Proton rms-radius $R$: recent determinations from (e,e)

Ingo Sick

Disturbing

scatter of results
values between 0.84 and 0.92\,fm, with error bars of typically 0.01\,fm

Main problem: interpretation of data; smaller problem: difference between data

Goal of talk

go to bottom of discrepancies, understand causes for differences
analysis necessarily critical of published results
only one true value of radius (.... obviously mine!)
study requires very careful look at published results
which takes time ..... 

What to expect

not ’new’ result of my own, but review of other results
in the end: give average of results I have no qualms with

State right away: results do not fix discrepancy with $\mu H$
4 peculiarities of proton

consequences of fact that \( G(q) \sim \text{Dipole}, \) density \( \rho(r) \sim \text{exponential} \)

1. Large \( r \) and tail of \( \rho(r) \) do matter more than for \( A > 2 \)

illustration: study \( \left[ \int_0^{r_{\text{cut}}} \rho(r) \, r^4 \, dr / \int_0^{\infty} \rho(r) \, r^4 \, dr \right]^{1/2} \) as function of cutoff \( r_{\text{cut}} \)

![Graphs showing R(r_{\text{cut}})/R vs r_{\text{cut}}/R and \( \rho(r) \) vs r (fm)]

![Graphs showing R(r_{\text{cut}})/R vs r_{\text{cut}}/R and \( \rho(r) \) vs r (fm)]

to get 98% of rms-radius \( R \) must integrate out to \( r_{\text{cut}} \sim 3.2 \cdot R \sim 3 \text{ fm} \)

where \( \rho(r)/\rho(0) \sim 10^{-4} \)

\( \Rightarrow R \) sensitive to very large \( r \) where \( \rho(r) \) poorly determined

\( \Rightarrow \) large \( r \) affect \( G(q) \) at very low \( q \), below \( q_{\text{min}} \) \( \Rightarrow \) affects extrapolation to \( q = 0 \)

\( \Rightarrow \) requires great care in extrapolation to \( q = 0 \)
Large-$r$ contribution not measurable in practice

model study: start from $\rho(r) = \text{exponential density}$
determine $G(q)$ from Fourier transform
1. full $\rho(r)$
2. cut $\rho(r)$ for $r > 3\text{fm}$
3. renormalize 2. (as done in fits)

$\to$ serious question: uncertainties of order 2% at all reachable??

.... not without doing something sensible for $\rho$ at large $r$
2. Extremely large higher moments due to $\rho(r) \sim$ exponential

<table>
<thead>
<tr>
<th></th>
<th>$\langle r^4 \rangle / \langle r^2 \rangle^2$</th>
<th>$\langle r^6 \rangle / \langle r^2 \rangle^3$</th>
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<tr>
<td>naive estimate</td>
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<td>exponential density</td>
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<td>experimental value</td>
<td>4.32</td>
<td>64.2</td>
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</table>

fit of $q \leq 5 fm^{-1}$ data Bernauer 2010

large-$r$ contribution even worse than for exponential density

Consequence: at $q \sim 0.9 fm^{-1}$ of maximal sensitivity to $\langle r^2 \rangle$:

- contribution of $\langle r^4 \rangle \sim 15\%$ of finite-size effect, even $\langle r^6 \rangle$ contributes 4%
- even at $q^2=0.6$, where finite size effect only 0.077, $\langle r^4 \rangle$ contributes 10%

→ serious interference of higher moments

Wrong $\langle r^4 \rangle$ or wrong $\langle r^6 \rangle$ → wrong $R$

= short version why some determinations of $R$, discussed below, are wrong
Determination of higher moments: difficult


For such a density (form factor) the higher moments are increasing with order, i.e. $\langle r^4 \rangle = 2.5 \, R^4$, $\langle r^6 \rangle = 11.6 \, R^6$ etc, hence giving a large contribution to $G(q)$.

The consequence: there is no $q$-region where the $R^2$ term dominates the finite size effect to $>98\%$ and the finite size effect is sufficiently big compared to experimental errors to allow a, say, $2\%$ determination of the rms-radius. There is also no region of $q$ where the $r^4$-moment can be determined accurately without getting into difficulty with the $r^6$-term.

e tc

Consequence

in fit of low-$q$ data uncertainty of the highest moments ($N=4,6$) large these moments affect extracted $\langle r^2 \rangle$

Way out

fit data to highest $q$'s with polynomial of large $N$
above uncertainty affects only the moments $\langle r^{2N} \rangle$ with the largest $N$
but produces significant $\langle r^2 \rangle$, $\langle r^4 \rangle$, ...

done e.g. by Bernauer et al. (2010), finds $\langle r^4 \rangle$ quoted above could use this $\langle r^4 \rangle$ as input in low-$q$ fit with polynomial .... if absolutely want to do polynomial fit

BUT: polynomial fit = bad idea, see below
3. Traditional for p and d: parameterize $G(q)$, not $\rho(r)$

creates new problems

data sensitive to $R$ at $q > q_{\text{min}}$, want slope of $G(q = 0)$

Sensitivity to $R$: explored via notch-test

maximum sensitivity to $R$ at $q \sim 0.8 \text{fm}^{-1}$ where $G(q) \sim 0.8$

note: corresponds to 0.01 - 0.04 $(\text{GeV}/c)^2$

extrapolation of $1 - G(q) < 0.2$ model-dependent

difficulty enhanced by floating normalization, i.e. floating "1"
4. Parameterization of $G(q)$ without considering $\rho(r)$ often generates nonsense

Pade fit of Bernauer data, $G(q) = \frac{1 + a_1 q^2}{1 + b_1 q^2 + b_2 q^4 + b_3 q^6}$

$q < 2fm^{-1}$, covers full region sensitive to $R$
excellent $\chi^2$ (as low as Spline fit)

get $R \sim 1.48fm$ (Phys.Rev.C 89(2014)012201)

Why discuss 0.84 fm vs. 0.88 fm if 1.48 fm fits perfectly low-\(q\) data?

This is not a rhetorical question! Cannot be answered by ignoring it.
Understanding of $R = 1.48 \text{ fm}$

Split fit into two contributions $G = G_1 + G_2$:

$G_1 = \text{Pade for } q^2 > 0.06 \text{ plus dashed line for } q^2 < 0.06$

$G_2 = \text{Pade} - G_1$

$G_1$ has 'normal' $q=0$ slope, norm of 0.995

$G_2 \sim e^{-q^2/(0.02 \text{ fm}^2)}$

Corresponds to $\rho \sim e^{-r^2/(200 \text{ fm}^2)}$

$G_2$ leads to large rms-radius despite small norm of $\sim 0.005$

Choice of parameterization of $G(q)$ implies choice of $\rho(r)$

Harmless-looking $G(q)$ (e.g. Pade) can correspond to outrageous $\rho(r)$

For sensible $R$ must study behavior of $\rho(r)$ at large $r$

To avoid nonsense not visible in $G(q)$

Together with point 1: use physics to constrain $\rho(r \gg) \rightarrow$ different talk

Concentrate today on fits without large-$r$ constraint
10 general considerations on recent fits of (e,e) data

- discuss occasionally with recent examples
- points 1-10 may seem trivial, but are all too often ignored

1. $G(q)$ as polynomial in $q^2$: $1 - q^2\langle r^2\rangle/6 + q^4\langle r^4\rangle/120 - \ldots$

- used by many authors in past
- in 2014 quantitatively studied by Kraus *et al.*
  - parameterized $G(q) \rightarrow$ pseudo data $\pm 0.4\% \rightarrow$ power series fit $\rightarrow R_{fit}$
  - always gives low $R_{fit}$, and $R_{fit}$ depends strongly on $q_{max}$

\[ e.g. \text{ for } Q^2 = 0.03 \text{ (typical) and linear fit} \]
\[ \text{defect according to figure } = 0.04 \text{ fm} \]
\[ \text{as large as discrepancy (e,e)...$\mu X$} \]

must be dumb to use low-order polynomial
AMT Fit

$r_{fit} - r_{actual}$ (fm)

$Q_{max}^2$ (GeV$^2$)
Understanding of low $R$’s

discuss for ”very low-q” linear fit in $q^2$
corresponds to assumption $\langle r^4 \rangle = 0$

how can produce $\langle r^2 \rangle \neq 0$ but $\langle r^4 \rangle = 0$?

What would a physicist think?
Understanding of low $R$’s

discuss for ”very low-q” linear fit in $q^2$
corresponds to $\langle r^4 \rangle = 0$
how can produce $\langle r^2 \rangle \neq 0$ with $\langle r^4 \rangle = 0$?

What would a physicist think?

would try to think how corresponding $\rho(r)$ would look like:
positive inside, negative at very large $r$
then negative tail can compensate positive part in $\langle r^4 \rangle$ to yield $\langle r^4 \rangle = 0$
given $r^6$ weight in $\langle r^4 \rangle$-integral

But: negative tail also impacts $\langle r^2 \rangle$
will yield too small value for $R$

remember: $\langle r^4 \rangle$ contributes $\sim$15% of finite-size effect
at $q$ of maximal sensitivity to $R$

= physical explanation of above results of Kraus et al.

= general argument why truncated polynomial (also higher order) generates problems
Illustration of problems with low-order polynomial fits

recent fit of Higinbotham et al.
Mainz80+Saskatoon data
$q_{\text{max}}^2 = 0.8\text{fm}^{-2}$ for $R$ with smallest $\delta R$

$G(q) = a_0(1 + a_1q^2)$

Find $R = 0.844 \pm 0.009\text{fm}$
conclude that is compatible with $\mu X$ result

Wrong, as a trivial back-of-the-envelope estimate shows! For $q^2=0.8$

\[
\begin{align*}
    q^2R^2/6 & = 0.094 \\
    q^4\langle r^4 \rangle/120 & = 0.0138
\end{align*}
\]

$\rightarrow q^2$ contribution wrong by $\sim 14.7\%$
$\rightarrow R^2$ is wrong by $\sim 14.7\%$

... and this sort of analysis is claimed to provide insight on radius-puzzle!

Another illustration: recent fit by Griffioen, Carlson, Maddox

use $G(q) = 1 - q^2\langle r^2 \rangle/6 + q^4\langle r^4 \rangle/120$
$Q^2 \leq 0.02\text{GeV}^2$, ”Bernauer” data, $R = 0.850 \pm 0.019\text{fm}$ compatible with $\mu X$?

find contribution $q^4$-term 0.0018
experimental $\langle r^4 \rangle$ yields 0.011

$\Delta \sim 15\%$ error in $\langle r^2 \rangle$. $R=0.850$ simply wrong!
Another demonstration of effect of $\langle r^4 \rangle$

$\langle r^2 \rangle$ vs $\langle r^4 \rangle$ for different densities $\delta(r - c)$, exponential, gaussian
Griffioen et al.

$R$ linear function of $\langle r^4 \rangle / \langle r^2 \rangle^2$
extrapolate to true $\langle r^4 \rangle$ (Bernauer high-$q$ fit)
get $R \sim 0.88 \text{fm}$
2. Poles of parameterized $G(q)$ cause problems

some parameterization have poles at $q > q_{max}$

- power series
- inverse polynomials
- some Pade

.....

poles $\rightarrow$ oscillations in $\rho(r)$ out to extremely large $r$

**consequences**

- large-$r$ contributions can have adverse effects on $R$
- cannot judge if large-$r$ behavior of $\rho(r)$ sensible

**example:** IP fit Bernauer jump at $N=10$ due to close pole

**note:** choice of $N \rightarrow$ arbitrariness of $R$
3. Two-photon effects

PWIA relation $\sigma \leftrightarrow G(q)$ complicated by 2-$\gamma$ exchange TPE at low $q$ mainly Coulomb distortion, well under control.

At $q$ of maximum sensitivity to $R$ \[ 1 - G(q) \sim 0.2, \] so TPE$\sim$0.01 do matter!

Inappropriate TPE:
- no corrections
- corrections for point nucleus (McKinley-Feshbach)
- phenomenological corrections (not determined at low $q$)

For valid result on $R$ must use valid TPE
4. Good $\chi^2$

$R$ valid only if data is fit with good $\chi^2$

Remember: one standard deviation $\sigma$ corresponds to $\Delta \chi^2 = 1$

many analyses take cavalier-attitude about $\chi^2$
accept $\chi^2$ of (say) 1.5 per degree of freedom
while good fits give 1.1

With (typically) 500 data points

difference 1.1 $\leftrightarrow$ 1.5 corresponds to $\Delta \chi^2$ of 200
this difference corresponds to $14\sigma$!
which is not acceptable when discussing a $5\sigma$ difference $(e,e) \leftrightarrow \mu H$
Important distinction: absolute value of $\chi^2$ is not the main issue

- depends on optimism of experimentalist assigning $\delta \sigma$
- depends on eventual rescaling of $\delta \sigma$

Really relevant: comparison of fits to same $\sigma \pm \delta \sigma$

- if fit ”A” gives significantly larger $\chi^2$ than fit ”B”
- then fit ”A” has systematic differences to data
- then fit ”A” must be discarded

Many published fits have $\chi^2 \gg \chi^2_{\text{min}}$, hence are irrelevant

Illustration

- recent fit of Higinbotham et al.
- use dipole form factor
- fit Mainz80+Saskatoon+Stanford+JLab data

- fit has reduced $\chi^2$ of 1.25
- find $R = 0.849 \pm 0.006 \text{fm}$

- conclude that $R$ is compatible with $\mu X$
But $\chi^2$ is much too large

take one of my fits of world data (603 data points)
find reduced $\chi^2$ of 0.96, not 1.25

Consequence: systematic deviation of Higibotham dipole fit from cross section data

solid line shows change of low-$q$ slope to $R = 0.88\, fm$
fits data!

the dipole "fit" (dotted line) is simply wrong
A direct comparison of cross section ratios of "fit" and fit

Higinbotham dipole

MD
Another illustration: CF fit of Griffioen et al.

fit Bernauer data
$q_{\text{max}} = 5 \text{fm}^{-1}$

get reduced $\chi^2$ of 1.61 (+pole ...)
find $R = 0.8389 \pm 0.0004 \text{fm}$

conclude that agrees with $\mu X$

But Bernauer data can be fit with reduced $\chi^2$ of 1.14 shown years ago

For 1400 data points difference 1.61 .. 1.14 is $\Delta \chi^2 = 660!$

Who in his right mind would call that a ”fit”?

5. Correct treatment of data

some data sets are floating
i.e. Bernauer data have 31 norm factors for 34 data sets
this property must be respected in the fits
even if this is a nuisance
ignoring this (out of laziness) invalidates the results
6. $G_e(q)$ and $G_m(q)$ are correlated
   the (e,e) cross sections depend on $G_e$ and $G_m$
   both have to be determined from the data
   use of convenient recipe for $G_m$: $G_e$ not reliable, must be discarded

7. Use of relevant data

   $R$ is determined mainly by cross sections at low $q$
   omission of $\sigma$-data yields $R$ not coming from (e,e)
   then $R$ determined by other input
   cannot be claimed to come from (e,e)

   statement seems trivial, but relevant in practice (see VDM-discussion)

8. Choice of $q_{max}$

   fit of (e,e) involves choice of $q_{max}$ of data employed
   choice a priori arbitrary
   must explore consequences
   radii depend strongly on $q_{max}$ (see below for more detail)
   trivial example already shown in figure by Kraus et al.

   $R$ without exploration of $q_{max}$-dependence not valid
9. Model-dependence due to choice of fit-function

some authors use 1- parameter fits at low $q$
power of $q^2$, linear in $z$, single dipole, ....
then $\langle r^4 \rangle / \langle r^2 \rangle$ is fixed by fit-function, not data
$\langle r^4 \rangle \neq$ true value known from fit of data over whole $q$-range
then $\langle r^2 \rangle$ must compensate for wrong $\langle r^4 \rangle \rightarrow$ wrong value of $R$
also gives larger $\chi^2$

Illustration: low-$q$ fits of Horbatsch+Hessels

fit Bernauer data, $q_{max} \sim 1.6 fm^{-1}$
use 1-parameter dipole resp. 1-parameter linear function in $z$
find $R = 0.842(2)$ resp. $0.888(1) fm$

fit-function fixes $\langle r^4 \rangle$ to $1.244 fm^4$ resp. $2.15 fm^4$. True value $2.58 fm^4$ (fit to all data)

explains both low and discrepant radii

is reason why $\chi^2/dof = 1.11$ instead of 1.03
for 761 points $\Delta \chi^2 = 61! \sim 8\sigma's$

'Fits' 8$\sigma$’s from minimum are irrelevant when discussing 5$\sigma$ difference (e,e)$\leftrightarrow \mu X$!
10. Use of polarization data

\( G_e \) and \( G_m \) are strongly coupled
best separation needed for accurate \( G_e \)
must include polarization data measuring \( G_e/G_m \)

in same vein:
use *world* data, not only biased selection
*world* without Bernauer: 600 data points
for \( q < 1 \text{fm}^{-1} \sim 90 \) points
Bernauer: 1400 points
use of *all* data improves in particular L/T separation

Summary of recent fits to (e,e) data: see table next page

lists the various shortcomings
the most serious ones:
  power series in \( q^2 \)
  poor \( \chi^2 \)
  \( \langle r^4 \rangle/\langle r^2 \rangle \) fixed by parameterization
  \( q_{max} \) dependence ignored
## Summary of recent fits

<table>
<thead>
<tr>
<th></th>
<th>poles/singularities</th>
<th>polynomial in (q^2)</th>
<th>incorrect/phen./no 2(\gamma)</th>
<th>too large (\chi^2)</th>
<th>data mistreated</th>
<th>(G_M(q)) assumed</th>
<th>no relevant (e,e) data</th>
<th>(q_{max})-dependence ignored</th>
<th>no PT data</th>
<th>(\langle r^4\rangle/\langle r^2\rangle) fix, or (\langle r^6\rangle=0)</th>
<th>(\rho(r))</th>
<th>(q_{max}) (fm)</th>
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<tbody>
<tr>
<td>Adamuscin (VDM) [?]</td>
<td>(x)</td>
<td>(x)</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
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<td>(x)</td>
<td>?</td>
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<td>X</td>
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<td>(x)</td>
<td>?</td>
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</table>

CM=conformal mapping, BI=Bayesian Inference, TC=Tail Constraint, k=kink, –=no \(\rho(r)\)
Three important observations

1. Dependence of extracted $R$ on $q_{\text{max}}$

   dependence exhibited for polynomial series at very low $q$
   trivial, discussed already above

   fits with large $q_{\text{max}}$, $5 \text{ fm}^{-1} - 10 \text{ fm}^{-1}$, tend to give $R > 0.88 \text{ fm}$

   $q_{\text{max}}$ dependence found by various authors, show result of Lee et al.

How can these large $q$’s affect $R$?
rms-radius is ’measured’ at $q = 0$!

This behavior calls for explanation!
Understanding: effect of large-\(r\) tail of \(\rho(r)\)

remember: rms-radius sensitive to \(r\) due to \(r^4\) weight
large \(r\) affect low \(q\) and curvature of \(G(q)\) below \(q_{min}\)

Data up to large \(q\) fix \textit{shape} of \(\rho(r)\) including large-\(r\) tail
this reduces arbitrariness of shape of fitted \(G(q)\) at low \(q\)
this leads to more reliable extrapolation from \(q_{min}\) to \(q = 0\)

For demonstration study \(\rho(r, Q_{max}) = \ldots \frac{1}{r} \int_{0}^{Q_{max}} G(q) \sin(qr) q \, dq\)

Important observation: to fix \(\rho(r)\) at the larger \(r\) must include \(G(q)\) at the higher \(q\)’s

Fits with maximal \(q_{max}\) yield the most reliable extrapolation to \(q = 0\)
Importance of high $q$ data for $dG/dq^2(q = 0)$ of fit is not a contradiction

low-$q$ data important to fix $G(q)$ in region where data sensitive to $R$

high-$q$ data fix shape of $\rho(r)$ i.e. shape of $G(q)$ needed for extrapolation to $q = 0$

Or more simply said:

rms-radius depends strongly on density at large $r$: $R^2 = \ldots \int \rho(r)r^4dr$
to fix (implicitly) this density need $G(q)$ up to large $q$

2. Radii from VDM fits always low, typically in 0.84 fm region

always much too large $\chi^2$
needs understanding

Basic assumption of VDM

leads a priori to form factor $G(q) = \sum_i a_i/(1 + q^2 \gamma_i)$, $\rho(r) = \sum a_i e^{-\gamma_i r}/r$

$\gamma_i^{-1}$ = masses squared of vector mesons

The promise: VDM could fix problem with large-$r$ behavior
tail $\sim e^{-\gamma r}/r$ is given by physics
Complication

'pole' closest to physical region (responsible for low $q$) is *not* a pole
it is a cut starting at $4m^{2}_\pi$
accounts for interaction with pion tail of N (triangle diagram)

Strength distribution in cut: difficult to come by
determined by Hoehler *et al.* in 1976 using dispersion relations
only partly updated

Generic problem of VDM analyses
$\chi^2$ is too large
systematic differences to data at low $q$: fit Mergel *et al.*

![Graph](image)

Difference to true $R$
can be trivially read off figure
\pm same for all VDM fits
since Hoehler’s time
Reason for too large $\chi^2$

VDM spectral function is too strong a constraint
has not enough flexibility to allow good fit of (e,e)

Demonstration of constraining role of spectral function

VDM analysis of Adamuscin et al.
want to fit (e,e) data over largest $q^2$-range ($<0,>0$)

Difficulty: TPE in (e,e) cross sections ($G_e$ from L/T and PT disagree)
'solution': omit $\sigma$, fit only polarization transfer PT data
result: $R = 0.848 \pm 0.007\, fm$
amazingly small error bar!

BUT...

PT data measure only ratio $G_e/G_m$
contain no information whatsoever on charge radius

Conclusion

spectral function all by itself fixes $R$ to amazing precision
adding relevant (e,e) data, $\sigma$, only leads to bad $\chi^2$!
explains the problem of VDM analyses since Hoehler’s time
find always poor $\chi^2$ and $0.84\, fm$ + systematic deviation from data
Further complication: large-\( q \) behavior of \( G(q) = \sum \) monopoles

falls like \( q^{-2} \)
leads to \( 1/r \) singularity of \( \rho(r) \sim e^{-\kappa r}/r \)
also contradicts QCD which wants fall-off faster than \( q^{-4} \)
\textit{and} does not allow to even qualitatively fit data

Solution in standard VDM analyses

different ’fixes’ employed
effect in the end:
multiply \( \sum \) monopoles with dipole form factor
with mass \( \Gamma^{-1} \) substantially greater than the \( \gamma_i \)’s

In practice: dipole contributes 0.05 – 0.2 fm of rms-radius

this contribution is purely phenomenological (choice of dipole, choice of \( \Gamma \))
[theory+phenomenology] no better than [phenomenology alone]
removes hope to get from VDM better fix on \( \rho \) in tail

What remains: too large \( \chi^2 \) of VDM fits

systematic deviations from (e,e)
cannot claim to ”fit” (e,e) data
can only say to ”use” (e,e) data
Disgression

VDM adherents claim that analytic structure of $G(q)$ important could be, e.g. poles at $q > q_{\text{max}}$ certainly are detrimental

Test if analytic structure would really change results

use VDM-type form factor:
sum of monopoles times dipole = M·D-parameterization

$$G(q) = \sum_i a_i/(1 + q^2 \gamma_i) \quad 1/(1 + q^2 \Gamma)^2$$

with free $a_i$, $\gamma_i$, with VDM-constraint $\gamma_i^{-1} > 4m^2_{\pi}$

Result of fit of world data up to $12\text{fm}^{-1}$

variation of handpicked $\gamma_i$'s not even needed
fit of parameters $a_i$ enough
$\chi^2$ as low as other best fits (SOG, Laguerre) of same data
within a $\Delta \chi^2$ of $<1!$ ($\chi^2=587$ for 603 data points)

$R$ differs by $0.002\text{fm}$

$\rightarrow G_{\text{SOG}}, G_{\text{Laguerre}}$ not different from $G_{\text{MD}}$
SOG, Laguerre analytic structure equivalent to VDM

M·D parameterization optimal for (partial) control of large-$r$ density
3. Conformal Mapping approaches could be helpful, but...

flexible expansions of $G(q)$ in terms of $q$ not optimal
expansion parameter can become large, $>1$
variable transformation could help

Standard choice $z = \frac{\sqrt{t_c} - t - \sqrt{t_c - t_0}}{\sqrt{t_c} - t + \sqrt{t_c + t_0}}$, $t = q^2$

most often with $t_0 = 0$ and $t_c = 4m^2_\pi$
yields expansion parameter $z < 1$, see figure

Claimed to 'linearize' extrapolation: not really true
CM does ease multi-parameter expansion of $G(z)$

$q^{-4}$ fall-off of $G(q)$ justifies bound on size of coefficients of $G(z)$ (Hill, Paz)

avoids divergent results due to over-fitting of data involving large coefficients

but some care is still needed

see e.g. $G(q)$ of Lee et al., contradicts justification for bound

Even in terms of $z$, use of power-series is highly unwise

Better choice: Borisyuk

$G(q) = \text{polynomial in } z \text{ times dipole in } q$

then $q^{-4}$ fall-off guaranteed, polynomial remains of order 1
Typical parameterizations used in above table

**VDM, powers in $q^2$ many authors**
- already discussed, gives wrong $R$

**Inverse Polynomials, Continued Fraction, some Pade many authors**
- have often singularities above $q_{\text{max}}$
- have often no $\rho(r)$, cannot judge whether large-$r$ tail sensible

**[N][N+2]Pade with $b_i > 0$ Arrington, Sick**
- OK, but $\rho(r)$ requires effort

**Conformal Mapping + power-series in $z$ Lee**
- $G(q \to \infty) \to$ constant value, no $\rho(r)$

**Conformal Mapping + power series in $z$ times dipole Borisyuk**
- OK, $\rho(r)$ requires effort

**Sum sigmoid functions times dipole Graczyk**
- OK, $\rho(r)$ requires effort

**Dipole times spline modification at larger $q$’s Horbatsch**
- fixes $\langle r^4 \rangle / \langle r^2 \rangle$, wrong

**Sum-of-Gaussians, Laguerre polynomial, M·D Sick**
- OK
Average of $R$-values

use analyses without serious deficiencies, large $q_{max}$
take brute-force average without considering quoted $\delta R$

Adjustments applied

Borisyuk

does not include polarization transfer data
when I omit PT data, $R$ increases by $0.017 fm$, apply this correction

Graczyk

uses phenomenological TPE, not justified at low $q$
correct $R$ for difference to standard TPE

<table>
<thead>
<tr>
<th>1. author</th>
<th>adjustment</th>
<th>rms-radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernauer</td>
<td></td>
<td>$0.879 \pm 0.008 fm$</td>
</tr>
<tr>
<td>Borisyuk</td>
<td>$-0.017 fm$</td>
<td>$0.895 \pm 0.012 fm$</td>
</tr>
<tr>
<td>Graczyk</td>
<td>$0.009 fm$</td>
<td>$0.890 \pm 0.003 fm$</td>
</tr>
<tr>
<td>Lee</td>
<td></td>
<td>$0.904 \pm 0.015 fm$</td>
</tr>
<tr>
<td>Sick (+tail constraint)</td>
<td></td>
<td>$0.886 \pm 0.008 fm$</td>
</tr>
<tr>
<td>Average (unweighted)</td>
<td></td>
<td>$0.891 \pm 0.009 fm$</td>
</tr>
</tbody>
</table>

estimated standard deviation from the average $R$: $0.009 fm$

Outliers: $0.879$ (Bernauer data), $0.904$ (power series?)
Fits

Adamuscin: VDM
Bernauer: power series, inverse polynomials, spline, B, 1.14
Borisyuk: power series in \( z \) times dipole, W, 0.75
Graczyk: sum of dipoles·sigmoid functions, W, \( \chi^2 \) not known
Griffioen: power series \( q^4 \), ”B”, 1.0
Griffioen: continued fraction (pole), B, 1.6
Higinbotham: power series \( q^2 \), S+M, 0.7
Higinbotham: dipole, S+M, 1.25
Horbatsch: dipole, B, 1.11
Horbatsch: 1-parameter fit in \( z \), B, 1.11
Lee: power series in \( z \), W \( \delta \sigma \) rescaled, 0.64
Lee: power series in \( z \), B \( \delta \sigma \) rescaled, 0.82
Lorenz: VDM, B, 2.2
Lorenz, continued fraction (pole), B, <1.6
Sick: SOG, Laguerre polynomials, M·D, W, 0.98
Sick: SOG, Laguerre polynomials, M·D, W, 1.35 (norm fixed)
Overall: even results without obvious flaws differ somewhat
remaining model dependence of extrapolation, $q_{\text{max}}$, choice of data

For more reliable $R$ respect 3 insights

1. Fit data to largest $q$ possible

   this determines $\rho(r)$ including large-$r$
   this fixes shape of $G(q)$ below $q_{\text{min}}$ needed for extrapolation to $q = 0$

2. Use parameterization accessible in both $q$- and $r$-space

   SOG, Laguerre, M·D (as in VDM fits)
   then can check implied behavior of $\rho(r)$ at large $r$
   can check whether physically sensible
   can fit data + eventual large-$r$ constraint simultaneously

3. Constrain large-$r$ behavior via physics knowledge

   $\rho(r \gg)$ is given by wave function of least-bound Fock state: $n + \pi^+$
   shape (not absolute norm) given by known removal energy
   can be imposed at asymptotic $r$ where $\rho(r) < 1\%$ of $\rho(\text{center})$

   this fixes most of present problems with $R$

See PRC 89 (14) 012201