Phase diagram and thermodynamics from the VePQM model: how to improve the approximation?

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Collaborators: Zsolt Szép, György Wolf

results partially based on: PRD87 (2013) no.1, 014011; PRD93 (2016) no.11, 114014
Overview

1. Introduction
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2. The model
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   - Field equations
   - Meson masses
   - Parametrization at $T = 0$

4. Results
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   - Pressure and derived quantities
   - Critical endpoint
   - Isentropic curves

5. Summary
Envisaged phase diagram of QCD

Effective models help revealing the rich phase structure at large $\mu_B$. The success of an effective model depends on: d.o.f used, implemented resummation, parametrization of the model ...

- At $\mu_B = 0$ $T_c = 153(3)$ MeV
- Is there a CEP?
- The $T$-dependence of thermodynamical quantities like pressure, interaction measure, quark density is known from lattice only at $\mu_B = 0$.
- At which $\mu_B$ is there the phase boundary for $T = 0$?
- In medium changes of masses and widths

Details of the phase diagram are heavily studied theoretically (Lattice, EFT), and experimentally (RHIC, LHC, FAIR, NICA)
Motivation, chiral symmetry

Chiral symmetry, chiral models

If the quark masses are zero (chiral limit) \( \implies \) QCD invariant under the following global transformation (chiral symmetry):

\[
U(3)_L \times U(3)_R \simeq U(3)_V \times U(3)_A = SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A
\]

- \( U(1)_V \) \( \rightarrow \) baryon number conservation (exact symmetry of nature)
- \( U(1)_A \) \( \rightarrow \) connected to axial anomaly
- \( U(3)_L \times U(3)_R \rightarrow \) broken down to \( U(1)_V \times SU(2)_V \) if \( m_u = m_d \neq m_s \)
  \( \rightarrow \) or to \( U(1)_V \) if \( m_u \neq m_d \neq m_s \) (realized in nature)

Since QCD is very hard to solve \( \rightarrow \) low energy effective models \( \rightarrow \) reflecting the global symmetries of QCD \( \rightarrow \) degrees of freedom: observable particles instead of quarks and gluons

Linear realization of the symmetry \( \rightarrow \) linear sigma model
Vector meson extended Polyakov - Quark - Meson (VePQM) model

**Lagrangian I.**

$\mathcal{L}$ constructed based on linearly realized global $U(3)_L \times U(3)_R$ symmetry and its explicit breaking

$$
\mathcal{L} = \text{Tr}[(D_\mu \Phi)^\dagger(D_\mu \Phi)] - m_0^2 \text{Tr}(\Phi^\dagger \Phi) - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 \\
+ c_1 (\text{det} \Phi + \text{det} \Phi^\dagger) + \text{Tr}[H(\Phi + \Phi^\dagger)] - \frac{1}{4} \text{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) \\
+ \text{Tr} \left[ \left( \frac{m_1^2}{2} 1 + \Delta \right) (L_{\mu}^2 + R_{\mu}^2) \right] + i \frac{g_2}{2} (\text{Tr}\{L_{\mu\nu}[L^\mu, L^\nu]\} + \text{Tr}\{R_{\mu\nu}[R^\mu, R^\nu]\}) \\
+ \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_{\mu}^2 + R_{\mu}^2) + h_2 \text{Tr}[(L_{\mu} \Phi)^2 + (\Phi R_{\mu})^2] + 2 h_3 \text{Tr}(L_{\mu} \Phi R_{\mu} \Phi^\dagger) \\
+ \bar{\Psi} i \gamma_\mu D^\mu \Psi - g_F \bar{\Psi} (\Phi_S + i \gamma_5 \Phi_{PS}) \Psi,
$$

$$
D^\mu \Phi = \partial^\mu \Phi - ig_1 (L^\mu \Phi - \Phi R^\mu) - ieA^\mu_e [T_3, \Phi], \\
L^{\mu \nu} = \partial^\mu L^\nu - ieA^{\mu \nu}_e [T_3, L^\nu] - \{ \partial^\nu L^\mu - ieA^{\nu \mu}_e [T_3, L^\mu] \}, \\
R^{\mu \nu} = \partial^\mu R^\nu - ieA^{\mu \nu}_e [T_3, R^\nu] - \{ \partial^\nu R^\mu - ieA^{\nu \mu}_e [T_3, R^\mu] \}, \\
D^\mu \Psi = \partial^\mu \Psi - i G^\mu \Psi, \quad \text{with} \quad G^\mu = g_s G^\mu_a T_a.
$$

+ Polyakov loop potential
the matter and external fields are

\[ \Phi = \sum_{i=0}^{8} (\sigma_i + i\pi_i) T_i, \quad H = \sum_{i=0}^{8} h_i T_i \quad \text{for } T_i : U(3) \text{ generators} \]

\[ R^\mu = \sum_{i=0}^{8} (\rho^\mu_i - b^\mu_i) T_i, \quad L^\mu = \sum_{i=0}^{8} (\rho^\mu_i + b^\mu_i) T_i, \quad \Delta = \sum_{i=0}^{8} \delta_i T_i \]

\[ \Psi = (u, d, s)^T \]

non strange – strange base:

\[ \xi_N = \sqrt{2/3}\xi_0 + \sqrt{1/3}\xi_8, \]
\[ \xi_S = \sqrt{1/3}\xi_0 - \sqrt{2/3}\xi_8, \quad \xi \in (\sigma_i, \pi_i, \rho^\mu_i, b^\mu_i, h_i) \]

broken symmetry: non-zero condensates \( \langle \sigma_N/S \rangle \equiv \bar{\sigma}_N/S \)
**Mesonic particle content**

- **Vector** and **Axial-vector** meson nonets

  \[ V^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N + \rho^0}{\sqrt{2}} & \rho^+ & K^{*+} \\ \rho^- & \frac{\omega_N - \rho^0}{\sqrt{2}} & K^{*0} \\ K^{*-} & K^{*0} & \omega_S \end{pmatrix}^\mu \]

  \[ A^\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{f_{1N} + a_1^0}{\sqrt{2}} & a_1^+ & K_1^+ \\ a_1^- & \frac{f_{1N} - a_1^0}{\sqrt{2}} & K_1^0 \\ K_1^- & K_1^0 & f_{1S} \end{pmatrix}^\mu \]

  \( \rho \rightarrow \rho(770), K^* \rightarrow K^*(894) \)

  \( \omega_N \rightarrow \omega(782), \omega_S \rightarrow \phi(1020) \)

  \( a_1 \rightarrow a_1(1230), K_1 \rightarrow K_1(1270) \)

  \( f_{1N} \rightarrow f_1(1280), f_{1S} \rightarrow f_1(1426) \)

- **Scalar** (\( \sim \bar{q}_i q_j \)) and **pseudoscalar** (\( \sim \bar{q}_i \gamma_5 q_j \)) meson nonets

  \[ \Phi_S = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sigma_N + a_0^0}{\sqrt{2}} & a_0^+ & K_0^{*+} \\ a_0^- & \frac{\sigma_N - a_0^0}{\sqrt{2}} & K_0^{*0} \\ K_0^{*-} & K_0^{*0} & \sigma_S \end{pmatrix} \]

  \[ \Phi_{PS} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\eta_{N+} + \pi^0}{\sqrt{2}} & \pi^+ & K^+ \\ \pi^- & \frac{\eta_{N-} - \pi^0}{\sqrt{2}} & K^0 \\ K^- & K^0 & \eta_S \end{pmatrix} \]

  Unknown assignment mixing in the \( \sigma_N - \sigma_S \) sector

  \( \pi \rightarrow \pi(138), K \rightarrow K(495) \)

  Mixing: \( \eta_N, \eta_S \rightarrow \eta(548), \eta'(958) \)

  **Spontaneous symmetry breaking:** \( \sigma_{N/S} \) acquire nonzero expectation values \( \Phi_{N/S} \)

  Fields shifted by their expectation value: \( \sigma_{N/S} \rightarrow \sigma_{N/S} + \Phi_{N/S} \)
## Structure of scalar mesons below 2 GeV

<table>
<thead>
<tr>
<th></th>
<th>Mass (MeV)</th>
<th>width (MeV)</th>
<th>decays</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0(980)$</td>
<td>$980 \pm 20$</td>
<td>$50 - 100$</td>
<td>$\pi \pi$ dominant</td>
</tr>
<tr>
<td>$a_0(1450)$</td>
<td>$1474 \pm 19$</td>
<td>$265 \pm 13$</td>
<td>$\pi \eta, \pi \eta', K \bar{K}$</td>
</tr>
<tr>
<td>$K_s(800) = \kappa$</td>
<td>$682 \pm 29$</td>
<td>$547 \pm 24$</td>
<td>$K\pi$</td>
</tr>
<tr>
<td>$K_s(1430)$</td>
<td>$1425 \pm 50$</td>
<td>$270 \pm 80$</td>
<td>$K\pi$ dominant</td>
</tr>
<tr>
<td>$f_0(500) = \sigma$</td>
<td>$400-550$</td>
<td>$400 - 700$</td>
<td>$\pi \pi$ dominant</td>
</tr>
<tr>
<td>$f_0(980)$</td>
<td>$980 \pm 20$</td>
<td>$40 - 100$</td>
<td>$\pi \pi$ dominant</td>
</tr>
<tr>
<td>$f_0(1370)$</td>
<td>$1200-1500$</td>
<td>$200 - 500$</td>
<td>$\pi \pi \approx 250, K \bar{K} \approx 150$</td>
</tr>
<tr>
<td>$f_0(1500)$</td>
<td>$1505 \pm 6$</td>
<td>$109 \pm 7$</td>
<td>$\pi \pi \approx 38, K \bar{K} \approx 9.4$</td>
</tr>
<tr>
<td>$f_0(1710)$</td>
<td>$1722 \pm 6$</td>
<td>$135 \pm 7$</td>
<td>$\pi \pi \approx 30, K \bar{K} \approx 71$</td>
</tr>
</tbody>
</table>

Scalar $\bar{q}q$ nonet content: 1 $a_0$, 1 $K_0^*$, and 2 $f_0$s $\rightarrow$ 40 possible assignments

Result of a $T = 0$ parametrization: $a_0^{\bar{q}q} \rightarrow a_0(1450), K_0^{*, \bar{q}q} \rightarrow K_0^*(1430)$

Considering only $\bar{q}q$ states is somehow unrealistic because most probably scalars are mixtures of $\bar{q}q$, four-quark states and glueballs

$f_0(500), f_0(980), a_0(980), K_0^*(800)$ could be predominantly four-quark states

$f_0(1710)$ could be predominantly glueball
Features of our approach

- D.O.F’s: – scalar, pseudoscalar, vector, and axial-vector nonets
  - $u, d, s$ constituent quarks ($m_u = m_d$)
  - Polyakov loop variables $\Phi, \bar{\Phi}$ with $U_{\log}^{YM}$ or $U_{\log}^{glue}$

- no mesonic fluctuations in the grand potential, only fermionic ones

$$Z = e^{-\beta V \Omega(T, \mu_q)} = \int_{\text{PBC}} \Pi_a D\xi_a \int_{\text{APBC}} \Pi_f Dq_f D\bar{q}_f^\dagger \exp \left[ -\int_0^\beta d\tau \int_V d^3x \left( \mathcal{L} + \mu_q \sum_f q_f^\dagger q_f \right) \right]$$

approximated as $\Omega(T, \mu_q) = U_{\text{meson}}^{\text{tree}}(\langle M \rangle) + \Omega^{(0)}_{\bar{q}q}(T, \mu_q) + U_{\log}(\Phi, \bar{\Phi})$ with $\bar{\mu}_q = \mu_q - iG_4$

$$e^{-\beta V \Omega^{(0)}_{\bar{q}q}} = \int_{\text{APBC}} \Pi_{f,g} Dq_g D\bar{q}_f^\dagger \exp \left\{ \int_0^\beta d\tau \int_x q_f^\dagger \left[ (i\gamma_0 \vec{\gamma} \cdot \vec{\nabla} - \frac{\partial}{\partial \tau} + \bar{\mu}_q) \delta_{fg} - \gamma_0 M_{fg} \right]_{\xi_a=0} q_g \right\}$$

- quarks not coupled to the (axial)vectors $\rightarrow$ tree-level (axial)vector masses

- fermionic vacuum and thermal fluctuations included in the (pseudo)scalar curvature masses used to parameterize the model

- 4 coupled $T/\mu_B$-dependent field equations for condensates: $\phi_N, \phi_S, \Phi, \bar{\Phi}$

- thermal contribution of $\pi, K, f_0^L$ included in the pressure, however their curvature mass contains no mesonic fluctuations
Polyakov loops in Polyakov gauge

Polyakov loop variables: $\Phi(\vec{x}) = \frac{\text{Tr}_c L(\vec{x})}{N_c}$ and $\bar{\Phi}(\vec{x}) = \frac{\text{Tr}_c \bar{L}(\vec{x})}{N_c}$ with

$$L(\vec{x}) = \mathcal{P} \exp \left[ i \int_0^\beta d\tau G^4(\vec{x}, \tau) \right]$$

→ signals center symmetry ($\mathbb{Z}_3$) breaking at the deconfinement transition

low $T$: confined phase, $\langle \Phi(\vec{x}) \rangle, \langle \bar{\Phi}(\vec{x}) \rangle = 0$
high $T$: deconfined phase, $\langle \Phi(\vec{x}) \rangle, \langle \bar{\Phi}(\vec{x}) \rangle \neq 0$

- Polyakov gauge: $G^4(\vec{x}, \tau) = G^4(\vec{x}')$, plus gauge rotation to diagonal form in color space
- further simplification: $\vec{x}$-independence

$$L = e^{i\beta G^4} = \text{diag}(a, b, c) \left( \begin{array}{c} 1 \\ \in SU(3)_{\text{color}} \end{array} \right); \quad a, b, c \in \mathbb{Z}$$

→ use this to calculate partition function of free quarks on constant gluon background
Inclusion of the Polyakov loop modifies the Fermi-Dirac distribution function

\[
\begin{aligned}
f(E_p - \mu_q) &\rightarrow f_\Phi^+(E_p) = \\
&= \frac{(\bar{\Phi} + 2\Phi e^{-\beta(E_p-\mu_q)}) e^{-\beta(E_p-\mu_q)} + e^{-3\beta(E_p-\mu_q)}}{1 + 3 (\bar{\Phi} + \Phi e^{-\beta(E_p-\mu_q)}) e^{-\beta(E_p-\mu_q)} + e^{-3\beta(E_p-\mu_q)}} \\
f(E_p + \mu_q) &\rightarrow f_\Phi^-(E_p) = \\
&= \frac{(\Phi + 2\bar{\Phi} e^{-\beta(E_p+\mu_q)}) e^{-\beta(E_p+\mu_q)} + e^{-3\beta(E_p+\mu_q)}}{1 + 3 (\Phi + \bar{\Phi} e^{-\beta(E_p+\mu_q)}) e^{-\beta(E_p+\mu_q)} + e^{-3\beta(E_p+\mu_q)}}
\end{aligned}
\]

\(\Phi, \bar{\Phi} \rightarrow 0 \Rightarrow f_\Phi^\pm(E_p) \rightarrow f(3(E_p \pm \mu_q))\)

\(\Phi, \bar{\Phi} \rightarrow 1 \Rightarrow f_\Phi^\pm(E_p) \rightarrow f(E_p \pm \mu_q)\)

three-particle state appears: mimics confinement of quarks within baryons

at \(T = 0\) there is no difference between models with and without Polyakov loop:

\[\Theta(3(\mu_q - E_p)) \equiv \Theta((\mu_q - E_p))\]

H. Hansen et al., PRD75, 065004
VePQM model – Polyakov loop

Polyakov - loop potential

“Color confinement” \[ \langle \Phi \rangle = 0 \rightarrow \text{no breaking of } \mathbb{Z}_3 \]

“Color deconfinement” \[ \langle \Phi \rangle \neq 0 \rightarrow \text{spontaneous breaking of } \mathbb{Z}_3 \]

H. Hansen et al., PRD75, 065004 (2007)

Form of the potential:

- Polynomial: \( U_{\text{Poly}}^{\text{YM}} \)
- Logarithmic: \( U_{\text{YM}} \)
- Improved Polyakov loop potential (logarithmic): \( U_{\text{glue}} \)
Form of the potential

I.) Simple polynomial potential invariant under $\mathbb{Z}_3$ and charge conjugation: R.D. Pisarski, PRD 62, 111501

$$
\mathcal{U}_{\text{poly}}^{\text{YM}}(\Phi, \overline{\Phi}) = -\frac{b_2(T)}{2} \Phi \overline{\Phi} - \frac{b_3}{6} (\Phi^3 + \overline{\Phi}^3) + \frac{b_4}{4} (\Phi \overline{\Phi})^2
$$

with

$$
b_2(T) = a_0 + a_1 \frac{T_0}{T} + a_2 \frac{T_0^2}{T^2} + a_3 \frac{T_0^3}{T^3}
$$

II.) Logarithmic potential coming from the $SU(3)$ Haar measure of group integration

$$
\mathcal{U}_{\text{log}}^{\text{YM}}(\Phi, \overline{\Phi}) = -\frac{1}{2} a(T) \Phi \overline{\Phi} + b(T) \ln \left[ 1 - 6\Phi \overline{\Phi} + 4 (\Phi^3 + \overline{\Phi}^3) - 3 (\Phi \overline{\Phi})^2 \right]
$$

with

$$
a(T) = a_0 + a_1 \frac{T_0}{T} + a_2 \frac{T_0^2}{T^2}, \quad b(T) = b_3 \frac{T_0^3}{T^3}
$$

$\mathcal{U}^{\text{YM}}(\Phi, \overline{\Phi})$ models the free energy of a pure gauge theory
$\rightarrow$ the parameters are fitted to the pure gauge lattice data
VePQM model – Polyakov loop

Improved Polyakov loop potential

Previous potentials describe successfully the first order phase transition of the pure $SU(3)$ Yang–Mills

$\leftrightarrow$ taking into account the gluon dynamics (quark polarization of gluon propagator) $\rightarrow$ QCD glue potential

$\leftrightarrow$ can be implemented by changing the reduced temperature

$$
t_{\text{glue}} \equiv \frac{T - T_{\text{glue}}^c}{T_{\text{glue}}^c}, \quad t_{\text{YM}} \equiv \frac{T_{\text{YM}}^c - T_{\text{YM}}^c}{T_{\text{YM}}^c}
$$

$$
t_{\text{YM}}(t_{\text{glue}}) \approx 0.57 t_{\text{glue}}
$$

$$
\frac{U_{\text{glue}}}{T^4}(\Phi, \bar{\Phi}, t_{\text{glue}}) = \frac{U_{\text{YM}}}{(T_{\text{YM}})^4}(\Phi, \bar{\Phi}, t_{\text{YM}}(t_{\text{glue}}))
$$

L. M. Haas et al., PRD 87, 076004 (2013)
Four coupled field equations are obtained by extremizing the grand potential

\[ \Omega(T, \mu_q) = U_{\text{tree meson}}(\langle M \rangle) + \Omega_{\bar{q}q}^{(0)\text{vac}} + \Omega_{\bar{q}q}^{(0)T}(T, \mu_q) + U_{\log}(\Phi, \bar{\Phi}) \]

using \[ \frac{\partial \Omega_H}{\partial \phi_N} = \frac{\partial \Omega_H}{\partial \phi_S} = \frac{\partial \Omega_H}{\partial \Phi} = \frac{\partial \Omega_H}{\partial \bar{\Phi}} = 0 \quad E_f^\pm(p) = E_f(p) \mp \mu_q, \quad E_f^2(p) = p^2 + m_f^2 \]

1) \[ -\frac{1}{T^4} \frac{d}{d\Phi} U(\Phi, \bar{\Phi}) + \frac{6}{T^3} \sum_{f=u,d,s} \int \frac{d^3p}{(2\pi)^3} \left( \frac{e^{-\beta E_f^-(p)}}{g_f^-(p)} + \frac{e^{-2\beta E_f^+(p)}}{g_f^+(p)} \right) = 0 \]

2) \[ -\frac{1}{T^4} \frac{d}{d\Phi} U(\Phi, \bar{\Phi}) + \frac{6}{T^3} \sum_{f=u,d,s} \int \frac{d^3p}{(2\pi)^3} \left( \frac{e^{-\beta E_f^+(p)}}{g_f^+(p)} + \frac{e^{-2\beta E_f^-(p)}}{g_f^-(p)} \right) = 0 \]

3) \[ m_0^2 \phi_N + \left( \lambda_1 + \frac{\lambda_2}{2} \right) \phi_N^3 + \lambda_1 \phi_N \phi_S^2 - \frac{c_1}{\sqrt{2}} \phi_N \phi_S - h_0N + \frac{3}{2} g_F (\langle \bar{q}_u q_u \rangle_T + \langle \bar{q}_d q_d \rangle_T) = 0 \]

4) \[ m_0^2 \phi_S + (\lambda_1 + \lambda_2) \phi_S^3 + \lambda_1 \phi_N^2 \phi_S - \frac{\sqrt{2}}{4} c_1 \phi_N^2 - h_0S + \frac{3}{\sqrt{2}} g_F \langle \bar{q}_s q_s \rangle_T = 0 \]

renormalized fermion tadpole: \[ m_{u,d} = \frac{g_F}{2} \phi_N \quad \text{and} \quad m_s = \frac{g_F}{\sqrt{2}} \phi_S \]

\[ \langle \bar{q}_f q_f \rangle_T = 4m_f \left[ -\frac{m_f^2}{16\pi^2} \left( \frac{1}{2} + \ln \frac{m_f^2}{M_0^2} \right) + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_f(p)} (f^-(p) + f^+(p)) \right] \]
Curvature masses

It can be shown that curvature mass is the sum of the tree-level mass and the one-loop self-energy calculated at vanishing external momenta.

\[
M^2_{i,ab} = \left. \frac{\partial^2 \Omega(T, \mu_f)}{\partial \varphi_i, a \partial \varphi_i, b} \right|_{\text{min}} = m^2_{i,ab} + \Delta_0 m^2_{i,ab} + \Delta_T m^2_{i,ab},
\]

\[
m^2_{i,ab} \rightarrow \text{tree-level mass matrix},
\]

\[
\Delta_0/ T m^2_{i,ab} \rightarrow \text{fermion vacuum/thermal fluctuation},
\]

\[
\Delta_0 m^2_{i,ab} = \left. \frac{\partial^2 \Omega^{\text{vac}}_{qq}}{\partial \varphi_i, a \partial \varphi_i, b} \right|_{\text{min}} = -\frac{3}{8\pi^2} \sum_{f=u,d,s} \left[ \left( \frac{3}{2} + \log \frac{m_{f}}{M^2} \right) m^2_{f,a} m^2_{f,b} + m^2_{f} \left( \frac{1}{2} + \log \frac{m^2_{f}}{M^2} \right) m^2_{f,ab} \right],
\]

\[
\Delta_T m^2_{i,ab} = \left. \frac{\partial^2 \Omega^{\text{th}}_{qq}}{\partial \varphi_i, a \partial \varphi_i, b} \right|_{\text{min}} = 6 \sum_{f=u,d,s} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_f(p)} \left[ (f_f^+(p) + f_f^-(p)) \left( m^2_{f,ab} - \frac{m^2_{f,a} m^2_{f,b}}{2E^2_f(p)} \right) \right.
\]

\[
+ (B_f^+(p) + B_f^-(p)) \frac{m^2_{f,a} m^2_{f,b}}{2TE_f(p)} \right],
\]

where \( m^2_{f,a} \equiv \partial m^2_{f}/\partial \varphi_i, a, m^2_{f,ab} \equiv \partial^2 m^2_{f}/\partial \varphi_i, a \partial \varphi_i, b \).
Determination of the parameters

14 unknown parameters \((m_0, \lambda_1, \lambda_2, c_1, m_1, g_1, g_2, h_1, h_2, h_3, \delta_S, \Phi_N, \Phi_S, g_F)\) → determined by the min. of \(\chi^2\):

\[
\chi^2(x_1, \ldots, x_N) = \sum_{i=1}^{M} \left[ \frac{Q_i(x_1, \ldots, x_N) - Q_i^{\text{exp}}}{\delta Q_i} \right]^2,
\]

\((x_1, \ldots, x_N) = (m_0, \lambda_1, \lambda_2, \ldots), Q_i(x_1, \ldots, x_N) \rightarrow \text{from the model}, Q_i^{\text{exp}} \rightarrow \text{PDG value}, \delta Q_i = \max\{5\% \text{, PDG value}\}\)

multiparametric minimalization → MINUIT

- PCAC → 2 physical quantities: \(f_\pi, f_K\)
- Curvature masses → 16 physical quantities:
  \(m_{u/d}, m_s, m_\pi, m_\eta, m_\eta', m_K, m_\rho, m_\Phi, m_{K^*}, m_{a_1}, m_{f_1^H}, m_{K_1}, m_{a_0}, m_{K_{S}}, m_{f_0^L}, m_{f_0^H}\)
- Decay widths → 12 physical quantities:
  \(\Gamma_{\rho \to \pi \pi}, \Gamma_{\Phi \to KK}, \Gamma_{K^* \to K \pi}, \Gamma_{a_1 \to \pi \gamma}, \Gamma_{a_1 \to \rho \pi}, \Gamma_{f_1 \to KK^*}, \Gamma_{a_0}, \Gamma_{K_{S} \to K \pi}, \Gamma_{f_0^L \to \pi \pi}, \Gamma_{f_0^L \to KK}, \Gamma_{f_0^H \to \pi \pi}, \Gamma_{f_0^H \to KK}\)
- Pseudocritical temperature \(T_c\) at \(\mu_B = 0\)
Parametrization at $T = 0$

**Result of the parametrization**

- 40 possible assignments of scalar mesons to the scalar nonet states
- 3 values of $M_0$ are used $\implies$ 120 cases to investigate
  for each case $5 \cdot 10^4 - 10^5$ configurations are used for the $\chi^2$ minimization
- lowest $\chi^2$ obtained for $M_0 = 0.3$ GeV

  $\chi^2 = 18.57$ and $\chi^2_{\text{red}} \equiv \frac{\chi^2}{N_{\text{dof}}} = 1.16$

  assignment: $a_0^q \rightarrow a_0(980)$, $K^*_0, a_0(980), f_0^L, a_0(980), f_0^H, a_0(980)$

  problems: $m_{a_0} < m_{K^*_0}$, $m_{f_0^{H/L}}$ too light

- by minimizing also for $M_0$ we obtain using $U_{\text{YM}}^{\log}(\Phi, \bar{\Phi})$ with $T_0 = 182$ MeV:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$\phi_N$ [GeV]</td>
<td>0.1411</td>
<td>$g_1$</td>
<td>5.6156</td>
</tr>
<tr>
<td>$\phi_S$ [GeV]</td>
<td>0.1416</td>
<td>$g_2$</td>
<td>3.0467</td>
</tr>
<tr>
<td>$m_0^2$ [GeV$^2$]</td>
<td>$2.3925 \times 10^{-4}$</td>
<td>$h_1$</td>
<td>27.4617</td>
</tr>
<tr>
<td>$m_1^2$ [GeV$^2$]</td>
<td>$6.3298 \times 10^{-8}$</td>
<td>$h_2$</td>
<td>4.2281</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$-1.6738$</td>
<td>$h_3$</td>
<td>5.9839</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>23.5078</td>
<td>$g_F$</td>
<td>4.5708</td>
</tr>
<tr>
<td>$c_1$ [GeV]</td>
<td>1.3086</td>
<td>$M_0$ [GeV]</td>
<td>0.3511</td>
</tr>
<tr>
<td>$\delta_S$ [GeV$^2$]</td>
<td>0.1133</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The presented results are obtained with this set of parameters.
Temperature dependence of order parameters and masses

**Results**

- ch. partners \((\pi, f_0^L), (\eta, a_0)\) and \((K, K_0^*)\) become degenerate at high \(T\)
- \(U(1)_A\) not restored, axial partners \((\pi, a_0)\) and \((\eta, f_0^L)\) not become degenerate
**Mass pattern in the $\eta$, $\eta'$ sector**

- our pattern: $m_\eta \leq m_{\eta N} < m_{\eta S} \leq m_{\eta'}$ and similarly $m_{f_0} \leq m_{\sigma N} < m_{\sigma S} \leq m_{f_0}$ and also $a_0$ degenerates with $\eta$
- in contrast to the pattern obtained w/o the inclusion of (axial)vector mesons
  
  Schaefer & Wagner, PRD79, 014018 (QM) and Tiwari, PRD88, 074017 (PQM)

- in the FRG study of Rennecke & Schaefer, arXiv:1610.08748 (w/o (axial)vector mesons)
  
  - LPA: $a_0$-meson degenerates with $\eta'$-meson
  - LPA'+Y: $a_0$-meson degenerates with $\eta$-meson
Calculation of thermodynamical quantities

pressure: \( p(T, \mu_q) = \Omega_H(T = 0, \mu_q) - \Omega_H(T, \mu_q) \)

entropy density: \( s = \frac{\partial p}{\partial T} \), quark number density: \( \rho_q = \frac{\partial p}{\partial \mu_q} \)

energy density: \( \epsilon = -p + Ts + \mu_q \rho_q \), speed of sound: \( c_s^2 = \frac{\partial p}{\partial \epsilon} \),

We include mesonic thermal 1-loop contribution to the pressure:

\[ p_{\text{meson}} = -\Omega^{1\text{-loop}, T}_{\text{meson}} = -n_b T \int \frac{d^3 q}{(2\pi)^3} \ln \left( 1 - e^{-\beta E_b(q)} \right) \]

where, \( E_b(q) = \sqrt{q^2 + m^2_b} \), meson multiplicities: \( n_\pi = 3, n_K = 4, n_{\pi^0} = 1 \)

comparing with the lattice \( \longrightarrow \)

subtracted condensate: \( \Delta_I, S = \frac{\Phi_N - \frac{h_N}{h_S} \cdot \Phi_S \big|_T}{\Phi_N - \frac{h_N}{h_S} \cdot \Phi_S \big|_{T=0}} \)

scaled interaction measure: \( I/T^4 = (\epsilon - 3p)/T^4 \)
Pressure and derived quantities

**t-dependence of the condensates compared to lattice results**

- **The subtracted chiral condensate**

\[
\Delta l, s = \left. \left( \Phi_N - \frac{h_N}{h_S} \cdot \Phi_S \right) \right|_T
- \left. \left( \Phi_N - \frac{h_N}{h_S} \cdot \Phi_S \right) \right|_T = 0
\]

- **lattice result shows a very smooth transition**
- **our result is completely off**
- **renormalization of the Polyakov loop could explain part of the discrepancy**

**Andersen et al., PRD92, 114504**

**Polyakov loop expectation values**
Normalized pressure and the effects of meson contributions

- we used $U_{\text{glue}}$ with $T_{c}^{\text{glue}} = 270$ MeV
- pions dominate the pressure at small $T$
- contribution of the kaons is important
- at high $T$ the pressure overshoots the lattice data of Borsányi et al., JHEP 1011, 077 (2010)

- overshooting increases with decreasing $T_{c}^{\text{glue}}$
Scaled interaction measure, speed of sound and $p/\epsilon$
Critical endpoint

$T - \mu_B$ Phase Diagram

- we used $U_{\text{log}}^{\text{glue}}$ with $T_c^{\text{glue}} = 210$ MeV
- curvature $\kappa$ at $\mu_B = 0$ obtained from the fit $\frac{T_c(\mu_B)}{T_c(\mu_B=0)} = 1 - \kappa \left( \frac{\mu_B}{T_c(\mu_B=0)} \right)^2$

$\kappa = 0.0193$ obtained, close to the lattice value $\kappa = 0.020(4)$ of Cea et al., PRD93, 014507
CPE obtained with other methods

- Dyson-Schwinger equation
  \[ (\mu_B, T)_{\text{CEP}} \approx (3.0, 0.9) T_c(\mu = 0) \]
  Roberts et al., PRL106 (2011) 172301
  \[ (\mu_B, T)_{\text{CEP}} \approx (660, 97) \text{ MeV for } N_f = 2 + 1 \]
  \[ (\mu_B, T)_{\text{CEP}} = (504, 115) \text{ MeV for } N_f = 2 + 1 \]
  Fischer et al., PRD90 (2014) 034022

- FRG study of Rennecke & Schaefer, arXiv:1610.08748
  \[ (\mu_B, T)_{\text{CEP}} = \begin{cases} (795, 44) \text{ MeV} & \text{LPA} \\ (765, 46) \text{ MeV} & \text{LPA + Y} \\ (705, 61) \text{ MeV} & \text{LPA’ + Y} \end{cases} \]
Analytic $T_c(\mu_B)$ in two flavored $L\sigma M$

Obtained in the chiral limit of $N_f = 2$ $L\sigma M$ at LO of $1/N_\pi$
expansion (w/o vector mesons)  
A. Patkós et al., PLB582 (2004) 179

in this ideal gas approximation the $T_c(\mu_B)$ phase boundary is given

$$m_R^2 + \left( \frac{\lambda_R}{72} + \frac{g_F^2}{4} \right) T_c^2 + \frac{g_F^2}{12\pi^2} \mu_B^2 = 0 \quad \text{(ellipse curve, since } m_R^2 < 0)$$

using the parametrization $\mu_B = q T_c$, (as in the case of lattice)

$$T_c(q) = \sqrt{\frac{-m_R^2}{\lambda_R/72 + g_F^2/4 \left( 1 + \frac{q^2}{3\pi^2} \right)}}$$

⇒ an educated guess for the curve of the phase boundary

$$f(x) = \frac{a}{\sqrt{1 + bx}} \quad \text{with} \quad x = (\mu_B/T_c)^2$$
Using the educated guess in the current model

Fit on the \((\mu_q, T_c(\mu_q))\) points of the VePQM model

In view of the simple approximation used, there is no surprise that the fit is good.

Would be interesting to see whether SDE or FRG shows deviation from
\[ f(x) = \frac{a}{\sqrt{1+bx}} \] due to improved resummation and the inclusion of fluctuations.
The model

**$T - \rho_B$ Phase Diagram**

Our model

- The $T - \rho_B$ phase diagram of our model is closer to that of the nuclear liquid-gas PT than to those obtained in other chiral models.
Isentropic trajectories in the $T - \mu_B$ plane (I.)

In our model, where $\mu_B^{\text{CEP}} > 850\text{MeV}$

- same qualitative behavior of the isentropic trajectories for $\mu_B \leq 400\text{ MeV}$
- indication that in the lattice result there is no CEP in this region of $\mu_B$

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Günther et al., arXiv:1607.02493
Isentropic trajectories in the $T - \mu_B$ plane (II.)

**PNJL with vector interaction**

- Effective models show a different behavior of the isentropic trajectories close to CEP compared to the small $\mu_B$ case.

- No indication for CEP from the continuum extrapolated lattice results analytically continued to the $\mu_B \leq 400$ MeV region.

**lattice (analytic continuation)**

- Bellwied et al., PLB751 (2015) 559
How can we improve the approximation?

ever example case → the Yukawa model:

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - V_m(\varphi) + \bar{\Psi}(i\not\partial - g \varphi)\Psi \]

Effective potential of current approximation:

\[ V_{\text{eff}} = V_m(\Phi) + i \text{Tr} \log (i S_0^{-1}) ; \quad \Phi: \text{vev}, S_0: \text{fermion prop.} \]

Possible improvement:

\[ V_{\text{eff}} = V_m(\Phi) + i \text{Tr} \log (i S_0^{-1}) - \frac{i}{2} \text{Tr} \log (i D^{-1}) ; \quad D: \text{meson prop.} \]

Further improvement:

\[ V_{\text{eff}} = V_m(\Phi) + i \text{Tr} \log (i S_0^{-1}) - \frac{i}{2} \text{Tr} \log (i G^{-1}) \]

\[ i G^{-1}(x, y) = i D^{-1}(x, y) - \Pi(x, y) ; \quad \Pi(x, y) = ig^2 \text{Tr} S_0^2(x, y) \]
The thermodynamics of the extended PQM model was studied with similar parametrization as in Parganlija et al., PRD 87, 014011.

40 possible assignments of scalars to the nonet states were investigated. Lowest $\chi^2$: all scalar masses below 1 GeV

For the best set of parameters a CEP was found in the $\mu_B - T$ plane. A self-consistent treatment of quarks would most probably decrease $\mu_B^{\text{CEP}}$ and increase $T_c^{\text{CEP}}$.

$T$ and $\mu_B$ dependence of various thermodynamical observables measured on the lattice is qualitatively reproduced with an improved Polyakov loop potential.

Comparison of isentropic curves with lattice results suggests that CEP with small $\mu_B$ value is unfavorable ($\mu_B^{\text{CEP}} < 400$ MeV)

The model and the approximation used could be improved by:
  – including four-quark states for a more reliable vacuum phenomenology
  – coupling the constituent quarks to the (axial)vectors
  – including mesonic fluctuations
  – treating the quarks self-consistently.
Backup slides
Spontaneous symmetry breaking

Interaction is approximately chiral symmetric, spectra is not
\[ \rightarrow \text{SSB:} \]
\[ \sigma_{N/S} \rightarrow \sigma_{N/S} + \bar{\sigma}_{N/S} \]
\[ \bar{\sigma}_{N/S} \equiv < \sigma_{N/S} > \]

For tree level masses we have to select all terms quadratic in the new fields. Some of the terms include mixings arising from terms like \( \text{Tr}[(D_\mu \Phi)\dagger(D^\mu \Phi)] \):

\[ \eta_N - f_{1N}^\mu : -g_1 \bar{\sigma}_N f_{1N}^\mu \partial_\mu \eta_N, \]
\[ \pi - a_1^\mu : -g_1 \bar{\sigma}_N (a_1^{\mu+} \partial_\mu \pi^- + a_1^{\mu0} \partial_\mu \pi^0) + \text{h.c.}, \]
\[ \eta_S - f_{1S}^\mu : -\sqrt{2}g_1 \bar{\sigma}_S f_{1S}^\mu \partial_\mu \eta_S, \]
\[ K_S - K^*_\mu : \frac{ig_1}{2} (\sqrt{2} \bar{\sigma}_S - \bar{\sigma}_N) (\bar{K}_\mu^* \partial^\mu K_S^0 + K_\mu^* \partial^\mu K_S^+) + \text{h.c.}, \]
\[ K - K_1^\mu : -\frac{g_1}{2} (\bar{\sigma}_N + \sqrt{2} \bar{\sigma}_S) (K_1^{\mu0} \partial_\mu \bar{K}^0 + K_1^{\mu+} \partial_\mu K^-) + \text{h.c.} \]
Mixing in the extended model

Mixing in the $N - S$ sector for $\sigma$ and $\pi$ $\rightarrow (m_\sigma^2)_{NS} \neq 0$, $(m_\pi^2)_{NS} \neq 0$ $\rightarrow$ resolved 2 dim. orthog. transf.

Mixing between nonets $\rightarrow \rho^\mu_a \leftrightarrow \sigma$ and $b^\mu_a \leftrightarrow \pi$
Resolved by the following field shifts:

\begin{align*}
    f^\mu_{1N/S} &\rightarrow f^\mu_{1N/S} + w_{f_{1N/S}} \partial^\mu \eta_{N/S}, \\
a^{\mu+,0}_{1} &\rightarrow a_{1}^{\mu+,0} + w_{a_1} \partial^\mu \pi^{+,0}, (+h.c.) \\
    K^{\mu+,0}_{1} &\rightarrow K_{1}^{\mu+,0} + w_{K_1} \partial^\mu K^{+,0}, (+h.c.) \\
    K^{\mu+,0}_{*} &\rightarrow K_{*}^{\mu+,0} + w_{K_*} \partial^\mu K_{S}^{+,0}, (+h.c.)
\end{align*}

Vanishing of the crossterms $\rightarrow$ determination of the $w_i$’s

After these shifts, $\pi$, $\eta_N$, $\eta_S$, $K$, and $K_S$ are not canonically normalized $\rightarrow$ field renormalization: $Z_\pi$, $Z_{\eta_N}$, $Z_{\eta_S}$, $Z_K$, $Z_{K_S}$

(for details see PRD87 (2013) no.1, 014011)
Dependence of $\mu_B^{\text{CEP}}$ on the width of the susceptibility at $\mu = 0$

Lattice results (fixed $a$):

$(\mu_B, T_c)^{\text{CEP}} = (725 \pm 35, 160 \pm 3.5)\text{MeV}$

$m_\pi \approx 2m_\pi^{\text{phys}} \rightarrow$ shown as '2.' on fig. Fodor & Katz, JHEP 0203:014,2002

$(\mu_B, T_c)^{\text{CEP}} = (360 \pm 40, 162 \pm 3)\text{MeV}$

$m_\pi = m_\pi^{\text{phys}} \rightarrow$ shown as '1.' on fig. Fodor & Katz, JHEP 0404:050,2004

Estimation by S. D. Katz: $\Delta T_c(\chi_{\bar{\psi}\psi}) \approx 0.5 - 1 \text{MeV}$ and $\Delta T_c(\chi_{\bar{\psi}\psi}) \approx 2 - 4 \text{MeV}$

$T_c(\chi_{\bar{\psi}\psi}) \approx 28 \text{MeV}$ at the physical point in the continuum limit \textit{Aoki et al.} (2006)

$\rightarrow$ higher $\mu_B^{\text{CEP}}$ can be expected in continuum limit
Normalized pressure as a function of $t$ at different $\mu_q$’s
Quark susceptibility and density versus $t$ at different $\mu_q$'s