Three easy exercises in off-shell string-inspired methods (OSSIM)

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Exercise 1. How do we know that off-shell string-inspired methods are gauge-invariant?

- The easy answer: OSSIM is always done in the background field method (BFM) Feynman gauge, equivalent to the gauge-invariant Pinch Technique (PT)—nothing left to do.

- Nevertheless, modifying a remark of Feynman (made about QED):

  \textit{It might be worthwhile to spend one's time expressing [the PT and OSSIM] in every physical and mathematical way possible.}

- So: We generalize OSSIM to arbitrary gauges and apply PT principles.

- The intrinsic PT may be related to ambiguities occurring in perturbative loops in all covariant gauges except the BFM Feynman gauge. These arise from discontinuities in the worldline current.
Exercise 2. Using Feynman-parameter technology directly

- Much of the motivation for OSSIM is to have a compact form for Feynman-parameter integrals, after momentum integrations are done.
- Simplicity and compactness are maximized with the string-inspired choice of Feynman parameters.
- Long ago general rules were given for giving the Feynman-parameter form of any Feynman graph with any momentum-dependent numerator, after momentum integrations, but never explored for non-Abelian gauge theory [JMC and Tiktopoulos, 1973].
- It turns out that, at least at one-loop level, these old methods—with the string-inspired Feynman parameters—are as simple and compact as OSSIM in the BFM Feynman gauge, and moreover easily accommodate gauge-dependent terms coming from outside this gauge.
Exercise 3: How is adjoint string breaking described?

The adjoint string breaks; how is this described?

1. For quarks, a well-known and straightforward extension of the old Schwinger calculation.
2. More complicated for gluons: Non-perturbative at every stage; must begin with the concept of a gluon mass in order to describe the critical chromoelectric field, and ensure gauge invariance.
3. All this amounts to a generalization of OSSIM to one-loop graphs of infinite order.
A quick review of the PT

- The PT is a method for combining Feynman graphs for off-shell Green’s functions, such as the gluon propagator, with parts of other graphs (pinch parts), so as to construct a gauge-invariant Green’s function.
- Papavassiliou and Binosi have shown that to all orders of perturbation theory the PT is equivalent to calculating off-shell Green’s functions in the background field method (BFM) Feynman gauge.
- The original PT starts from S-matrix elements and identifies pinch parts (missing a key propagator) through Ward identities; the intrinsic PT starts from the off-shell Green’s function and throws out parts that are identified with the original PT.
Figure: The original PT for the three-gluon vertex, using a six-quark S-matrix element. Graphs (c), (g), and (h) have pinch parts (missing the quark propagator) that contribute to this vertex.
Figure: Pinch parts come from longitudinal gluon momenta triggering Ward identities that replace quark lines by $\pm 1$. Note that these are contributions to the three-gluon vertex, and, interpreted as proper vertex parts, they have a factor of an inverse gluon propagator $\Delta^{-1}(p_1 + p_2)$. 
Kinematic preliminaries and OSSIM

- Both the PT and OSSIM use polarization vectors for off-shell background gluons of momentum $p$, satisfying $\epsilon(p) \cdot p = 0$ even though $p^2 \neq 0$ (for example, any gluon in Fig. 1(a)). Example: A background gluon of momentum $p$ couples to 
  \[ \bar{u}(p + q)\gamma_\mu u(q) \equiv \epsilon_\mu(p; q) \] where the spinors are on-shell.

- After pinching, the same type of polarization vectors are generated for Figs. 1(c), (g), (h).

- We always work in Euclidean space (spacelike momenta) for the off-shell Green’s functions; this is assured by using equal-mass Minkowski-space spinors.
Figure: The basic three-gluon vertex. The solid lines are quantum gluons; the wiggly line is a background gluon. We choose the momenta on the quantum lines to go in the same direction, so that $p + k_1 = k_2$. 
The three-gluon bare vertex is the sum of a convective, a spin, and a pinch part

The standard three-gluon vertex:

\[
\Gamma_{\alpha\mu\nu}(p, k_1, k_2) = (k_1 + k_2)_{\alpha}\delta_{\mu\nu} - (k_2 + p)_{\mu}\delta_{\alpha\nu} + (p - k_1)_{\nu}\delta_{\alpha\mu}.
\]

Decompose it into convective, spin, and pinch terms:

\[
\Gamma_{\alpha\mu\nu}(p, k_1, k_2) = \Gamma^C_{\mu\nu\alpha} + \Gamma^S_{\alpha\mu\nu} + \Gamma^P_{\alpha\mu\nu};
\]

\[
\Gamma^C_{\alpha\mu\nu} = (k_1 + k_2)_{\alpha}\delta_{\mu\nu}, \quad \Gamma^S_{\alpha\mu\nu} = -2p_{\mu}\delta_{\nu\alpha} + 2p_{\nu}\delta_{\mu\alpha};
\]

\[
\Gamma^P_{\alpha\mu\nu} = -k_2\nu\delta_{\mu\alpha} - k_1\mu\delta_{\nu\alpha}.
\]

\[
\Gamma^P \text{ does all the pinching; the Feynman rules for the BFM Feynman gauge are to set } \Gamma^P = 0 \text{ and } \Gamma \rightarrow \Gamma^F_{\alpha\mu\nu} \equiv \Gamma^C_{\alpha\mu\nu} + \Gamma^S_{\alpha\mu\nu}.
\]
Longitudinal gluon momenta trigger Ward or ST identities

- **Background gluon Ward identity:**
  \[ p_\alpha \Gamma^F_{\alpha \mu \nu}(p, k_1, k_2) = \Delta^{-1}_{\mu \nu}(k_2) - \Delta^{-1}_{\mu \nu}(k_1). \]
  \((\Gamma^S \text{ gives zero.})\)

- **Quantum gluon Slavnov-Taylor identity:**
  \[ k_1^\mu \Gamma_{\alpha \mu \nu}(p, k_1, k_2) = p^2 P_{\alpha \nu}(p) - k_2^2 P_{\alpha \nu}(k_2) \]
  \[ P_{\alpha \nu}(q) = \delta_{\alpha \nu} - \frac{q_\alpha q_\nu}{q^2} \]

- Inverse propagators on background lines signal terms that come from pinch parts and that should be dropped (intrinsic PT).
Ward identities in Feynman-parameter space

- In momentum space, Ward identities have an easily-recognized structure, but not so after momentum integrations have been done.
- It is not hard to guess that a Ward identity for a graph after momentum integrations have been done is a total derivative in Feynman parameters.
- Equivalently, with OSSIM it is a total derivative in proper times.
- **Warning:** There are total derivatives in OSSIM that are unrelated to the Ward identities used in the PT.
OSSIM today

- OSSIM yields expressions for off-shell Feynman graphs after momentum-space integrations, leaving Feynman-parameter integrations whose integrands are described by special algorithms.

- OSSIM results can be found directly in string theory in the limit $\alpha' \to 0$; the result is in the BFM Feynman gauge. There is no natural way to use any other gauge, which would be complicated.

- OSSIM can also be found directly in field theory, using Feynman-Schwinger proper-time methods. For simplicity, and for agreement with string theory, BFM Feynman gauge is always used.

- To my knowledge, OSSIM from field theory has only been applied at one loop, but with (in principle) arbitrarily many off-shell background gluons attached to a single quantum loop.
Proper-time OSSIM

The one-loop effective action for a scalar is:

\[ \Gamma_S\{B_\mu\} = -\frac{1}{2} \text{Tr} \log \Delta_F. \]

Here \( B_\mu \) is the background potential, a sum of plane waves.

OSSIM always uses Feynman gauge, which greatly simplifies the index structure for quantum gluons.

The building block is the Schwinger-Feynman proper-time propagator for a \textbf{scalar} quantum field:

\[ \Delta_F(x - y) = P \int_0^\infty ds e^{-m^2 s} \int_x^y \{dz_\mu\} \exp\{-\int_0^s d\tau [\frac{\dot{z}_\mu^2}{4} - ig \dot{z} \cdot B(z)]\} \]

where P means color/proper-time ordering (which we now suppress).

The coupling \( \dot{z} \cdot B \) induces one source in the path integral for each plane-wave insertion. \textbf{It is the same as using } \Gamma^C, \textbf{the convective vertex, in Feynman graphs.}
Spin is a complication

- Use Grassmann variables to account for the spin vertex (world-line supersymmetry); the scalar result is multiplied by a complicated factor in the $x_{ij}$.
- For any spin, delta-function terms arise from $\bar{G}_B$; the rule is to integrate by parts and ignore surface terms.
- The final result of OSSIM is that the Feynman-parameter integrals are cleverly organized.
- But how do we know the result is gauge-invariant?
Use Feynman gauge; add a spin coupling by writing

\[ \Delta_F^{-1} = D^2 - \Sigma_{\mu\nu} B^{\mu\nu} \]

\((D_\mu\) is the background covariant derivative). The standard choice for \(\Sigma_{\mu\nu}\) is

\[ \Sigma_{\mu\nu}^{(\alpha\beta)} = i[\delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha] \]

which couples the quantum and background gluons as in the BFM Feynman gauge. In Feynman graphs this amounts to the same coupling as given by the spin vertex \(\Gamma^S\). There is no pinch vertex \(\Gamma^P\).
Three easy exercises in off-shell string-inspired methods (OSSIM)

Integration over the worldline equals momentum-space integration

The path integral for the convective terms yields for a scalar quantum field a sum of terms:

\[
\Gamma\{B_{\mu_i}\} = \frac{(ig^2)^N \text{Tr}(t_{a_N} \cdots t_{a_1})}{16\pi^2} \int_0^\infty \frac{ds}{s^{3-N}} \int [dx_i] \exp\{s \sum_{i<j} p_i \cdot p_j G_B(x_{ij})\}
\]

\[
\times \exp\left[\sum_{i<j}^N (-i(p_i \cdot \epsilon_j - p_j \cdot \epsilon_i) \dot{G}_B(x_{ij}) + \epsilon_i \cdot \epsilon_j \ddot{G}_B(x_{ij}))\right]
\]

saving only the term multi-linear in the polarization vectors.

Here \(x_i = \tau_i/s\) are Feynman parameters, \(x_{ij} \equiv x_i - x_j\), and the \(G_B(x_{ij}) = |x_{ij}| - (x_{ij})^2\) are bosonic Green's functions on the circle.
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The path-integral current for worldline variables is a sum of discontinuous pieces

▶ For perturbation theory:

\[ B_\mu(z) = \sum_{i=1}^{N} t_{a_i} \epsilon_{i\mu}(p_i) e^{ip_i \cdot z} \]

▶ For a given ordering of the \( t_{a_i} \) (and \( \tau_i \)) the color trace factors out, and the source for \( z_\mu(\tau) \) is **discontinuous**:

\[ K_\mu(\tau) = \sum_{i=1}^{N} \delta(\tau - \tau_i)(\epsilon_{i\mu}\partial_{\tau_i} + ip_{i\mu}) \]

where the plane waves act at proper times \( \tau_i \).

▶ In the limit \( N \to \infty \), \( k_{i+1} - k_i = O(1/N) \) the current is smooth.
Exercise 1: Adding pinch vertices, to go beyond BFM Feynman gauge, generates ambiguities

- The results of adding pinch vertices will be gauge-dependent.
  - Apply the intrinsic PT. Of course, the result is Green’s functions in the BFM Feynman gauge, which is just the Feynman gauge with $\Gamma^P = 0$.
  - The $\Gamma^P$ terms generate terms that are ambiguous, when translated into worldline language.
  - Perhaps application of the PT in the worldline formalism has something to do with regularizing the ambiguities.
Extending OSSIM: Pinch vertices in worldline language

The convective vertex \( i \partial_\alpha \) for a background gluon of momentum \( p_\alpha \) is \( (k_1 + k_2)_\alpha \) where \( k_1, k_2 \) are the momenta on either side of the vertex and \( p + k_1 = k_2 \).

The gradient of a classical on-orbit action \( S \) with respect to its endpoints, \( \partial_\mu S \), yields \( \pm P_\mu \), the canonical momentum at the endpoint, so

\[
\partial_1 \mu \Delta F_0 (x_1 - x_2) = \langle -\frac{\dot{z}_\mu}{2} (\tau_1 = 0; x_1, x_2, s) \rangle = -ik_{1\mu}.
\]

The \( \dot{z}_\mu \) form is ambiguous; we need to specify \( k_1 \) or \( k_2 \) by \( \dot{z}_\mu (\tau_1 \pm \epsilon) \).

We also need terms quadratic in \( \dot{z}_\mu \) (e.g., \( k_i^2 \) coming from Slavnov-Taylor identities).
Ward identities are ambiguous too

- By momentum conservation, one should be able to replace the convective vertex $\Gamma^C = (k_1 + k_2)_{\alpha}$ by, e.g., $(2k_1 + p)_{\alpha} \rightarrow 2k_{1\alpha}$ (after multiplying by $\epsilon_{\alpha}(p)$).

- But now the Ward identity $p \cdot \Gamma^C = k_2^2 - k_1^2$ only works if $p^2 = 0$! And this is not required, either in OSSIM or in the PT.

- The proper-time positions on either side of a vertex are continuous, but the velocities are discontinuous.

- Using $\Gamma^C$ resolves the ambiguity by effectively averaging the velocities at the discontinuity (cf. Fourier transforms).
Maybe resolving ambiguities is as simple as dropping the pinch vertex

- With Feynman graphs, a pinch term is perfectly unambiguous and ordinarily would be kept.
- With proper-time methods, every perturbative background gluon causes a jump in $\dot{z}_\mu(\tau)$ and a potential ambiguity.
- The convective vertex $\Gamma^C$ is unambiguous and symmetric in the quantum gluons on either side of the background-gluon vertex, but **not so** for the pinch vertex $\Gamma^P$. 
Ambiguities and the intrinsic PT for proper-time methods

▶ A typical Slavnov-Taylor identity for the intrinsic PT:

\[-k_1 \cdot (k_2 + p) \delta_{\nu \alpha} + \cdots = (p - k_2) \cdot (p + k_2) \delta_{\nu \alpha} + \cdots = (p^2 - k_2^2) \delta_{\nu \alpha} + \cdots\]

▶ Replace \( k_i \) by \((-i/2) \dot{z}(\tau_i \pm \epsilon)\), note that \( \dot{z}(\tau_1 \pm \epsilon) \cdot \dot{z}(\tau_1 \pm \epsilon) \) is ambiguous.

▶ By momentum conservation, \( k_1 \cdot k_2 = \frac{1}{2}(k_1^2 + k_2^2 - p^2) \).

▶ The intrinsic PT demands that \( p^2 \) be dropped. Does this have anything to do with regularizing ambiguities in proper time, for example, \( [\dot{z}(\tau_1)]^2 \rightarrow \frac{1}{2} [\dot{z}(\tau_1 - \epsilon)^2 + \dot{z}(\tau_1 + \epsilon)^2] \)?
Exercise 2: Extending OSSIM the old-fashioned way

- Use old topological rules for expressing an arbitrary Feynman graph, with momentum-dependent numerators, after momentum-space integrations are done, plus string-inspired Feynman parameters.
- At least at one loop and in the BFM Feynman gauge this is certainly no more complex than standard OSSIM methods. For example, a term linear in the integration momentum \( k \) is replaced by

\[
k \rightarrow \sum_j x_{ij} p_j,
\]

the power of which is that this replacement holds for any value of the index \( i \) (with momentum conservation plus \( x_{ij} = x_i - x_j \)).
- Using this formulation amounts to resumming some exponential forms in standard OSSIM.
Figure: The natural Feynman parameters $\alpha_i$ are related to the Koba-Nielsen variables $x_i$ by $[i < j]$: $x_{ij} \equiv x_i - x_j = \sum_{k=i}^{j-1} \alpha_k$. 
Exercise 3: The nature of the gluon mass and adjoint string breaking

- Lattice, continuum studies yield a gluon mass—not just an ordinary mass, because the gluon can’t propagate indefinitely.
- Conventional PT Schwinger-Dyson equations give an ordinary mass, because there is no adjoint string breaking (infinite order in $g$).
- The signature of an unconventional mass on the lattice is compromised by other effects:
  - Finite-distance effects
  - Minkowski regime inaccessible
  - Mass is momentum-dependent
A model of adjoint string breaking

1. Use PT-OSSIM to study the generalized Schwinger\(^1\) instability for **massive** quantum gluons.
2. Use a background electric gauge field for the adjoint string from flux tube (or gluon-chain model).
3. Use as input for a study (to be done) of gluon propagation.

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\(^1\)Not the Nielsen-Olesen instability, which is cured by a gluon mass.
Semi-phenomenological proper-time methods for massive gluons

- Replace the massless OSSIM propagator, $\Delta_F^{-1}$, by adding a constant mass.

$$\Delta_F^{-1} = D^2 + m^2 - \Sigma_{\mu\nu} B^{\mu\nu}$$

because

1. Ward identities (the basic PT tool) are preserved.
2. The Brodsky-Binger supersymmetry relation, relating three-gluon vertices with spin 0, 1/2, and 1 in the loop, is preserved, with a common mass for all spins.
3. A lengthy investigation (unpublished) confirms the utility of this phenomenology

- Result: An adjoint string decay width

$$\Gamma = \frac{2m}{\pi^2} \left( \frac{g^2}{4\pi} \right)^2 \exp\left[ -2\pi^2/g^2 \right] \approx 0.0015m \text{ at } \frac{g^2}{4\pi} \approx \frac{1}{2}; \approx m \text{ at } \frac{g^2}{4\pi} \approx \pi.$$
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The way it’s done

▶ Do it the Schwinger way, with a few changes for group-theoretic numbers.
▶ Or do a simple tunneling estimate that gives the same answer, based on:
  1. $\sigma_A \approx 2\sigma_F \approx m^2$ based on extended gluon-chain model [Trento 2009].
  2. Assume constant chromoelectric field $\varepsilon$ across a distance $1/m$:

    \[
    \text{Flux} : \left( \frac{\pi}{m^2} \right) \varepsilon = g \quad \text{plus} \quad \sigma_F = \left( \frac{\pi}{m^2} \right) \frac{\varepsilon^2}{2} = m^2 \left( \frac{g^2}{2\pi} \right) \rightarrow \frac{g^2}{4\pi} \approx \frac{1}{2}
    \]

  3. Maximum tunneling rate for zero-momentum gluon pairs $\rightarrow$ adjoint string breaks at length $\ell = \frac{2}{m} \approx .7 \ Fm$.
  4. All these numbers are nicely consistent with known values: $\sigma_F \approx 0.19 \ GeV^2$, $m \approx 0.6 \ GeV$, $\frac{\tilde{g}^2(m^2)}{4\pi} \approx \frac{1}{2}$.
▶ Seeing effects associated with adjoint string breaking might well be within the grasp of lattice simulations.
Summary

1. We extend proper-time OSSIM to include gauge-variant terms, to which the intrinsic PT can apply.

2. We suggest that the intrinsic PT for extended perturbative OSSIM is equivalent to an algorithm for smoothing path integrals in the presence of a discontinuous source.

3. We reformulate and extend OSSIM with old and so far little-used topological rules for Feynman parameter integrals.

4. We use a phenomenological extension of OSSIM to massive gluons, leading to a picture of adjoint string breaking.