Integral Transform Approaches to Continuum

Summary:

- General remarks on integral transform approaches
- Critical review of various kernels for different purposes
- Wavelet kernels: results of a model study
Integral transform (IT)

\[ \Phi(\sigma) = \int d\omega \ K(\omega,\sigma) \ S(\omega) \]

One **IS NOT** able to calculate \( S(\omega) \) (the quantity of direct physical meaning) but **IS** able to calculate \( \Phi(\sigma) \).
In order to obtain $S(\omega)$ one needs to invert the transform

Problem:
The “inversion” of $\Phi(\sigma)$ may be problematic

$$\Phi(\sigma) = \int d\omega \ K(\omega,\sigma) \ S(\omega)$$

One IS NOT able to calculate $S(\omega)$
(the quantity of direct physical meaning)
but IS able to calculate $\Phi(\sigma)$
In many physical problems the difficult quantity $S(\omega)$ is

$$S_{ab}(\omega) = \sum_n \langle a \mid n \rangle \langle n \mid b \rangle \delta(E_n - \omega)$$

$$H \mid n \rangle = E_n \mid n \rangle$$

$$\langle a \mid a \rangle, \langle b \mid b \rangle < \infty$$
In many physical problems the difficult quantity $S(\omega)$ is

$$S_{ab}(\omega) = \sum_{n} <a|n><n|b> \delta(E_n - \omega)$$

Scattering states

Energies in the continuum

$$H|n> = E_n|n>$$

$$<a|a>, <b|b> < \infty$$
In many physical problems the difficult quantity $S(\omega)$ is

$$S_{ab}(\omega) = \sum_n \langle a | n \rangle \langle n | b \rangle \delta(E_n - \omega)$$

Using

$$\lim_{\eta \to 0} \eta (x - \alpha - i\eta)^{-1} = \mathcal{P}(x - \alpha - i\eta)^{-1} + i \pi \delta(x - \alpha)$$

and closure

$$\sum_n |n\rangle \langle n| = I$$

$$S_{ab}(\omega) = \frac{1}{\pi} \text{Im} \left\{ \langle a | \frac{1}{(H - \omega - i\eta)} | b \rangle \right\}$$
In perturbation induced inclusive reactions the cross section is proportional (Fermi Golden Rule)

\[ |a\rangle = |b\rangle = \Theta |0\rangle \]

\[ S(\omega) = \sum_n |\langle n | \Theta | 0 \rangle|^2 \delta(\omega - E_n + E_0) \]
\[ H \left| n \right> = E_n \left| n \right> \]
In non perturbative reactions the crucial quantity is

\[ S_{\alpha\beta}(\omega) = \sum_n <\phi_\beta|n>|n|\phi_\alpha>\delta(E_n - \omega) \]

\[ \phi_{\alpha,\beta} = \mathcal{V}_{\alpha,\beta} \chi_{\alpha,\beta} \]

\( \mathcal{V} \) is the sum of the potentials between particles belonging to different fragments

\[ \chi = \text{“channel functions” (with proper antisymmetrization)} \]
$H$ is the Hamiltonian of the 8-body system

Lithium-6 (Li-6) $\nu_{\alpha}$ Deuterium (H-2)

$\nu$\n
Alpha Particle (He-4) $\nu_{\beta}$ Alpha Particle (He-4)

Lithium-6 – Deuterium Reaction
In non-perturbative reactions the crucial quantity is

\[ S_{\alpha\beta}(\omega) = \sum_n <\phi_\beta | n > < n | \phi_\alpha > \delta(E_n - \omega) \]

\( \chi \) = “channel functions” (with proper antisymmetrization)

\[ \phi_{\alpha,\beta} = \mathcal{V}_{\alpha,\beta} \chi_{\alpha,\beta} \]

\( \mathcal{V} \) is the sum of the (short range) potentials between particles belonging to different fragments

**non trivial part**

\[ T_{\beta\alpha}(E) = <\phi_\beta | \chi_\alpha > + <\phi_\beta | (E - H + i\eta)^{-1} | \phi_\alpha > \]

**trivial part**

\[ \mathcal{P} \int d
\mathcal{V} (E - \omega + i\eta)^{-1} S_{\alpha\beta}(\omega) - i\pi S_{\alpha\beta}(E) \]

V.D.Efros 1985
In perturbation induced ($\Theta$) inclusive reactions the crucial quantity is

\[ S(\omega) = \sum_n |<n|\Theta|0>|^2 \delta (E_n - E_0 - \omega) \]

In non perturbative reactions the crucial quantity is

\[ S_{\alpha\beta}(\nu) = \sum_n <\phi_\beta|n><n|\phi_\alpha> \delta (E_n - \nu) \]
\[ S(\omega) = \sum_n |\langle n|\Theta|0\rangle|^2 \delta(\omega - E_n + E_0) \]

\[ \Phi(\sigma) = \int S(\omega) K(\omega,\sigma) \, d\omega = \]

1) integrate in \( d\omega \) using delta function

2) Use \( \sum_n |n > < n| = I \)
\[ \Phi (\sigma) = \int S(\omega) K(\omega, \sigma) \, d\omega = \left\langle 0 \mid \Theta^+ K(H-E_0, \sigma) \Theta \mid 0 \right\rangle \]
The calculation of *ANY* transform seems to require, *in principle*, only the knowledge of the ground state!

\[
\Phi(\sigma) = \int S(\omega) K(\omega, \sigma) \, d\omega = \langle 0 | \Theta^+ K(\mathcal{H}-E_0, \sigma) \Theta | 0 \rangle
\]
The calculation of **ANY** transform seems to require, *in principle*, only the knowledge of the ground state!

However,

\[ K(H-E_0,\sigma) \] can be quite a complicate operator.

So, how to calculate this mean value?

\[
\Phi(\sigma) = \langle 0 | \Theta^+ K(H-E_0,\sigma) \Theta | 0 \rangle
\]
If we had to deal with a “confined” system one could represent $H$ on **bound states eigenfunctions** $|\nu>$

$$\langle 0| \Theta^+ K(H-E_0, \sigma) \Theta |0 \rangle =$$

$$\sum_{\mu \nu} \langle 0| \Theta^+ |\mu \rangle \langle \mu | K(H_{\mu \nu} - E_0, \sigma) |\nu \rangle |\nu \rangle \Theta |0 \rangle$$
If we had to deal with a “confined” system one could represent $H$ on bound states eigenfunctions $|\nu>$

$$\langle 0|\Theta^+ K(H-E_0,\sigma) \Theta|0\rangle = \sum_{\mu\nu} \langle 0|\Theta^+ |\mu\rangle \langle \mu| K(H_{\mu\nu} -E_0,\sigma)|\nu\rangle \langle \nu| \Theta|0\rangle$$

After diagonalizing $H_{\mu\nu}$ the transform would be simply

$$\sum_{\lambda} K(\varepsilon_{\lambda} - E_0,\sigma) \left| \langle \lambda| \Theta|0\rangle \right|^2$$
If we had to deal with a "confined" system one could represent $H$ on bound states eigenfunctions $|\nu>$. 

$$\langle 0 | \Theta^+ K(H-E_0, \sigma) \Theta | 0 \rangle =$$

$$\sum_{\mu \nu} \langle 0 | \Theta^+ | \mu \rangle \langle \mu | K(H_{\mu \nu} - E_0, \sigma) | \nu \rangle \langle \nu | \Theta | 0 \rangle$$

After diagonalizing $H_{\mu \nu}$ the transform would be simply

$$\sum_{\lambda} K(\varepsilon_\lambda - E_0, \sigma) | \langle \lambda | \Theta | 0 \rangle |^2$$

(Up to convergence!)
However, a nucleus is NOT “confined”!
The nuclear $H$ has positive energy eigenstates and therefore, in general, CANNOT be represented on b.s. eigenfunctions $|\nu>$

*(Continuum discretization approximation)*
THE GOOD NEWS:

The representation of \( H \) on \textbf{finite norm functions} \( |\nu> \) and therefore the calculation of the transform via

\[
\Phi(\sigma) = \sum_\lambda K(\epsilon_\lambda - E_0, \sigma) |\lambda| \Theta |0> |^2
\]

is \textbf{allowed} for \textbf{specific kernels} \( K(\omega, \sigma) \)!

\textbf{No approximation!}
Conditions required:

1) \( \int S(\omega) \, d\omega < \infty \) \implies \( \int S(\omega) \, d\omega = \langle 0 | \Theta^+ \Theta | 0 \rangle \)

2) \( K(\omega, \sigma) \) is a real positive definite function (or linear combination)

3) \( \Phi(\sigma) = \int S(\omega) K(\omega, \sigma) \, d\omega < \infty \)
In fact: if $K(\omega,\sigma)$ is a real positive definite function

$$K(\omega,\sigma) = \kappa^* (\omega,\sigma) \kappa (\omega,\sigma)$$
In fact: if $K(\omega, \sigma)$ is a real positive definite function

$$K(\omega, \sigma) = \kappa^*(\omega, \sigma)\kappa(\omega, \sigma)$$

$$\Phi(\sigma) = \langle 0 | \Theta^+ \kappa^+(H-E_0, \sigma) \kappa(H-E_0, \sigma) \Theta | 0 \rangle$$

$$\langle \tilde{\Psi} | \tilde{\Psi} \rangle$$
In fact: if $K(\omega,\sigma)$ is a real positive definite function

$$K(\omega,\sigma) = \kappa^*(\omega,\sigma)\kappa(\omega,\sigma)$$

$$\Phi(\sigma) = \langle 0|\Theta^+\kappa^+(H-E_0,\sigma)\kappa(H-E_0,\sigma)\Theta|0\rangle \quad < \infty \quad \text{(see req.N.3)}$$

$|\tilde{\Psi}\rangle$ has finite norm and therefore

**can be** expanded on **b.s.** functions !!
Moreover, if $\Theta |0\rangle$ has finite norm:

(see condition N.1)
\[ \Phi (\sigma) = \sum_{\lambda} K(\varepsilon_{\lambda} - E_0, \sigma) |\langle \lambda | \Theta | 0 \rangle|^2 \]
Summarizing:

Any integral transform
\[ \Phi(\sigma) = \int d\omega \ K(\omega,\sigma) \ S(\omega) \]

of a structure function \( S(\omega) \) such that

1) \( \int S(\omega) \ d\omega < \infty \)

And with a kernel \( K(\omega,\sigma) \) such that

2) \( K(\omega,\sigma) \) is a real positive definite function
(or linear combination)

3) \( \Phi(\sigma) = \int S(\omega) \ K(\omega,\sigma) \ d\omega < \infty \)
... can be calculated by diagonalizing the H matrix represented on finite norm functions

\[ \Phi(\sigma) = \sum_{\lambda} K(\varepsilon_{\lambda} - E_0, \sigma) |\langle \lambda | \Theta | 0 \rangle|^2 \]

(Upto convergence!)
A side remark on the notation: in

$$\Phi (\sigma) = \int d\omega \ K(\omega,\sigma) \ S(\omega)$$

$\sigma$ can also indicate a set of parameters $\sigma_1, \sigma_2, \ldots$. 
Some examples:

- **Fourier Transform? NO!** the kernel $\text{Exp} \left( i \omega \sigma \right)$ is not a real function
  (in this case $\sigma$ represents the real time $t$)
Some examples:

- **Fourier** Transform?  NO! the kernel $\text{Exp} \,(i \, \omega \, \sigma)$ is not a real function
  ( in this case $\sigma$ represents the real time $t$ )

- **Laplace** transform?  YES! the kernel $\text{Exp}(- \, \omega \, \sigma)$ is real and $\Phi \,(\sigma) < \infty$
  ( in this case $\sigma$ represents the imaginary time $\tau = i t$, is generally evaluated with MC methods)
Some examples:

- **Fourier** Transform? **NO!** the kernel $\text{Exp} (i \omega \sigma)$ is not a real function
  (in this case $\sigma$ represents the real time $t$)

- **Laplace** transform? **YES!** the kernel $\text{Exp}(-\omega \sigma)$ is real and $\Phi (\tau) < \infty$
  (in this case $\sigma$ represents the imaginary time $\tau = it$, is generally evaluated with MC methods)

- **Stieltjes** transform? **YES!** the kernel: $1 / (\omega + \sigma)$

Some examples:

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- **Laplace** transform? YES! the kernel $Exp(-\omega \sigma)$ is real and $\Phi (\tau) < \infty$
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- **Stieltjes** transform? YES! the kernel: $1 / (\omega + \sigma)$

- **Lorentz** transform? YES! the kernel: $[ (\omega - \sigma_1)^2 + \sigma_2^2 ]^{-1}$
Some examples:

- **Fourier Transform?** NO! the kernel $\exp(\imath \omega \sigma)$ is not a real function.
  (in this case $\sigma$ represents the real time $t$)

- **Laplace transform?** YES! the kernel $\exp(-\omega \sigma)$ is real and $\Phi(\tau) < \infty$.
  (in this case $\sigma$ represents the imaginary time $\tau = \imath t$, is generally evaluated with MC methods)

- **Stieltjes transform?** YES! the kernel $1/(\omega + \sigma)$

- **Lorentz transform?** YES! the kernel: $\left[(\omega - \sigma_1)^2 + \sigma_2^2\right]^{-1}$

- **Sumudu transform?** YES! the kernel: $\left(e^{-\mu \omega/\sigma_1} - e^{-\nu \omega/\sigma_1}\right)^{\sigma_2}$
  it has been evaluated with MC methods
Some examples:

.....

- *Moment* transform? YES or NO! The Kernel $\omega^\sigma$ ($\sigma$ integer)

  is a real positive definite function, however, $\Phi (\sigma)$ may be $\infty$ for some $\sigma$
Some examples:

.....

- Moment transform? YES or NO! The Kernel $\omega^{\sigma}$ ($\sigma$ integer)
  is a real positive definite function, however, $\Phi (\sigma)$ may be $\infty$ for some $\sigma$

- Other kernels ???
Which is the best kernel?
Let's remember:

\[ \Phi (\sigma) = \int d\omega \ K(\omega,\sigma) \ S(\omega) \]

In order to obtain \( S(\omega) \) one needs to invert the transform.

Problem:

Sometimes the “inversion” of \( \Phi (\sigma) \) may be problematic.
The Laplace Kernel:

\[ \Phi (\sigma) = \int e^{-\omega \sigma} S(\omega) \, d\omega \]

In Condensed Matter Physics  In QCD  In Nuclear Physics

\( \sigma = \tau = \) it imaginary time!

\( \Phi (\tau) \) is calculated with Monte Carlo Methods
and then inverted with methods based on Bayesian theorem (MEM)
\[ \Phi (\sigma) = \int d\omega \ e^{-\omega \sigma} S(\omega) \]

It is well known that the numerical inversion of the Laplace Transform can be problematic.
Illustration of the problem:

Laplace transform
Illustration of the problem:

\[ S \]

\[ \Phi \]

Laplace transform

Numerical errors
Illustration of the problem:

Laplace transform

Numerical errors
a “good” Kernel has to satisfy two requirements

1) one must be able to calculate the integral transform

2) one must be able to invert the transform minimizing uncertainties
The Lorentz kernel:

$$K(\omega, \sigma_1, \sigma_2) = \left[ (\omega - \sigma_1)^2 + \sigma_2^2 \right]^{-1}$$

It is a representation of the $\delta$-Function!

$$\Phi(\sigma_1, \sigma_2) = \int \left[ (\omega - \sigma_1)^2 + \sigma_2^2 \right]^{-1} S(\omega) \, d\omega$$
Notice:

\[ S(\omega) = \frac{1}{\pi} \text{Im} \left[ \langle 0 | \Theta^+ (H - E_0 - \omega - \imath \eta)^{-1} \Theta | 0 \rangle \right] \]

Green F. with poles on the real axis

\[ \Phi (\sigma_R, \sigma_I) = \frac{1}{\pi} \text{Im} \left[ \langle 0 | \Theta^+ (H - E_0 - \sigma_R - \imath \sigma_I)^{-1} \Theta | 0 \rangle \right] \]

Green F. with poles on the complex plane !!

(Of course, when \( \sigma_I = \eta \to 0 \) \( \Phi (\sigma_R, \eta) \) coincides with \( S(\omega = \sigma_R) \) !!)
How can one easily understand why the inversion is much less problematic?

Lorentz transform

blurred, but still distinguishable
How can one easily understand why the inversion is \textbf{much less} problematic?

\begin{itemize}
  \item Lorentz transform
  \item Numerical errors
  \item blurred, but still distinguishable also with errors!
\end{itemize}
How can one easily understand why the inversion is much less problematic?

Inversion: “regularization method” at fixed width
**Benchmark TEST** on the Triton:

$S(\omega)$ is the Dipole Photoabsorption Cross Section
Many successful applications

See reports:
V. D. Efros, W. Leidemann, G. Orlandini, N. Barnea
“The Lorentz Integral Transform (LIT) method and its applications to perturbation induced reactions”

W. Leidemann, G. Orlandini
“Modern ab initio approaches and applications in few-nucleon physics with $A \geq 4$”
Progress in Particle and Nuclear Physics 68 (2013) 158–214
6-Body total photodisintegration

Theory:
LIT+ EIHH

S. Bacca et al. PRL89(2002)052502

soft mode

classical GT mode
Exper. LIT of the photoabsorption cross section of $^{16}$O

$\sigma_i = 10$ [MeV]

+ Ahrens et al. 1972

shaded area: LIT of data (conserving the total area)
LIT of the photoabsorption cross section of $^{16}\text{O}$

$\sigma_i = 10 \text{ [MeV]}$

CCSD with N3LO

N=10
N=14
N=18

shaded area: LIT of data (conserving the total area)
Comparison between the Exp. and Theor. LIT's

S. Bacca, N. Barnea, G. Hagen, G.O., T. Papenbrock
Phys.Rev.Lett. 111 122502 (1913)
LIT of $^3$He photodisintegration at different width $\sigma_I$ of the Lorentzian kernel.

Remember

$$\Phi(\sigma) = \sum_{\lambda} K(\epsilon_{\lambda}, \sigma_R, \sigma_I) |\langle \lambda | \Theta | 0 \rangle|^2$$

30 hyperspherical
31 hyperradial
$\Rightarrow$ 930 basis states

$b = 0.6 \text{ fm}$
A proper change of basis allows energies (peaks) in the interesting region

\[ ^3\text{He} + \gamma \longrightarrow d + p \]

W. Leidemann

dashed curves: error estimate of inversion
The Stieltjes Kernel:

$$K(\omega, \sigma) = (\omega + \sigma)^{-1}$$
Illustration of the problem: 
Same as Laplace!
However, it may be useful for another purpose:
In fact:

\[ \lim_{\sigma \to 0} \Phi(\sigma) = \int S(\omega) \omega^{-1} \, d\omega = 2\alpha_\Theta \]

“generalized polarizability”

e.g. electric polarizability, magnetic susceptibility, compressibility etc... depending on $\Theta$
Recent results on $\alpha_\text{D}$ with $\text{D} = \text{D}$

(El. Dipole Polarizability)
Electric Dipole Polarizability as limit of the Stieltjes transform for $\sigma \rightarrow 0$

$\Phi_D(\sigma)$

$^{16}\text{O}$

M. Miorelli et al. nucl.th-arXiv 1604-05381

b.s. expansion: Coupled Cluster

(non hermitian) Lanczos diagonalization
Interesting correlation with the proton charge radius

Role of 3b-force

G. Hagen et al.
New Kernels?
What about “wavelets”?
A wavelet Kernel is an oscillating function but with a "window".

It has 2 parameters:

- $\sigma_2$ drives the frequency of the oscillation
- $\sigma_1$ drives the position of the window over the $\omega$ range

**discrete**

**continuous**
A wavelet Kernel is an oscillating function but with a "window". It has 2 parameters:

\( \sigma_2 \) drives the frequency of the oscillation

\( \sigma_1 \) drives the position of the window over the \( \omega \) range

They combine the power of the Fourier Kernel (in detecting frequencies of oscillations) and the Lorentz Kernel (in picking the information around specific \( \omega \) ranges)
A wavelet Kernel is an oscillating function but with a "window".

It has 2 parameters:

\[ \sigma_1 \] drives the frequency of the oscillation

\[ \sigma_2 \] drives the position of the window over the \( \omega \) range

Since wavelets are orthonormal functions in principle their inversion is straightforward!

[ linear combination of \( \Phi(\sigma_1, \sigma_2) \) ]
A model study
(discrete wavelets)

Our model $S(\omega)$

A wavelet kernel

$$K(\omega, \sigma_1, \sigma_2)$$
Model $S(\omega)$ and reconstructed from wavelet transform: identical!

Francesco Turro
Another model study
(narrow resonance, discrete wavelets)

Our model $S(\omega)$

A wavelet kernel

$K(\omega, \sigma_1, \sigma_2)$
Model $S(\omega)$ and reconstructed from wavelet transform:

again identical!

Francesco Turro
Which information has been used to reconstruct $S(\omega)$???
Which information has been used to reconstruct $S(\omega)$?

Values of $K(\omega, \sigma_1, \sigma_2)$ with different widths:

$$\sigma_2 = \frac{1}{2^J}, \quad J = 1-5$$

Namely a lot of different resolutions up to $\sigma_2 = 0.03$!!!
This may not be possible with diagonalization in realistic cases!
Hyp. on smallest “resolution” (low density of $\varepsilon_{\lambda}$):

- Resonance width
- Wavelet widths
Hp. on smallest “resolution” (higher density of $\varepsilon_\lambda$):
conclusions

Integral transform approaches are very powerful ab initio methods for cross sections in the continuum
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  ab initio methods for cross sections in the
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- In many cases they reduce continuum problem in
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- Good kernels are representations of the delta-
  functions (the width of the delta-function representation
  represents the resolution of the problem)

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- Good kernels are representations of the delta-
  functions (the width of the delta-function representation
  represents the resolution of the problem)

- Wavelets may be an interesting alternative

- Still a lot to be explored !!!
Thank you for your attention!
non trivial part:

\[ \langle \phi_{\beta} \mid (E - H + i \eta)^{-1} \phi_{\alpha} \rangle = \]
non trivial part:

\[
< \phi_\beta | (E - H + i \eta)^{-1} \phi_\alpha > = \\
\]

Insert completeness of eigenstates \(|n>\) of \(H\):

\[
\sum_n |n><n| = 1 \\
\]

\[
= \sum_n < \phi_\beta | n > < n | (E - H + i \eta)^{-1} \phi_\alpha > = 
\]
non trivial part:

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\[ = \int d\omega \sum_n \delta (E_n - \omega) (E - \omega + i \eta)^{-1} \langle \phi_\beta | n > < n | \phi_\alpha \rangle = \]
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\[
= \int d\omega \sum_n \delta (E_n - \omega) (E - \omega + i \eta)^{-1} < \phi_\beta | n > < n | \phi_\alpha > =
\]

\[
= \int d\omega (E - \omega + i \eta)^{-1} F_{\alpha \beta} (\omega) =
\]
non trivial part:

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\[ = \int d\omega (E - \omega + i \eta)^{-1} F_{\alpha\beta}(\omega) = \]

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Insert completeness of eigenstates \(|n>\) of H: \[ \sum_n |n><n| = 1 \]

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\[ = \int d\omega \sum_n \delta (E_n - \omega) (E - \omega + i \eta)^{-1} < \phi_\beta | n > < n | \phi_\alpha > = \]

\[ = \int d\omega (E - \omega + i \eta)^{-1} F_{\alpha\beta}(\omega) = \]

\[ = \mathcal{P} \int d\omega (E - \omega + i \eta)^{-1} F_{\alpha\beta}(\omega) - i \pi F_{\alpha\beta}(E) \]

the problem reduces to calculate the function \( F_{\alpha\beta}(\omega) \)
\[ \Phi (\omega, \Gamma) = \int d\omega \left[ (\omega - \omega_0)^2 + \Gamma^2 \right]^{-1} \sum_n \delta (E_n - \omega) \langle \phi_\beta | n > < n | \phi_\alpha > \]

\[ = \left[ (E_n - \omega_0)^2 + \Gamma^2 \right]^{-1} \sum_n \langle \phi_\beta | n > < n | \phi_\alpha > = \]

\[ = \sum_n \langle \phi_\beta | \left[ E_n - \omega_0 - i\Gamma \right]^{-1} | n > < n | E_n - \omega_0 + i\Gamma \right]^{-1} | \phi_\alpha > = \]

\[ = \sum_n \langle \phi_\beta | \left[ H - \omega_0 - i\Gamma \right]^{-1} | n > < n | H - \omega_0 + i\Gamma \right]^{-1} | \phi_\alpha > = \]

\[ = \langle \phi_\beta | \left[ H - \omega_0 - i\Gamma \right]^{-1} | H - \omega_0 + i\Gamma \right]^{-1} | \phi_\alpha > = \]

\[ = \langle \psi_\beta | \psi_\alpha > \quad \text{where } \psi_\beta \text{ and } \psi_\alpha \text{ satisfy} \]

\[ [H - \omega_0 - i\Gamma] \psi_\alpha = | \phi_\alpha > \]

\[ [H - \omega_0 - i\Gamma] \psi_\beta = | \phi_\beta > \]