Real-Time Simulation of Quantum Systems Driven by Dissipation

Uwe-Jens Wiese

Albert Einstein Center for Fundamental Physics
Institute for Theoretical Physics, Bern University

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Collaboration:
Debasish Banerjee (DESY Zeuthen),
Fu-Jiun Jiang (NTNU, Taipei), Mark Kon (Boston University),
Stephan Caspar, Florian Hebenstreit,
Manes Hornung, Franziska Schranz (Bern)
Members of the Collaboration

Debasish Banerjee  Fu-Jiun Jiang  Mark Kon

Stephan Caspar  Florian Hebenstreit
Outline

Quantum Simulation

Why is Simulating Real-Time Dynamics so Hard?

Dissipation from Measurement Processes

Simulating Purely Dissipative Real-Time Dynamics

Simulation of Transport between two Magnetization Reservoirs

Cooling into a Bose-Einstein Condensate as a Dark State

Conclusions
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Richard Feynman’s vision of 1982

“I’m not happy with all the analyses that go with just the classical theory, because nature isn’t classical, dammit, and if you want to make a simulation of nature, you’d better make it quantum mechanical, and by golly it’s a wonderful problem, because it doesn’t look so easy.”
Ion traps as a digital quantum computer

Franklin Medal 2010: I. Cirac, D. Wineland, P. Zoller
Bose-Einstein condensation in ultra-cold atomic gases

Eric Cornell, Carl Wieman, Wolfgang Ketterle, 1995
Ultra-cold atoms in optical lattices as analog quantum simulators

Transition from a superfluid to a Mott insulator

Optical lattice quantum simulation of quantum spin systems

Antiferromagnetic precursors of high-\( T_c \) superconductors

LaCuO

YBaCuO

- Cu\(^{2+},Cu^{3+}\)
- O\(^2-\)
- Y\(^{3+}\)
- Ba\(^{2+}\)

Dimensions:
- \( 11.6802 \) Å
- \( 3.8872 \) Å
- \( 3.8227 \) Å
The Hubbard Model for doped antiferromagnets

\[ H = -t \sum_{\langle xy \rangle} (c_x^\dagger c_y + c_y^\dagger c_x) + U \sum_x (c_x^\dagger c_x - 1)^2, \quad c_x = \begin{pmatrix} c_{x\uparrow} \\ c_{x\downarrow} \end{pmatrix} \]

reduces to the Heisenberg model at half-filling for \( U \gg t \)

\[ H = J \sum_{\langle xy \rangle} \vec{S}_x \cdot \vec{S}_y \]

Important open question:
Can the Hubbard model at low temperature be quantum simulated?
The Hubbard Model for doped antiferromagnets

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Important open question:
Can the Hubbard model at low temperature be quantum simulated?
Digital quantum simulation of Kitaev’s toric code with trapped ions using engineered dissipation

- Precisely controllable many-body quantum device, which can execute a prescribed sequence of quantum gate operations.
- State of simulated system encoded as quantum information.
- Dynamics is represented by a sequence of quantum gates, following a stroboscopic Trotter decomposition.

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Real-time path integral describing a quench from $H_0$ to $H$

$$p_{\rho_0} [m(t)] = \frac{1}{Z_0} \text{Tr}[\exp(-\beta H_0) \exp(iHt) | m(t)\rangle \langle m(t) | \exp(-iHt)]$$

$$= \frac{1}{Z_0} \sum_{[n_0,n]} \exp(-S_{0E}[n_0]) \exp(i(S_R[n] + iS_I[n])) \delta_{n(t),m(t)}$$

$$= \frac{Z}{Z_0} \langle \cos(S_R) \delta \rangle, \quad Z_0 = \sum_{[n_0]} \exp(-S_{E}[n_0])$$

Path integral for a corresponding Euclidean ensemble

$$Z = \sum_{[n_0,n]} \exp(-S_{E}[n_0] - S_I[n])$$

Large error to signal ratio:

$$\Delta p_{\rho_0} [m(t)] = \frac{Z}{Z_0} \sqrt{\langle \cos^2(S_R) \delta^2 \rangle - \langle \cos(S_R) \delta \rangle^2} \approx \frac{Z}{Z_0} \sqrt{\langle \delta \rangle / 2}$$

$$\frac{\Delta p_{\rho_0} [m(t)]}{p_{\rho_0} [m(t)]} = \frac{\sqrt{\langle \delta \rangle / 2}}{\langle \cos(S_R) \delta \rangle} = \frac{Z \sqrt{\langle \delta \rangle / 2}}{Z_0 p_{\rho_0} [m(t)]} \sim \frac{\exp(\Delta f V t) \sqrt{\langle \delta \rangle / 2}}{p_{\rho_0} [m(t)]}$$
Real-time evolution of the density matrix of an isolated quantum system

\[
\partial_t \rho(t) = i [\rho(t), H(t)], \quad \rho(t) = U(t, t_0) \rho(t_0) U(t_0, t),
\]

\[
U(t_0, t) = \mathcal{T} \exp \left( -i \int_{t_0}^t dt' \ H(t') \right)
\]

Why is this so difficult to compute?
Real-time evolution of the density matrix of an isolated quantum system

\[ \frac{\partial t}{\partial t} \rho(t) = i[\rho(t), H(t)], \quad \rho(t) = U(t, t_0)\rho(t_0)U(t_0, t), \]

\[ U(t_0, t) = \mathcal{T} \exp \left( -i \int_{t_0}^{t} dt' H(t') \right) \]

Why is this so difficult to compute?

It should be easier to compute the real-time evolution when the system is under observation.
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Measurement process of an observable $O_k$

$$O_k |\lambda\rangle = o_k |\lambda\rangle, \quad P_{o_k} = \sum_\lambda |\lambda\rangle\langle\lambda|,$$

$$P_{o_k}^2 = P_{o_k}, \quad \text{Tr} P_{o_k} = g_{o_k}, \quad \sum_{o_k} P_{o_k} = 1$$

Evolution of the density matrix after one measurement

$$\rho_{o_1} = P_{o_1} \rho P_{o_1}, \quad \rho' = \sum_{o_1} \rho_{o_1} = \sum_{o_1} P_{o_k} \rho P_{o_1},$$

$$\text{Tr} \rho' = \sum_{o_1} \text{Tr}(P_{o_1} \rho P_{o_1}) = \sum_{o_1} \text{Tr}(\rho P_{o_1}) = 1$$

Evolution of the density matrix after $N$ measurements

$$\rho_{o_1,o_2,\ldots,o_N} = P_{o_N} U(t_N, t_{N-1}) \ldots U(t_3, t_2) P_{o_2} U(t_2, t_1) P_{o_1} \rho$$

$$\times P_{o_1} U(t_1, t_2) P_{o_2} U(t_2, t_3) \ldots U(t_{N-1}, t_N) P_{o_N},$$

$$\rho' = \sum_{o_1,o_2,\ldots,o_N} \rho_{o_1,o_2,\ldots,o_N}$$
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Path integral with a Schwinger-Keldysh contour

\[
p_{\rho_0 f}(o_1, o_2, \ldots, o_N) = \sum_i p_i \langle ii | (P_{o_1} \otimes P_{o_1}^*) (P_{o_2} \otimes P_{o_2}^*) \cdots (P_{o_N} \otimes P_{o_N}^*) | ff \rangle =
\]

\[
\sum_i p_i \sum_{n_1, n'_1} \cdots \sum_{n_{N-1}, n'_{N-1}} \prod_{k=1}^N \langle n_{k-1} | P_{o_k} | n_k \rangle \langle n'_{k-1} | P_{o_k} | n'_k \rangle^*,
\]

\[
\langle n_0 n'_0 | = \langle ii |, \quad | n_N n'_N \rangle = | ff \rangle
\]
Antiferromagnetic spin $\frac{1}{2}$ quantum Heisenberg model, 
$H = J \sum \langle xy \rangle \vec{S}_x \cdot \vec{S}_y$, driven by measurements of the total spin $S \in \{0, 1\}$ of adjacent spin pairs $\vec{S} = \vec{S}_x + \vec{S}_y$

$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

![Graph](image-url)
Continuous monitoring described by a Lindblad process

\[
\partial_t \rho = i[\rho, H] + \frac{1}{\varepsilon} \sum_{k, o_k} \left( L_{o_k} \rho L_{o_k}^\dagger - \frac{1}{2} L_{o_k}^\dagger L_{o_k} \rho - \frac{1}{2} \rho L_{o_k}^\dagger L_{o_k} \right)
\]

\[
= \gamma \sum_k \left( \sum_{o_k} P_{o_k} \rho P_{o_k} - \rho \right)
\]

Lindblad or Kraus quantum jump operators

\[
L_{o_k} = \sqrt{\varepsilon \gamma} P_{o_k}, \quad (1 - \varepsilon \gamma N) \mathbb{1} + \sum_{k, o_k} L_{o_k}^\dagger L_{o_k} = \mathbb{1}
\]

Equilibration of the Fourier modes of the magnetization in a dissipative process that “measures” $\vec{S}_x \cdot \vec{S}_y$

$$\tilde{S}(p) = \sum_x S_x^3 \exp(ip_1 x_1 + ip_2 x_2)$$

Equilibration of the Fourier modes of the magnetization in dissipative processes that “measure” $S^1_x S^1_y$ or $S^+_x S^+_y + S^-_x S^-_y$

Equilibration times

\[ \langle |S(p)|^2 \rangle \]

\[ \langle |S(p)|^2 \rangle \]

\[ \gamma^{-1} \]

\[ \gamma^{-1} \]

[\gamma T(p)]^{-1}

[\gamma T(p)]^{-1}

\[ |\alpha| \]

\[ |\alpha| \]
Staggered susceptibility $\langle M_s^2 \rangle / L^4$ and Binder ratio $\langle M_s^4 \rangle / \langle M_s^2 \rangle^2$

Staggered magnetization $M_s$ and length scale $\xi = c/(2\pi \rho_s)$,

$\langle M_s(t)^2 \rangle = M_s(t)^2 L^4 / 3 \sum_{n=0}^3 c_n (\xi(t)/L)^n$
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Diffusion of uniform magnetization through a hole

Diffusion of staggered magnetization through a hole
Diffusive currents derived from microscopic dynamics

\[ \vec{\nabla} \rho_u = \gamma \vec{j}_u, \quad \vec{\nabla} \rho_s = \gamma \vec{j}_s \]

Continuity equations

\[ \partial_t \rho_u + \vec{\nabla} \cdot \vec{j}_u = 0, \quad \partial_t \rho_s + \vec{\nabla} \cdot \vec{j}_s = 0 \]

Resulting diffusion equations

\[ \partial_t \rho_u = \gamma \Delta \rho_u, \quad \partial_t \rho_s = \gamma \Delta \rho_s. \]
Analytic solution of 1-d diffusion equation
Relaxation rate as a function of hole size $L$

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Cooling of $(3 + 1)$-d hard-core bosons

\[ H = J \sum_{\langle xy \rangle} (S_x^+ S_y^- + S_x^- S_y^+) \]

driven by non-Hermitean Lindblad operator

\[ L_1 = (S_x^+ S_y^+)(S_x^- - S_y^-) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad L_1^\dagger \neq L_1, \]

\[ L_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad L_2^\dagger = L_2 \]

S. Caspar, F. Hebenstreit, UJW, in preparation.
Momentum modes of 2-point correlation function

\[ C(p) = \sum_x (S_0^+ S_x^- + S_0^- S_x^+) \exp(i p x) \]

\[ V=16^3, \ T_0/J=100 \]

\[ \begin{align*}
\bullet & \quad (0,0,0) \\
\cdot & \quad (0,0,\pi/8) \\
\ast & \quad (0,0,\pi/4) \\
\ast & \quad (0,0,\pi/2) \\
\bullet & \quad (0,0,\pi) \\
\ast & \quad (\pi,\pi,\pi)
\end{align*} \]
Initial equilibration by diffusion $\tau \sim |p|^{-2}$
Long-time behavior of the condensate fraction

![Graph showing the long-time behavior of the condensate fraction. The graph plots $f_0/V$ against $\gamma t$, with three distinct curves indicating different behaviors at various $\gamma t$ values.]
Ultimate equilibration (by defect annihilation?) $\tau \sim L^3$
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- Real-time simulations of some large open quantum entirely driven by dissipation or by measurement processes are sign-problem-free and can be performed using importance sampling quantum Monte Carlo.

- Such simulations have allowed us to study the time-dependence of different dissipative processes which is slowed down by conserved quantities.

- Transport processes in dissipation driven strongly correlated large open quantum spin systems lead to diffusion of magnetization or staggered magnetization from one reservoir to another.

- Lindblad processes with non-Hermitean quantum jump operators which describe cooling of bosons into a dark state can also be simulated. Different momentum modes of the Bose-Einstein condensate equilibrate at different time scales.