QCD Equation of state and critical end-point estimates from lattice QCD

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HotQCD collaboration
Major Themes from Lattice

![Diagram showing phase transitions and critical points in strongly interacting matter. The diagram includes labels for critical points and phase transitions at various temperatures and baryon chemical potentials. Notable points include $\mu_B = 2T_f$ and $\mu_B = 3T_f$.](image-url)
In view of the RHIC Beam Energy Scan-II in 2019-20 it is important to have control over the Equation of State for $\mu_B / T \leq 3$. 

- $\mu_B = 2T_f$ 
- $\mu_B = 3T_f$
Major Themes from Lattice

- In view of the RHIC Beam Energy Scan-II in 2019-20 it is important to have control over the Equation of State for $\mu_B / T \leq 3$.
- Measure the curvature of chiral and freezeout curves expected from QCD thermodynamics.
Major Themes from Lattice

- In view of the RHIC Beam Energy Scan-II in 2019-20 it is important to have control over the Equation of State for $\mu_B/T \leq 3$.

- Measure the curvature of chiral and freezeout curves expected from QCD thermodynamics.

- Look for possible existence and bracket the position of critical end-point in the phase diagram.
Technique we follow: Taylor expansion

- One of the methods to circumvent sign problem at finite $\mu$:
  Taylor expansion of physical observables around $\mu = 0$ in powers of $\mu / T$
  [Bi-Swansea collaboration, 02]

\[
\frac{P(\mu_B, T)}{T^4} = \frac{P(0, T)}{T^4} + \left( \frac{\mu_B}{T} \right)^2 \frac{\chi_2^B(0, T)}{2 T^2} + \left( \frac{\mu_B}{T} \right)^4 \frac{\chi_4^B(0)}{4!} + \ldots
\]

- The radius of convergence of the series will give the location of singularities. Otherwise it is analytic in $\mu_B / T$. [Gavai& Gupta, 03]

- For a given $\mu_B / T$ how many $P_n$ are needed for convergence?
Technique we follow: Taylor expansion

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[Bi-Swansea collaboration, 02]

\[
\frac{P(\mu_B, T)}{T^4} = \frac{P(0, T)}{T^4} + \left( \frac{\mu_B}{T} \right)^2 \frac{\chi^B_2(0, T)}{2T^2} + \left( \frac{\mu_B}{T} \right)^4 \frac{\chi^B_4(0)}{4!} + \ldots
\]

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The fluctuations of conserved charges can be expressed in terms of Quark no. susceptibilities (QNS).

QNS $\chi_{ij}$'s can be written as derivatives of the Dirac operator.

Example:

$$\chi_{u2}^u = \frac{T}{V} \langle Tr(D_u^{-1}D_u'' - (D_u^{-1}D_u')^2) + (Tr(D_u^{-1}D_u'))^2 \rangle.$$  

$$\chi_{11}^{us} = \frac{T}{V} \langle Tr(D_u^{-1}D_u'D_s^{-1}D_s') \rangle.$$  

Higher derivatives $\rightarrow$ more inversions

Inversion is the most expensive step on the lattice!

Why extending to higher orders so difficult?

- Matrix inversions increasing with the order
- Delicate cancellation between a large number of terms for higher order QNS.
Challenges for Lattice computations

- The fluctuations of conserved charges can be expressed in terms of Quark no. susceptibilities (QNS).
- QNS $\chi_{ij}$'s can be written as derivatives of the Dirac operator.
  
  Example: 
  \[
  \chi^u_2 = \frac{T}{V} \langle \text{Tr} (D^{-1}_u D''_u - (D^{-1}_u D'_u)^2) + (\text{Tr} (D^{-1}_u D'_u))^2 \rangle.
  \]
  
  \[
  \chi^{us}_{11} = \frac{T}{V} \langle \text{Tr} (D^{-1}_u D'_u D^{-1}_s D'_s) \rangle.
  \]
- Higher derivatives $\rightarrow$ more inversions
  
  Inversion is the most expensive step on the lattice!

- Introduce $\mu$ such that it appears as a linear term multiplying the conserved number \[\text{[Gavai & Sharma, 1406.0474]}\] as in the continuum, a limit of conventional $e^\mu$ procedure \[\text{[Hasenfratz & Karsch, 83]}\].
- Divergences exist for $\chi_n$, $n \leq 4$. No divergences for $\chi_6$ and beyond.
- New algorithms for calculating inversions developed \[\text{[F. de Forcrand, B. Jaeger, 1710.07305]}\].
The fluctuations of conserved charges can be expressed in terms of Quark no. susceptibilities (QNS).

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Example:

$\chi_u^{11} = \frac{T}{V} \langle \text{Tr} \left( D_u^{-1} D_u'' - (D_u^{-1} D_u')^2 \right) + \left( \text{Tr}(D_u^{-1} D_u') \right)^2 \rangle$.

$\chi_{us}^{11} = \frac{T}{V} \langle \text{Tr} \left( D_u^{-1} D_u' D_s^{-1} D_s' \right) \rangle$.

Higher derivatives $\rightarrow$ more inversions

Inversion is the most expensive step on the lattice!

Calculate $n_B$ in imaginary $\mu$ and extract higher order fluctuations

[M D’Elia, M-P Lombardo, 02]

Current state of the art: 6th order fluctuations known with very good precision  
[Gunther et. al, 1607.02493,M. D’elia et. al., 1611.08285] even 8th order known with reasonably good precision.  
[M. D’Elia et. al., 1611.08285, more details in talk by C. Bonati]
Our Set-up

- $V = N^3 a^3$, $T = \frac{1}{N_T a}$. We use $N_T = 6, 8, 12, 16$ lattices for $\chi_{2,4}$ and $N_T = 6, 8$ for higher order fluctuations.

- Box size: $m_\pi V^{1/3} > 4$

- Input $m_s$ physical and $m^G_\pi = 160$ MeV for $T > 175$ MeV and $m^G_\pi = 140$ MeV for $T \leq 175$ MeV.

- Calculating explicitly the lowest eigenvalues improves performance of the fermion inverter.
EoS in the constrained case

- In most central heavy-ion experiments typically:
  \[ n_S = 0 , \ \text{Strangeness neutrality}, \]
  \[ \frac{n_Q}{n_B} = \frac{n_P}{n_P + n_N} = 0.4. \]
  [Bi-BNL collaboration, 1208.1220]

- For lower \( \sqrt{s} \) collisions: Need to understand baryon stopping!
- Imposes non-trivial constraints on the variation of \( \mu_S \) and \( \mu_Q \).
- Possible to vary them by only varying \( \mu_B \) through
  \[
  \mu_S = s_1\mu_B + s_3\mu_B^3 + s_5\mu_B^5 + \ldots
  \]
  \[
  \mu_Q = q_1\mu_B + q_3\mu_B^3 + q_5\mu_B^5 + \ldots
  \]
Central values of $P_4$, $P_6$ already deviate from Hadron Resonance gas model at $T > 145$ MeV $\rightarrow$ need to analyze the errors on $P_6$ better.

$P_6$ has characteristic structure at $T > T_c$ $\rightarrow$ remnant of the chiral symmetry due to the light quarks. Effects of $U_A(1)$ anomaly?

Essentially non-perturbative $\rightarrow$ cannot be predicted within Hard Thermal Loop perturbation theory.
EoS in the constrained case

- The EoS for the constrained case is well under control for $\mu_B/T \sim 2.5$ with $\chi_6$.
- Full parametric dependence for $N_B$ on $T$ available in arxiv: 1701.04325.
- Expanding to $\mu_B/T = 3$, need to calculate $\chi_8$!
Summary for the EoS

- Continuum estimates from two different fermion discretization agree for $\mu_B/T \leq 2$.
  [Bielefeld-BNL-CCNU collaboration, 1701.04325, Borsanyi et. al, 1606.07494].
- Steeper EoS for RHIC energies compared to LHC energy.
Baryon number density

- $\chi_6$ contribution is $30$-times larger than in pressure.

\[
\frac{N(\mu_B)}{T^3} = \frac{\mu_B}{T} \chi^B_2(0) + \frac{1}{2} \left( \frac{\mu_B}{T} \right)^4 \chi^B_4(0) + \frac{1}{4!} \left( \frac{\mu_B}{T} \right)^6 \chi^B_6(0) + \ldots
\]

- Strongly sensitive to the singular part of $\chi^B_6$.

- For strangeness neutral system, effect is milder.
Curvature of freeze-out line

- The lines of constant $f \equiv \epsilon$ or $p$ is characterized as:

$$T_f(\mu_B) = T_0 \left( 1 - \kappa_2^f \left( \frac{\mu_B}{T_0} \right)^2 - \kappa_4^f \left( \frac{\mu_B}{T_0} \right)^4 \right)$$

- For $145 \leq T \leq 165$ MeV: $0.0064 \leq \kappa_2^P \leq 0.0101$, $0.0087 \leq \kappa_2^\epsilon \leq 0.012$.

- Consistent with the curvature of the chiral 'crossover' transition curve $0.0066(7)$ to $0.013(3)$. [arxiv:1011.3130, 1507.03571, 1507.07510, 1508.07599]

- For $\mu_B/T \leq 2$ the contribution from $\kappa_4$ to $T_f(\mu_B)$ within errors of $\kappa_2$. 

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Different LCP’s agree within 2 MeV for $\mu_B/T \leq 2$ for 3 initial choices of $T_0$.

For lines $P = \text{const}$, the entropy density changes by 15% $\rightarrow$ better description of LCP for viscous medium formed in heavy-ion collisions.

[Bi-BNL-CCNU collaboration, 1701.04325].

STAR results give a steeper curvature.

[arXiv:1412.0499].

Agreement with the recent ALICE results. [arXiv:1408.6403].

Consistent with phenomenological models if a higher $T_f \sim 165$ MeV is assumed [Becattini et. al., 1605.09694].

However lattice studies show explicitly that the HRG breaks down!
QCD vs Hadron Resonance Gas model

- $T_f$ measured at ALICE is at the edge where lattice results deviate from HRG.
- For $T_f \sim 165$ MeV thermodynamic quantities deviate from HRG estimates more dramatically!

- Repulsive baryon interactions more important? Excluded volume calculations included in the standard statistical model increases $T_f$ for ALICE energies. 

\[ \text{[A. Andronic et. al., 16]} \rightarrow \text{Consistent with expected deviations from HRG model} \]
Including Van der Waal’s interaction for baryons + non-interacting mesons + resonances, new versions of HRG has been studied → significant deviation from non-interacting HRG.

Lattice data can constrain such models strongly! Currently none of these models are perfect to describe QCD at crossover region. For new developments

It would be important to resolve this 10 MeV spread in $T_f$ specially for CEP searches.
New diagnostics!

- Off-diagonal fluctuations are more sensitive to deviation from HRG and baryon interactions.
- $\chi_{31}^{BS} - \chi_{11}^{BS}$ already rules out a different freezeout $T_f$ for strangeness. 
  
  [Bielefeld-BNL-CCNU collaboration, 13].

- $\chi_{11}^{BS}/\chi_2^S$ shows $\sim15\%$ deviation between 155 and 165 MeV. Analysis with ALICE [A. Andronic et. al., 16] consistent with lattice at $T_c \sim 155$ MeV. Including $\Sigma^* \rightarrow N\bar{K}$ will make the ratio lower!

- Similar results at higher $\mu_B$ would be interesting! [A. Chatterjee et. al., STAR collaboration, Poster QM17]
The Taylor series for $\chi^B_2(\mu_B)$ should diverge at the critical point. On finite lattice $\chi^B_2$ peaks, ratios of Taylor coefficients equal, indep. of volume.

The radius of convergence will give the location of the critical point.

[Gavai & Gupta, 03]

Definition: $r_{2n} \equiv \sqrt{2n(2n - 1)} \left| \frac{\chi^B_{2n}}{\chi^B_{2n+2}} \right|$.  

- Strictly defined for $n \to \infty$. How large $n$ could be on a finite lattice?
- Signal to noise ratio deteriorates for higher order $\chi^B_n$. 

Critical-end point search from Lattice

- Current bound for CEP: $\mu_B/T > 2$ for $135 \leq T \leq 160$ MeV
  [Bielefeld-BNL-CCNU, 1701.04325].
- The $r_n$ extracted by analytic continuation of imaginary $\mu_B$ data
  [D'Elia et. al., 1611.08285] consistent with this bound.
- Results with a lower bound? [Datta et. al., 1612.06673, Fodor and Katz, 04] need to understand the systematics in these studies. Ultimately all estimates will agree in the continuum limit!
Preparing for BES-II runs: LQCD EoS important for hydrodynamic modeling of QGP. For $\mu_B/T < 2 \rightarrow \sqrt{s_{NN}} \geq 11$ GeV already under control with $\chi_6^B$. 
Outlook

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- Analysis of $\chi_8^B$ important to estimate the errors on the EoS measured with the sixth order cumulants and going towards $\mu_B/T = 3$. 
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The lines of constant $\epsilon, p$ consistent with LQCD estimates of curvature of chiral crossover line.

Higher order cumulants will also help in bracketing the possible CEP. Most LQCD calculations suggest $\mu_B(CEP)/T \geq 2$. 
For any order $n$, the artifacts $\sim \mathcal{O}(a^{n-4})$. 
Backup: Nature of the divergences for higher order susceptibilities

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