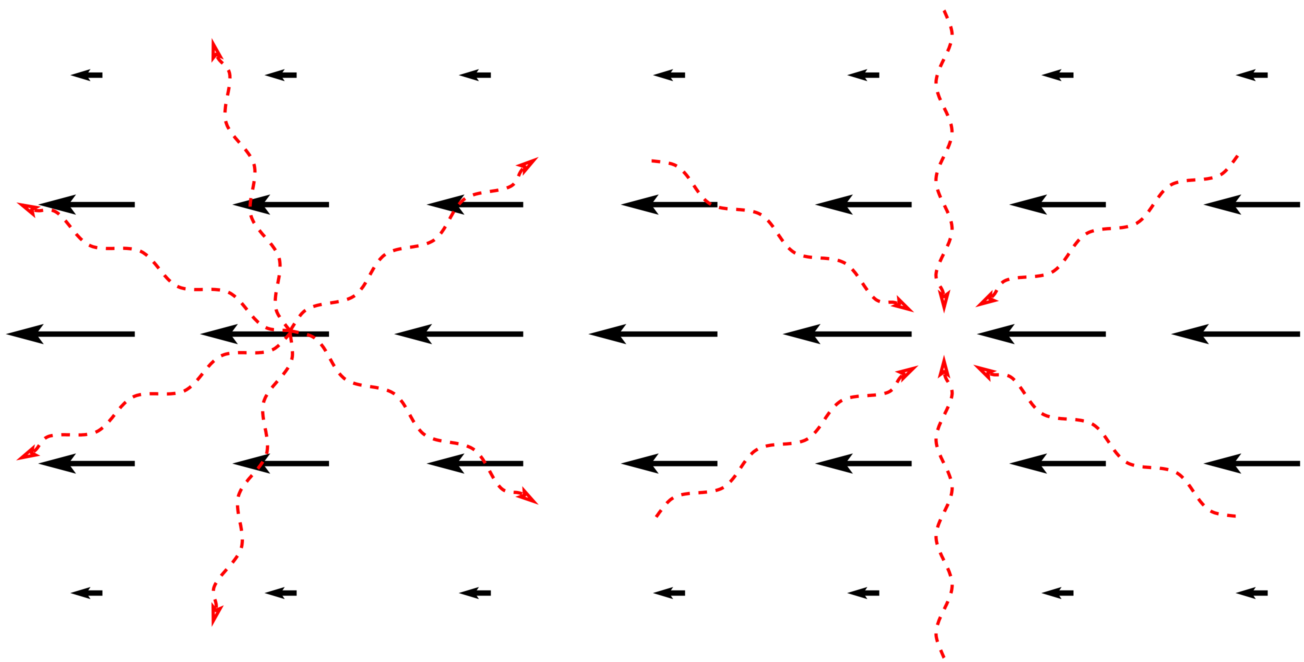


Leading-order diagram

straight-forward calculation

Intrepretation: How Hydro contributes to η

Consider shear flow:



Flow decays because x-momentum leaves (diffuses from) flowing region. One mechanism: propagation of hydro (sound) waves!

Computing Retarded TT Correlator

Straightforward application of Feynman rules:

$$G_{ra}^{xy,xy}(\omega)[\text{hydro}] = -i\omega\left(\frac{17Tk_{\max}}{120\pi^2\gamma_\eta}\right) + (i+1)\omega^{\frac{3}{2}}\,\frac{7+\left(\frac{3}{2}\right)^{\frac{3}{2}}T}{240\pi\gamma_\eta^{3/2}}$$

$k_{\max}$: k-scale above which hydro incorrect/inconsistent.

- $-i\omega$ term: extra contrib. to η
- $i\omega^{3/2}$: effective ω dependence of η.
- $\omega^{3/2}$: like τ_π but *wrong* ω dependence.

Lesson: η

Small η: freer propagation of sound, shear modes.
More efficient momentum transport, raising η.

Depends on $k_{\rm max}$. Where does hydro break down?
Scale where it's no longer self-consistent.

Safe guess: $k_{\rm max} < \tau_\pi^{-1}/2$. In $\mathcal{N}$=4 SYM, this is about $2T$.

- $\mathcal{N}$=4SYM: added η/s is $\sim 1/N_{\rm c}^2$.
- Weak coupling: $\eta_{\textrm{from hydro}} \sim \alpha^4$ while $\eta_{\rm tot} \sim \alpha^{-2}$
- Real QCD: $\frac{\eta}{s} = .16$: add 0.01. $\frac{\eta}{s} = .08$: add 0.036!

Lesson: τ_π

Weak coupling and large $N_{\rm c}$: comparing

$$N_{\rm c}^0 \alpha^3 T^{5/2}\,\omega^{3/2} \quad \text{vs} \quad N_{\rm c}^2 \alpha^{-4} T^2\,\omega^2$$

Deep IR, $\omega^{3/2}$ term wins, 2-order hydro breaks.
But scale where $\omega^{3/2}$ term takes over is $\omega \sim N_{\rm c}^{-4}\alpha^{14}T$.

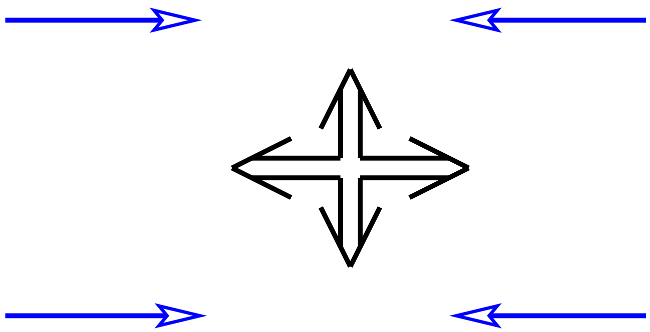
Check that ω where they equal is more IR than "your physics" and then use 2-order hydro!

- $N_{\rm c} = 3 = N_{\rm f}$ QCD, $T = 200\text{MeV}$, $\frac{\eta}{s} = .16$: $\omega \sim \frac{T}{20}$ Safe!
- $N_{\rm c} = 3 = N_{\rm f}$ QCD, $T = 200\text{MeV}$, $\frac{\eta}{s} = .08$: $\omega \sim 7T$ Problem!

Keep going: λ_1

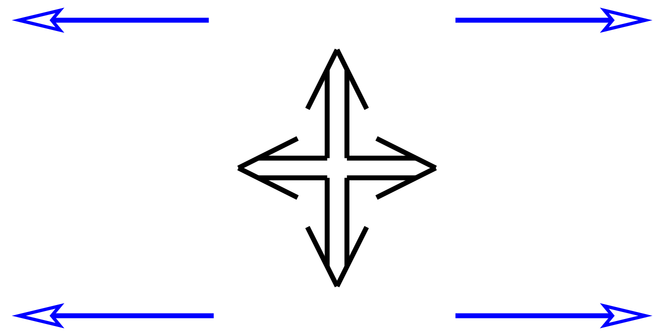
What is λ_1 anyway?

When you compress on an axis



Stress rises on that axis

When you expand

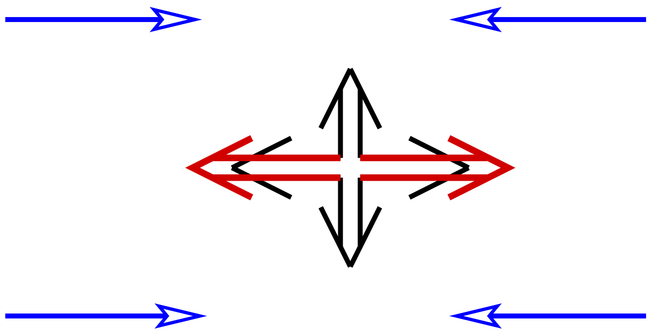


Stress falls on that axis

This is shear viscosity η.

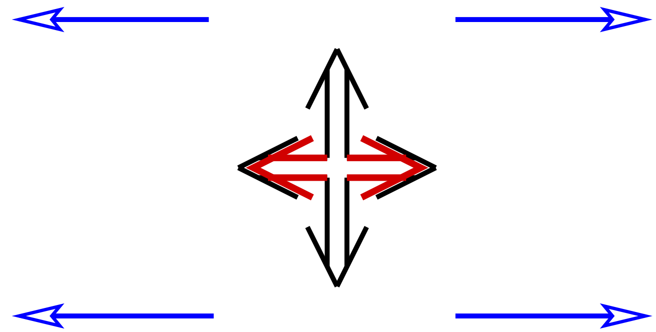
Now compress/expand more!

Compress more!

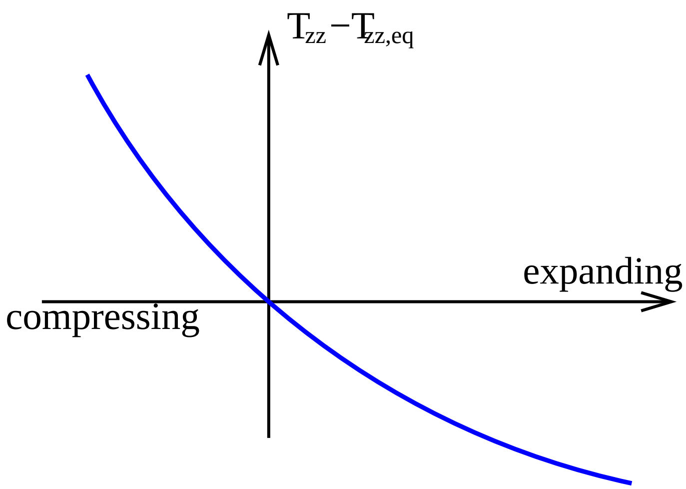


Stress rises MORE

Expand more!



Stress falls LESS



Expansion vs Stress:

Slope at 0 is $-\eta$

Curvature defines λ_1

λ_1: nonlin. response to (shear) stress.

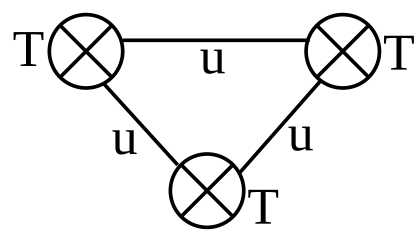
λ_1 from Hydro

Kubo relation for λ_1

$$\lambda_1 = \eta\tau_\pi - \lim_{\omega_1,\omega_2\to 0} \partial_{\omega_1}\partial_{\omega_2} G^{xy,xz,yz}_{\mathrm{fully-ret}}(-\omega_1-\omega_2,\omega_1,\omega_2)$$

Contact term gives $\lambda_{1,b}$.

Long-distance term:



Straightforward calculation,

$$G^{xy,xz,yz}_{\mathrm{fully-ret}} = \frac{(i-1)T}{240\pi\gamma_\eta^{\frac{3}{2}}} \frac{17+3\sqrt{6}}{7} \frac{\omega_1^{\frac{7}{2}}+\omega_2^{\frac{7}{2}} - (\omega_1+\omega_2)^{\frac{7}{2}}}{\omega_1\omega_2}$$

Again, $\omega^{3/2}$ behavior, divergent contrib. to λ_1.

λ_1: what we learned

Contribution to λ_1 is

- divergent
- complicated in ω_1, ω_2 structure

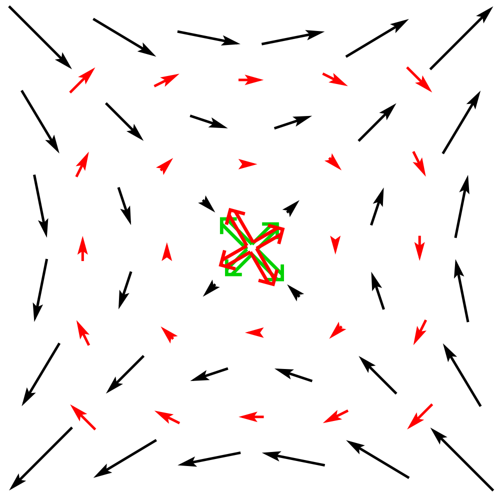
Structure which cannot be reproduced by any change to non-fluctuating hydro.

You cannot reproduce this sort of physics unless you include fluctuations in your hydro.

Though maybe λ_1 isn't very important??

What about λ_2?

What λ_2 means:
Rotating system (vorticity)
has shear-stress rotated
wrt. shear flow



Vorticity can exist in equilibrium: $\partial_{\vec{k}}$ not ∂_ω

$$\lambda_2 = -4 \lim_{\omega_1, k_2 \to 0} \partial_{\omega_1} \partial_{k_{2z}} G^{xy,xz,yt}_{\mathrm{fully-ret}}(.., \omega_1, \vec{k}_2)$$

This time long-distance correction is finite.

Conclusions

- $1/\sqrt{N}$ and/or fluctuation/dissipation tells you that hydro should have noise in it.
- Smaller by ~ 3 to 5 vs. initial state fluctuations
- Required to get right long-time tails
- Hydro fluctuations increase η. Limit how small η can be (in QCD, $\eta/s > 0.1$)
- Hydro fluct give divergent contributions to 2-time-deriv 2'nd order hydro coefficients τ_π, λ_1.
  This behavior captured if you include fluct.
  Cannot be captured if you don't.