Quarks and gluons in a magnetic field

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Outline of talk

• Motivation
• Ritus eigenfunction method for fermions in a constant magnetic field
  • asymptotic nature of series (in Landau levels)
• Summation and Schwinger phase
  • tree-level
  • nonperturbatively
• Gap equation
• Results - chiral condensate
• Summary
Motivation

- heavy ion collisions - charged particles, big velocities...

  → strong magnetic fields

- influences the quark chiral condensate, but

- strong magnetic fields vs. strong interaction
  (at most, similar scales - not good for approximations)

Ritus eigenfunction method

• choose \( \vec{B} = B\hat{e}_3, \quad h = QB \geq 0 \)

• Dirac operator \( D = i\partial_\mu \gamma^\mu - h\gamma^2 x_1 \)

• inverse propagator: \( \Gamma^{(0)}(x, y) = i[D - m]\delta(x - y) \)

• propagator: \( i[D - m]S^{(0)}(x, y) = \delta(x - y) \)

• constant magnetic field introduces Landau levels, with Hermite functions as eigenfunctions, and spin projection (Landau levels get connected to the spin)

• translational invariance is broken!
Ritus eigenfunction method

• Ritus’ solution (replace momentum modes with the Hermite function basis):

\[ \Gamma^{(0)}(x, y) = \sum_{n=0}^{\infty} \int \frac{d^3 \tilde{p}}{(2\pi)^3} E(x; \tilde{p}, n) \Gamma^{(0)}(\tilde{p}, n) \overline{E}(y; \tilde{p}, n) \]

• Ritus matrices connect spin and Landau levels

\[ E(x; \tilde{p}, n) = \hbar^{1/4} e^{-i\tilde{p} \cdot x} \left[ \psi_{n-1}(\varepsilon) \Sigma^+ + \psi_n(\varepsilon) \Sigma^- \right] \]

• spin projectors:

\[ \Sigma^\pm = \frac{1}{2} \left[ 1 \pm i\gamma^1 \gamma^2 \right] \]

• matrices are orthonormal and form a complete set

\[ \varepsilon = \sqrt{\hbar x_1} + \frac{p_2}{\sqrt{\hbar}} \quad \tilde{p}^\mu = (p_0, 0, p_2, p_3), \quad \overline{p}^\mu = (p_0, 0, 0, p_3) \]

Ritus eigenfunction method

- inverse propagator (function of eigenvalues)

\[-\iota \Gamma^{(0)}(\bar{p}, n) = \bar{p}_\mu \gamma^\mu - \sqrt{2n\hbar} \gamma^2 - m\]

- propagator

\[\iota S^{(0)}(\bar{p}, n) = \frac{\bar{p}_\mu \gamma^\mu - \sqrt{2n\hbar} \gamma^2 + m}{\bar{p}^2 - 2n\hbar - m^2 + \iota 0_+}\]

- (Landau levels appear in the denominator)
• general form for the chiral condensate

\[ \langle \bar{q}q \rangle = N_c \text{Tr}_d \, S(x, x) \]

\[ = N_c \frac{\hbar}{2\pi} \int \frac{d^2 \vec{p}}{(2\pi)^2} \text{Tr}_d \left\{ \Sigma^- S(\vec{p}, n = 0) + \sum_{n=1}^{\infty} S(\vec{p}, n) \right\} \]

• decomposing and projecting in terms of the Ritus matrices, a prefactor of \( h \) appears in all loop integrals (the propagator is a function of two momentum components and the Landau level: dimensions must be maintained)

• asymptotic expansion, not good for ‘small’ magnetic fields where we know that the quark has a nontrivial condensate!

• we have to sum up the Landau levels...
Summation and Schwinger phase

- tree-level inverse propagator
  \[ \Gamma^{(0)}(x, y) = \sum_{n=0}^{\infty} \int \frac{d^3 \tilde{p}}{(2\pi)^3} E(x; \tilde{p}, n) \Gamma^{(0)}(\tilde{p}, n) \overline{E}(y; \tilde{p}, n) \]

- contains the integral
  \[ I = \int dp_2 e^{ip_2(x_2-y_2)} \psi_a(\varepsilon) \psi_b(\tau) \]

- with
  \[ \varepsilon(\tau) = \sqrt{\hbar} x_1(y_1) + \frac{p_2}{\sqrt{\hbar}} \]

- “it can be shown that” ;-) this can be written (almost) in terms of transverse momenta and Laguerre polynomials
  \[ I \sim e^{i\Phi} \int d^2p_t e^{ip_t \cdot (\tilde{x} - \tilde{y})} f(\tilde{p}_t) \exp \left\{ -\frac{p_t^2}{\hbar} \right\} L_n^\alpha \left( 2\frac{p_t^2}{\hbar} \right) \]

• “it can be shown” ;-) that this can be written (almost) in terms of transverse momenta

\[ I \sim e^{i\Phi} \ldots \]

• where the Schwinger phase encodes the deviations from translational invariance

\[ \Phi = -\frac{\hbar}{2}(x_2 - y_2)(x_1 + y_1) \]

• the sums over the Laguerre polynomials are known, to give...

Summation and Schwinger phase

• the tree-level inverse propagator

\[-i \Gamma^{(0)}(x, y) = e^{i \Phi} \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left[ p_\mu \gamma^\mu - m \right] \]

• and similarly the tree-level propagator (small h)

\[i S^{(0)}(x, y) = e^{i \Phi} \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \left\{ \frac{[p_\mu \gamma^\mu + m]}{[p^2 - m^2 + i0^+]} + \frac{i \hbar \gamma^1 \gamma^2 \left[ \bar{p}_\mu \gamma^\mu + m \right]}{[p^2 - m^2 + i0^+]^2} \right\} \]

• reduction when magnetic field vanishes (unlike the Ritus decomposition),
• but no obvious relation between the two!
• up to the Schwinger phase, the momentum space expressions look promising...

Gorbar, Miransky, Shovkovy, Wang, PRD88 (2013) 025025 & 025043,
the strategy is to take a nonperturbative ansatz for the Ritus decomposed two-point functions (where the inverse can be found), with various spin components and see if we can sum to get similar expressions...

\[-i\Gamma(\vec{p}, n) = \Sigma^+ (\vec{p}_\mu \gamma\mu A - B) + \Sigma^- (\vec{p}_\mu \gamma\mu C - D) - \sqrt{2nh\gamma^2} E\]

(allow for different spin projections)

sum isn’t a problem for the inverse propagator:

\[-i\Gamma(x, y) = e^{i\Phi} \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \times \left\{ \Sigma^+ (\vec{p}_\mu \gamma\mu A - B) + \Sigma^- (\vec{p}_\mu \gamma\mu C - D) - \vec{p}_t \cdot \vec{\gamma} E \right\}\]

functions of two variables, \( A = A(\vec{p}^2, p^2_t) \)

reduction \( h \to 0 : (A, C, E) \to A, (B, D) \to B \)
Nonperturbatively

• corresponding propagator looks like...

\[ \nu S(\bar{p}, n) = \Sigma^+ \bar{p}_\mu \gamma^\mu \frac{\Delta_1 C - \Delta_2 D}{\Delta} + \ldots \]

\[ \Delta_1 = \bar{p}^2 AC - BD - 2nhE^2, \quad \Delta_2 = AD - BC, \quad \Delta = \Delta_1^2 - \bar{p}^2 \Delta_2^2 \]

• summed under approximation
  • neglect n-dependence of functions
    (keep explicit n factors)
  • expand denominator in (small h) \( \Delta_2 \)

• approximation retains the connection between spin structures
• in the end, the gap equation will determine the momentum dependence of the functions
Nonperturbatively

- approximated summed propagator

\[-iS(x, y) = e^{i\Phi} \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot (x-y)} \]

\[\times \left\{ \frac{\Sigma^+(\bar{p}_\mu \gamma^\mu C + D) + \Sigma^-(\bar{p}_\mu \gamma^\mu A + B) - \bar{p}_t \cdot \gamma E}{(p^2 AC - p_t^2 E^2 - BD + i0_+)} \right. \]

\[+ hE^2 \frac{\Sigma^+(\bar{p}_\mu \gamma^\mu C + D) - \Sigma^-(\bar{p}_\mu \gamma^\mu A + B)}{(p^2 AC - p_t^2 E^2 - BD + i0_+)^2} \]

\[-(AD - BC') \frac{\Sigma^+(\bar{p}_\mu \gamma^\mu D + \bar{p}^2 C) - \Sigma^-(\bar{p}_\mu \gamma^\mu B + \bar{p}^2 A)}{(p^2 AC - p_t^2 E^2 - BD + i0_+)^2} \right\} \]

- reduces to tree-level (all h), also to standard propagator in the absence of the magnetic field.
• rainbow truncation (chiral quarks)

\[ \Gamma(x, y) = \Gamma^{(0)}(x, y) + g^2 C_F \gamma^\mu S(x, y) \gamma^\kappa W_{\kappa\mu}(y, x) \]

• dressed (Landau gauge) gluon interaction

\[ \iota W_{\kappa\mu}(y, x) = \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (y-x)} t_{\kappa\mu}(q) \frac{G(q^2)}{q^2} \]

\[ g^2 \frac{G(q^2)}{q^2} = 4\pi^2 d \exp \left\{ \frac{q^2}{\omega^2} \right\} \times \begin{cases} q^2/\omega^2, & \text{I} \\ -1, & \text{II} \end{cases} \]

• Schwinger phase factorizes, so we can work in momentum space - but with functions of two variables

\begin{align*}
\text{I : } & \omega = 0.5 \text{ GeV}, \quad d = 16 \text{ GeV}^{-2} \\
\text{II : } & \omega = 0.5 \text{ GeV}, \quad d = 41 \text{ GeV}^{-2}
\end{align*}

functions at zero momentum
• reduction for vanishing magnetic field
  (type I explicitly matches earlier results)
• similar patterns for both interactions, since the gluon is unaffected by the magnetic field (under truncation)
Results

- condensate rises quadratically for small $h$
- and linearly for large $h$
- qualitative agreement is good, even for large $h$!

$I: \langle \bar{q}q \rangle_{h=0} = (-251 \text{ MeV})^3$

$II: \langle \bar{q}q \rangle_{h=0} = (-252 \text{ MeV})^3$

- relative increment:

$$r(h) = \frac{\langle \bar{q}q \rangle_h}{\langle \bar{q}q \rangle_{h=0}} - 1$$

D’Elia, J.Phys.Conf.Ser.432 (2013) 012004,
Bali, Bruckmann, Endrodi, Fodor, Katz, Schafer, PRD86 (2012) 071502,
Results

- lowest Landau level approximation:

\[ [\psi_{n-1}(\varepsilon)\Sigma^+ + \psi_n(\varepsilon)\Sigma^-] \to \psi_0(\varepsilon)\Sigma^- \]

- and the ‘mass function’ D/C dominates for large h! (because C decreases)
Summary

• quark gap equation (rainbow truncation) with a phenomenological gluon interaction in the presence of a constant magnetic field

• strong magnetic field vs. strong interaction
  - expansion in Landau levels is not suitable in this context

• we use an approximation to sum the Landau levels, ensuring the limit when the magnetic field vanishes

• in the presence of a constant magnetic field, the chiral quark condensate increases, quadratically for small field and linearly for large field