The QCD phase diagram in NJL-like models

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Phase diagram of strongly interacting matter:
From Lattice QCD to Heavy-Ion Collision Experiments,
27 November - 1 December, 2017
The modern sketch of HIC

Experiment
The picture of the heavy ion collision’s evolution

Figure 1: arXiv (nucl-th): 1304.3634
The QCD phase diagram

- chiral symmetry restoration (constituent quarks → current quarks);
- deconfinement;
The QCD phase diagram

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- deconfinement;

Do these transitions coincide?
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- deconfinement;

Do these transitions coincide?

Lattice QCD
Hands S. Contemp. Phys. 42, 209 [2001], $T_c = 0.17$ GeV (SU(2))
The Nambu-Jona-Lasinio model

The Lagrangian

\[ \mathcal{L}_{\text{NJL}} = \bar{q} (i\slashed{\partial} - \hat{m}_0 - \gamma_0 \mu) q + G_s \left[ (\bar{q}q)^2 + (\bar{q}i\gamma_5 \vec{\tau}q)^2 \right], \]

\( G_s \) the effective coupling strength,
\( \bar{q} \) and \( q \) - quark fields
\( \hat{m}_0 = \text{diag} (m_u^0, m_d^0) \), \( m_u^0 = m_d^0 \) - the current quark masses, \( \vec{\tau} \) - Pauli matrices SU(2).

We can:
The Nambu-Jona-Lasinio model

The Lagrangian

\[ \mathcal{L}_{\text{NJL}} = \bar{q} \left( i \partial_\tau - \hat{m}_0 - \gamma_0 \mu \right) q + G_s \left[ (\bar{q}q)^2 + (\bar{q}i \gamma_5 \vec{\tau} q)^2 \right] , \]

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We can:

- explain and describe spontaneous chiral symmetry broken as \( m_q = m_0 + \langle \bar{q}q \rangle \);
The Nambu-Jona-Lasinio model

The Lagrangian

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\( G_s \) the effective coupling strength, \( \bar{q} \) and \( q \) - quark fields, \( \hat{m}_0 = \text{diag} \left( m_u^0, m_d^0 \right) \), \( m_u^0 = m_d^0 \) - the current quark masses, \( \vec{\tau} \) - Pauli matrices \( SU(2) \).

We can:

- explain and describe spontaneous chiral symmetry broken as \( m_q = m_0 + < \bar{q}q > \);
- build the phase diagram with crossover at low chemical potential and 1st order transition at high chemical potential (\( m_0 \neq 0 \)),

The Nambu-Jona-Lasinio model

The Lagrangian

\[ \mathcal{L}_{NJL} = \bar{q} \left( i \partial - m_{0} - \gamma_{0} \mu \right) q + G_{s} \left[ (\bar{q}q)^{2} + (\bar{q}i\gamma_{5} \vec{\tau}q)^{2} \right], \]

\( G_{s} \) the effective coupling strength,
\( \bar{q} \) and \( q \) - quark fields
\( \hat{m}_{0} = \text{diag} \left( m_{u}^{0}, m_{d}^{0} \right), \) \( m_{u}^{0} = m_{d}^{0} \) - the current quark masses, \( \vec{\tau} \) - Pauli matrices \( SU(2) \).

We can:

- explain and describe spontaneous chiral symmetry broken as \( m_{q} = m_{0} + \langle \bar{q}q \rangle \);
- build the phase diagram with crossover at low chemical potential and 1st order transition at high chemical potential \( (m_{0} \neq 0) \),
- no deconfinement(local model)

The NJL model + Polyakov-loop

\[ \mathcal{L}_{PNJL} = \bar{q} (i \gamma_\mu D^\mu - \hat{m}_0 - \gamma_0 \mu) q + G_s \left[ (\bar{q}q)^2 + (\bar{q}i \gamma_5 \tau q)^2 \right] - \mathcal{U} (\Phi, \bar{\Phi}; T) \]


\( q = (q_u, q_d) \) quark fields,
\( \hat{m}_0 = \text{diag} (m^0_u, m^0_d) \) - current quark masses, \( m^0_u = m^0_d = m_0 \)
\( D^\mu = \partial^\mu - iA^\mu \) - covariant derivative,
\( A^\mu(x) = g A^\mu_a \frac{\lambda_a}{2} \), \( A^\mu_a \) the gauge field \( SU(3) \),
\( A^\mu = \delta^\mu_0 A^0 = -i \delta^\mu_4 A_4 \),
\( \lambda_a \) - Gell-Mann matrices,
\( G_s \) - scalar coupling strength.

The Polyakov field \( \Phi \) is determined as:

\[ \Phi[A] = \frac{1}{N_c} \text{Tr}_c L(\vec{x}') \]

\[ L(\vec{x}) = \text{P} \exp \left[ i \int_0^\beta d\tau A_4(\vec{x}, \tau) \right] \]

\[ \langle L(\vec{x}') \rangle = e^{-\beta \Delta F_Q(\vec{x})} \]

Effective potential can be constructed with following considerations:

- it should reproduce the lattice calculation in pure gauge sector
- it should reproduce \( \mathbb{Z}_3 \) symmetry breaking
The effective potential

Polynomial fit:

\[
\frac{U(\Phi, \bar{\Phi}; T)}{T^4} = -\frac{b_2(T)}{2} \bar{\Phi} \Phi - \frac{b_3}{6} (\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4} (\bar{\Phi} \Phi)^2,
\]

\[
b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3.
\]
The effective potential

Polynomial fit:

\[
\frac{U(\Phi, \bar{\Phi}; T)}{T^4} = -\frac{b_2(T)}{2}\Phi\bar{\Phi} - \frac{b_3}{6}(\Phi^3 + \bar{\Phi}^3) + \frac{b_4}{4}(\Phi\bar{\Phi})^2,
\]

\[b_2(T) = a_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2 + a_3 \left(\frac{T_0}{T}\right)^3.\]

Logarithmic fit:

\[
\frac{U(\Phi, \bar{\Phi}; T)}{T^4} = -\frac{1}{2}a(T)\Phi\bar{\Phi} + b(T)\ln \left[1 - 6\Phi\bar{\Phi} + 4(\Phi^3 + \bar{\Phi}^3) - 3(\Phi\bar{\Phi})^2\right],
\]

\[a(T) = \tilde{a}_0 + \tilde{a}_1 \left(\frac{T_0}{T}\right) + \tilde{a}_2 \left(\frac{T_0}{T}\right)^2, \quad b(T) = \tilde{b}_3 \left(\frac{T_0}{T}\right)^3.\]
The effective potential parametrization

\[ \Phi \rightarrow 1, \frac{p}{T^4} \rightarrow 1.75, \text{ where } T \rightarrow \infty \]

\[ \Rightarrow \tilde{a}_0 = 3.51 \text{ for logarithmic fit} \]

\[ 1.75 = \frac{a_0}{2} + \frac{b_3}{3} - \frac{b_4}{4} \text{ for polynomial fit} \]

\[ \frac{\partial U(\Phi, \bar{\Phi}, T)}{\partial \Phi} |_{\mu = 0} (\Phi = \bar{\Phi} \text{ at } \mu = 0) \Rightarrow \text{the mean square method } \Rightarrow a_i, b_i \]

A. V. Friesen et al., IJMP A27, 1250013 (2012)

### Parameters

<table>
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<th></th>
<th>(\tilde{a}_0)</th>
<th>(\tilde{a}_1)</th>
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Symmetries restoration and breaking: $Z_3$ symmetry

Effective potential and Polyakov loop field $\Phi$ for different temperatures.

The mean-field approximation

We can introduce the partition function

\[ \mathcal{Z}[\bar{q}, q] = \int \mathcal{D}\bar{q}\mathcal{D}q \exp \left\{ \int_0^\beta d\tau \int_v d^3x \left[ \mathcal{L}_{\text{NJL(PNJL)}} \right] \right\}. \]  

(1)

Then, using the mean-field approximation procedure, we get PNJL grand potential \((N_f = 2)\):

\[
\Omega(\Phi, \bar{\Phi}, m, T, \mu) = U(\Phi, \bar{\Phi}; T) + G\langle \bar{q}q \rangle^2 - 2N_cN_f \int \frac{d^3p}{(2\pi)^3} E_p - 2N_fT \int \frac{d^3p}{(2\pi)^3} \left[ \ln N^+_\Phi(E_p) + \ln N^-_{\Phi}(E_p) \right],
\]

where \(N^\pm_\Phi(E_p) = \left[ 1 + 3 \left( \Phi + \bar{\Phi} e^{-\beta E_p^\pm} \right) e^{-\beta E_p^\pm} + e^{-3\beta E_p^\pm} \right]\)

and \(E_p = \sqrt{p^2 + m^2}\) - quark energy; \(E_p^\pm = E_p \mp \mu\). The equations of motion

\[
\frac{\partial \Omega_{\text{MF}}}{\partial \sigma_{\text{MF}}} = 0, \quad \frac{\partial \Omega_{\text{MF}}}{\partial \Phi} = 0, \quad \frac{\partial \Omega_{\text{MF}}}{\partial \bar{\Phi}} = 0.
\]
Symmetries restoration and breaking: chiral symmetry

Crossover transition

\[ \frac{\partial < \bar{q}q >}{\partial T} \bigg|_{\mu = \text{const}} \]

1\textsuperscript{st} order transition: the quark susceptibility

\[ \frac{\chi_q(T, \mu)}{T^2} = \frac{\partial^2 (p/T^4)}{\partial (\mu/T)^2} \]
Phase diagram of PNJL model

Parameters: $m_0 = 5.5$ MeV, $\Lambda = 0.639$ GeV, $G_s = 5.5$ GeV$^{-2}$, $a_i, b_i, T_0 = 0.27$ GeV
PNJL with vector interaction

Introduction of vector interaction into model

\[
\mathcal{L}_{\text{PNJL}} = \bar{q} (i\gamma_\mu D^\mu - \hat{m}_0 - \gamma_0 \mu) q + G_s \left[ (\bar{q}q)^2 + (\bar{q}i\gamma_5 \vec{\tau}q)^2 \right] - G_v (\bar{q}\gamma_\nu q)^2 - U(\Phi, \bar{\Phi}; T)
\]

leads to re-normalization of chemical potential:

\[
\tilde{\mu} = \mu - 4G_v N_c N_f \int_\Lambda \frac{d^3 p}{(2\pi)^3} \frac{m}{E_p} \left[ f_\Phi^+ + f_\Phi^- \right].
\]

\[T_0 = 0.19 \text{ GeV}\]
Extended PNJL
It is possible to introduce a phenomenological dependence of $G_s(\Phi)$ and $G_v(\Phi)$:

$$
\tilde{G}_s(\Phi) = G_s[1 - \alpha_1 \Phi \Phibar - \alpha_2 (\Phi^3 + \Phibar^3)],
$$

$$
\tilde{G}_v(\Phi) = G_v[1 - \alpha_1 \Phi \Phibar - \alpha_2 (\Phi^3 + \Phibar^3)],
$$

with $\alpha_1 = \alpha_2 = 0.2$. (Y. Sakai et al PRD 82, 076003 (2010), P. de Forcrand, O. Philipsen NPB 642, 290(2002))

A. V. Friesen et al. IJMP A30 1550089 (2015)
Crossover curvature

It was suggested that critical curves for all physical quantities (chiral condensate, quark susceptibility, strange quark susceptibility, Polyakov loop) must meet at one point, which is the CEP (Kaczmarek O. et al. PRD 83, 014504 (2011)).

\[
\frac{T_c(\mu)}{T_c(0)} = 1 - k \left( \frac{\mu}{T_c(\mu)} \right)^2.
\]

Enrödi G. JHEP (4) 1, 2011; Cea P. PRD89, 074512 (2014)
The mesons correlations

The partition function:

\[
\mathcal{Z}[\bar{q}, q] = \int \mathcal{D}q \mathcal{D}\bar{q} \exp \left\{ \int_0^\beta d\tau \int_V d^3x \left[ \mathcal{L}_{NJL(PNJL)} \right] \right\}.
\]

During the mean field approximation procedure we neglect the part of integral which is responsible for correlations:

\[
\mathcal{Z}_{FL}[T, V, \mu] = \int \mathcal{D}\sigma \mathcal{D}\bar{\pi} \exp \left\{ - \left[ \int_0^\beta d\tau \int_V d^3x \frac{2\sigma\sigma_{MF} + \sigma^2 + \bar{\pi}^2}{4G_s} \right] + \text{Tr} \ln \left[ 1 - S_{MF}[m] (\sigma + i\gamma_5 \bar{\tau}\bar{\pi}) \right] \right\},
\]

which eventually can be rewritten as

\[
\mathcal{Z}_{FL}^{(2)}[T, V, \mu] = \left[ \det \left( D^{-1}_\sigma \right) \right]^{-\frac{1}{2}} \left[ \det \left( D^{-1}_\pi \right) \right]^{-\frac{3}{2}}
\]

with meson propagator

\[
D_M^{-1} = \frac{1}{2G_s} - \Pi_M(q_0, \vec{q}).
\]
The meson pressure:

\[
\Omega_M^{(2)}(T, \mu) = -\frac{N_M}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d\omega}{2\pi} f_B(\omega) \Phi_M(\omega, \vec{p}) = \\
-\frac{N_M}{2} \int \frac{d^3q}{(2\pi)^3} \int_0^{+\infty} \frac{d\omega}{\pi} \frac{d\omega}{d\omega} \left[ -\omega + T \ln[1 - e^{\beta(\omega-\mu)}] + T \ln[1 - e^{\beta(\omega+\mu)}] \right] \frac{d\Phi_M(\omega, \vec{q})}{d\omega}
\]

The phase shift

\[
\Phi_M = \frac{1}{2i} \ln S_M(\omega, \vec{q}) = \frac{1}{2i} \ln \frac{1 - 2G_s \Pi_M(\omega - i\eta, \vec{q})}{1 - 2G_s \Pi_M(\omega + i\eta, \vec{q})}.
\]

The phase shift is determined by polarization operator. In case pole approximation

\[
1 - 2G_s \Pi_M(\omega, \vec{q}) = (\omega^2 - E_M^2) \times g_{Mqq}^2,
\]

we have noninteracting mesons gas

\[
\Omega_M = \frac{N_M}{2} \int \frac{d^3q}{q} \left[ E_M + T \ln[1 - e^{-\beta(E_M-\mu)}] + T \ln[1 - e^{-\beta(E_M+\mu)}] \right].
\]

By introducing the width of mesons, we can take into account meson correlations

\[
\frac{d\Phi_R(s, T)}{ds} = A_R(s, T) = a_R \frac{M_M \Gamma_M}{(s - M_M)^2 + (M_M \Gamma_M)^2} = \frac{\pi}{\frac{\pi}{2} + \arctg \left( \frac{q^2 + M_M^2}{M_M \Gamma_M} \right)} \frac{M_M \Gamma_M}{(s - M_M)^2 + (M_M \Gamma_M)^2},
\]
D. Blaschke et al.: arXiv: 1612.09556; 1511.00338;
NJL-like models seems to be enough well to describe the structure of phase diagram; phase transition;
position of critical endpoint is parameters-dependent;
the drop-out of critical endpoint
- when vector interaction is included (PRD 73, 014019 (2006)) (exception nonlocal model D.Blaschke et al. 1207.4890)
other parameters which can effect on the phase diagram structure: flavours, electromagnetic field, correlations (more correct phase shift, feedback reaction)... etc
Thank you for attention