Quarkonium hybrids in effective field theory

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with: M. Berwein, N. Brambilla, and A. Vairo

ECT* Trento, February 2016.
In the last decade many new unexpected states have been found close or above threshold.

The states that do not fit Quarkonium potential models are called Exotics and labeled Xs, Ys and Zs.

This states are candidates for non traditional hadronic states, including four constituent quark or an excited gluon constituent.

Large experimental effort to study normal and Exotic quarkonium: BaBar, Belle2, BESIII, LHCb and Panda (under construction).
What are quarkonium Hybrids?

A quarkonium hybrid consists of $Q$, $\bar{Q}$ in a color octet configuration and a gluonic excitation $g$. 

Key Characteristics

- Heavy quarks are non-relativistic, dynamical time–scale set by the heavy quarks mass.
- Gluons are fast, dynamical time–scale set by $\Lambda_{\text{QCD}}$.
- The hierarchy between dynamical time–scales can be exploited to describe the system.
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Quarkonium hybrids are a similar system to diatomic molecules

- Slow degree–of–freedom: Nuclei → Heavy Quark
- Fast degree–of–freedom: Electrons → Gluons
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Born-Oppenheimer approximation

1. Solve the Schrödinger equation for the electrons with static nuclei. The electronic energy levels depend on the nuclei positions and are called static energies.

2. The molecular energy levels are obtained solving the Schrödinger equation for the nuclei with the static energies as background potential.
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Our Aim

- Systematize the ideas behind the Born–Oppenheimer approximation for Quarkonium hybrids using EFT techniques.
Symmetries of the static system

Static states classified by symmetry group $D_{\infty h}$

 Representations labeled $\Lambda^\sigma_{\eta}$

- $\Lambda$ rotational quantum number
  $|\hat{n} \cdot K| = 0, 1, 2 \ldots$ corresponds to $\Lambda = \Sigma, \Pi, \Delta \ldots$

- $\eta$ eigenvalue of $CP$:
  $g \hat{=} +1$ (gerade), $u \hat{=} -1$ (ungerade)

- $\sigma$ eigenvalue of reflections

- $\sigma$ label only displayed on $\Sigma$ states
  (others are degenerate)
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  (others are degenerate)

- The static energies correspond to the irreducible representations of $D_{\infty} h$.
- In the limit $r \to 0$ more symmetry: $D_{\infty} h \to O(3) \times C$. 
Lattice data on hybrid static energies

The gluonic static energies are eigenvalues of the NRQCD Hamiltonian in the static limit.

Non-perturbative object that has been computed on the Lattice.

The most recent data by Juge, Kuti, Morningstar, 2002 and Bali and Pineda 2003.

$\Sigma_g^+$ is the ground state potential that generates the standard quarkonium states.

The rest of the static energies correspond to excited gluonic states that generate hybrids.

The two lowest hybrid static energies are $\Pi_u$ and $\Sigma_u^-$, they are nearly degenerate at short distances.

Quenched and unquenched calculations for $\Sigma_g^+$ and $\Pi_u$ were compared in Bali et al 2000 and good agreement was found below string breaking distance.
Potential Non–Relativistic QCD

**Motivation**

- Quarkonium systems are non–relativistic bound states.
- **Multiscale system:** \( m \gg mv \gg mv^2 \), and \( \Lambda_{QCD} \). \( m \) is the heavy–quark mass, \( v \ll 1 \) the heavy quark velocity.
- We can exploit the scale hierarchies by building an Effective Field Theory (EFT).

**Matching procedure**

- Integrating out the \( m \) scale leads to the well known NRQCD. Caswell, Lepage 1986; Bodwin, Braaten and Lepage 1995. Since \( m \gg \Lambda_{QCD} \) the matching is perturbative.
- The degrees of freedom in pNRQCD are a color singlet \((S)\) and octet fields \((O^a)\) and the ultrasoft gluons.
- \( R \) is the CoM coordinate and \( r \) the relative coordinate of the quark pair.
- **Multipole expansion:** Since \( r \sim 1/mv \), integrating out \( mv \) implies a multipole expansion of the gluon fields.
Potential Non–Relativistic QCD

Let us start from weakly-coupled pNRQCD:

\[ \mathcal{L}_{\text{pNRQCD}} = \int d^3 r \, \text{Tr} \left[ S^\dagger \left( i \partial_0 + \frac{\nabla^2 r}{M} - V_s(r) \right) S + O^\dagger \left( i \partial_0 + \frac{\nabla^2 r}{M} - V_o(r) \right) O \right] + g V_A(r) \text{Tr} \left[ O^\dagger r \cdot E S + S^\dagger r \cdot E O \right] + \frac{g}{2} V_B(r) \text{Tr} \left[ O^\dagger r \cdot E O + O^\dagger O r \cdot E \right] - \frac{1}{4} G_\mu^a G^{\mu \nu} \]

Pineda, Soto 1998; Brambilla, Pineda, Soto, Vairo 2000

- Hierarchy of scales:
  - Weakly-coupled pNRQCD is valid for short distances: \( m v \sim 1/r \gg \Lambda_{QCD} \).
  - Heavy quarks being slower than gluons implies \( \Lambda_{QCD} \gg m v^2 \).

- Work plan:
  - Integrate out the light d.o.f.
Born-Oppenheimer EFT for QCD

The Hamiltonian density corresponding to the light d.o.f at leading order in the multipole expansion is

\[ \hat{h}_0(R) = \frac{1}{2} (E^a E^a - B^a B^a) \]

**Gluelump operators** $G^a$

- $G^a$ are a basis of color-octet eigenstates of $\hat{h}_0(R)$ with eigenvalues $\Lambda_\kappa$.

\[ \hat{h}_0(R) G^a_{i\kappa}(R) = \Lambda_\kappa G^a_{i\kappa}(R) \]

- $\Lambda_\kappa$ is called the gluelump mass and it is a nonperturbative quantity.

- $\kappa$ labels the $O(3) \times C$ representation ($K^{PC}$ quantum numbers).

The eigenstates of the octet sector Hamiltonian are

\[ |\kappa\rangle = O^a(r, R) G^a_{i\kappa}(R)|0\rangle, \]

We can expand the Lagrangian this basis by projecting into the subspace spanned by

\[ \int d^3r \, d^3R \sum_\kappa |\kappa\rangle \Psi_{i\kappa}(t, r, R) \]
Born-Oppenheimer EFT for QCD

- After projecting and integrating out $\Lambda_{QCD}$:

\[
\mathcal{L}_{BO}^o = \int d^3r \sum_{\kappa} \psi_{i\kappa}^\dagger(t, r, R) \left[ \left( i\partial_t + \frac{\nabla_r^2 M}{2} - V_0(r) - \Lambda_\kappa \right) \delta_{ij} - \sum_\lambda P_{i\kappa\lambda} b_{\kappa\lambda} r^2 P_{j\kappa\lambda} + \cdots \right] \psi_{j\kappa}(t, r, R) + \ldots
\]

The $P_{i\kappa\lambda}$ are projectors that select different polarizations of $\Psi_{i\kappa}$.

**NLO term: $b_{\kappa\lambda}$**

- At finite $r$ the eigenstates must be organized in representations of $D_{\infty h}$.
- Proportional to $r^2$ due to the multipole expansion.

\begin{itemize}
  \item $b_{\kappa\lambda}$ is a non-perturbative quantity.
  \item We obtain it from a fit to the lattice data.
  \item Breaks the $O(3) \times C \rightarrow D_{\infty h}$, $b_{\kappa\lambda} = b_{\kappa - \lambda}$.
  \item Responsible for the attractive part of the potential.
\end{itemize}
Born-Oppenheimer EFT for QCD

Defining the projected wavefunction as $\Psi_{\kappa\lambda} = P_{i\kappa\lambda} \Psi_i$ and $\Psi_i = \sum_{\lambda} P_{i\kappa\lambda} \Psi_{\kappa\lambda}$:

$$\mathcal{L}_{BO}^0 = \int d^3r \sum_{\kappa} \sum_{\lambda\lambda'} \Psi_{k\lambda}^\dagger (t, r, R) \left\{ \left[ i\dot{\Psi}_i - V_o(r) - \Lambda_k \right. ight.$$ 

$$- b_{\kappa\lambda} r^2 + \cdots \right\} \delta_{\lambda\lambda'} - P_{i\kappa\lambda} \nabla^2 r P_{i\kappa\lambda'} \right\} \Psi_{\kappa\lambda'}(t, r, R)$$

**Nonadiabatic coupling**

We have splitted the kinetic operator acting and the nonadiavatic coupling

$$P_{i\kappa\lambda} \nabla^2_r P_{i\kappa\lambda'} = \nabla^2_r + C_{\kappa\lambda\lambda'}$$

with

$$C_{\kappa\lambda\lambda'} = P_{i\kappa\lambda} \left[ \frac{\nabla^2_r}{M}, P_{i\kappa\lambda'} \right]$$

- The nonadiabatic coupling mixes states which are different projections of the same gluelump.
- States which are different projections of the same gluelump are degenerate in the limit $r \to 0$. 
Wilson loop matching

The static NRQCD Gluonic eigenstates $|\lambda; x_1, x_2\rangle^{(0)}$, where $\lambda$ is a representation of $D_{\infty,h}$, are unknown. Nevertheless, since

1. $|\lambda; x_1, x_2\rangle^{(0)}$ form a basis, then for any state

$$|X_\lambda\rangle = c_\lambda \ |\lambda; x_1, x_2\rangle^{(0)} + c_{\lambda'} \ |h'; x_1, x_2\rangle^{(0)} + \ldots$$

2. For large $T$ in the Euclidean time of lattice QCD the exponentials are suppressed and be dominated by the lowest static energy

$$E^{\text{light}}_\lambda (r) = \lim_{T \to \infty} \frac{i}{T} \log \langle X_\lambda, T/2 | X_\lambda, -T/2 \rangle .$$

3. $|X_\lambda\rangle$ just needs to have a non–vanishing overlap with the desired static state. A convenient choice for these $|X_\lambda\rangle$ gives the static energies in terms of Wilson loops

$$|X_\lambda\rangle = \chi(x_2) \phi(x_2, R) T^a P^a_{\lambda}(R) \phi(R, x_1) \psi^{\dagger}(x_1) |\text{vac}\rangle.$$
Gluonic excitation operators up to dim 3

<table>
<thead>
<tr>
<th>$\Lambda_{\eta I}^\sigma$</th>
<th>$K^{PC}$</th>
<th>$P_a^\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_u^-$</td>
<td>1$^{--}$</td>
<td>$\hat{r} \cdot B, \hat{r} \cdot (D \times E)$</td>
</tr>
<tr>
<td>$\Pi_u$</td>
<td>1$^{+-}$</td>
<td>$\hat{r} \times B, \hat{r} \times (D \times E)$</td>
</tr>
<tr>
<td>$\Sigma_{g'}^+$</td>
<td>1$^{--}$</td>
<td>$\hat{r} \cdot E, \hat{r} \cdot (D \times B)$</td>
</tr>
<tr>
<td>$\Pi_g$</td>
<td>1$^{--}$</td>
<td>$\hat{r} \times E, \hat{r} \times (D \times B)$</td>
</tr>
<tr>
<td>$\Sigma_g^-$</td>
<td>2$^{--}$</td>
<td>$(\hat{r} \cdot D)(\hat{r} \cdot B)$</td>
</tr>
<tr>
<td>$\Pi'_g$</td>
<td>2$^{--}$</td>
<td>$\hat{r} \times ((\hat{r} \cdot D)B + D(\hat{r} \cdot B))$</td>
</tr>
<tr>
<td>$\Delta_g$</td>
<td>2$^{--}$</td>
<td>$(\hat{r} \times D)^i(\hat{r} \times B)^i + (\hat{r} \times D)^j(\hat{r} \times B)^j$</td>
</tr>
<tr>
<td>$\Sigma_u^+$</td>
<td>2$^{+-}$</td>
<td>$(\hat{r} \cdot D)(\hat{r} \cdot E)$</td>
</tr>
<tr>
<td>$\Pi'_u$</td>
<td>2$^{+-}$</td>
<td>$\hat{r} \times ((\hat{r} \cdot D)E + D(\hat{r} \cdot E))$</td>
</tr>
<tr>
<td>$\Delta_u$</td>
<td>2$^{+-}$</td>
<td>$(\hat{r} \times D)^i(\hat{r} \times E)^i + (\hat{r} \times D)^j(\hat{r} \times E)^j$</td>
</tr>
</tbody>
</table>

- We can obtain the static energy in pNRQCD using the multipole expansion of $|X_\lambda\rangle$

  $$|X_\lambda\rangle \approx \left(Z_\lambda(r)O^a(\mathbf{r}, \mathbf{R}) + O(r)\right) |0\rangle = c_{\kappa\lambda} \left(O^a(\mathbf{r}, \mathbf{R}) P_{\kappa\lambda i} G_{i, \kappa}(R)\right) |0\rangle + c_{\kappa'\lambda'} \left(O^a(\mathbf{r}, \mathbf{R}) P_{\kappa'\lambda' i} G_{i, \kappa'}(R)\right) |0\rangle + \ldots$$

- The gluonic operator piece of $P^a_{\lambda}$ can be written in the basis $G^a$

  $$E^\text{light}_{\lambda}(r) = \lim_{T \to \infty} \frac{i}{T} \log \langle X_\lambda, T/2 | X_\lambda, -T/2 \rangle = V_o(r) + \Lambda_\kappa + b_{\kappa\lambda} r^2 + \ldots$$

  ▶
Lowest energy multiplet $\Sigma_u^--\Pi_u$

- The lowest mass gluelump has quantum numbers $1^+−$ and
  $\Lambda_{1^+−}^{RS} = 0.87 \pm 0.15$ GeV. Bali, Pineda 2004

- It generates the two lowest laying hybrid static energies $\Pi_u$ and $\Sigma_u^−$ which are
degenerate at short distances.

- The kinetic operator mixes them but not with other multiplets.

- Well separated by a gap of $\sim 1$ GeV from the next multiplet with the same CP.

Coupled radial equations for $\Sigma_u^--\Pi_u$

$$\begin{aligned}
\left[ -\frac{\partial_r^2}{m} + \frac{1}{mr^2} \left( \frac{l(l+1)+2}{2\sqrt{l(l+1)}} \right) + \frac{E^\text{light}}{E^\text{light}} \right] \begin{pmatrix}
\Psi_{N-\epsilon, \Sigma} \\
\Psi_{N-\epsilon, \Pi}
\end{pmatrix} &= \mathcal{E}_N \begin{pmatrix}
\Psi_{N-\epsilon, \Sigma} \\
\Psi_{N-\epsilon, \Pi}
\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\left[ -\frac{\partial_r^2}{m} + \frac{l(l+1)}{mr^2} + E^\text{light}_\Pi \right] \psi_{N-\epsilon, \Pi} &= \mathcal{E}_N \psi_{N-\epsilon, \Pi}
\end{aligned}$$

- The coupled Schrödinger equations can be solved numerically.
Hybrid state masses from $\sqrt{0.25}$


Solving the coupled Schrödinger equations we obtain

<table>
<thead>
<tr>
<th>GeV</th>
<th>$m_H$</th>
<th>$\langle 1/r \rangle$</th>
<th>$E_{\text{kin}}$</th>
<th>$P_\Pi$</th>
<th>$m_H$</th>
<th>$\langle 1/r \rangle$</th>
<th>$E_{\text{kin}}$</th>
<th>$P_\Pi$</th>
<th>$m_H$</th>
<th>$\langle 1/r \rangle$</th>
<th>$E_{\text{kin}}$</th>
<th>$P_\Pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>4.15</td>
<td>0.42</td>
<td>0.16</td>
<td>0.82</td>
<td>7.48</td>
<td>0.46</td>
<td>0.13</td>
<td>0.83</td>
<td>10.79</td>
<td>0.53</td>
<td>0.09</td>
<td>0.86</td>
</tr>
<tr>
<td>$H'_1$</td>
<td>4.51</td>
<td>0.34</td>
<td>0.34</td>
<td>0.87</td>
<td>7.76</td>
<td>0.38</td>
<td>0.27</td>
<td>0.87</td>
<td>10.98</td>
<td>0.47</td>
<td>0.19</td>
<td>0.87</td>
</tr>
<tr>
<td>$H_2$</td>
<td>4.28</td>
<td>0.28</td>
<td>0.24</td>
<td>1.00</td>
<td>7.58</td>
<td>0.31</td>
<td>0.19</td>
<td>1.00</td>
<td>10.84</td>
<td>0.37</td>
<td>0.13</td>
<td>1.00</td>
</tr>
<tr>
<td>$H'_2$</td>
<td>4.67</td>
<td>0.25</td>
<td>0.42</td>
<td>1.00</td>
<td>7.89</td>
<td>0.28</td>
<td>0.34</td>
<td>1.00</td>
<td>11.06</td>
<td>0.34</td>
<td>0.23</td>
<td>1.00</td>
</tr>
<tr>
<td>$H_3$</td>
<td>4.59</td>
<td>0.32</td>
<td>0.32</td>
<td>0.00</td>
<td>7.85</td>
<td>0.37</td>
<td>0.27</td>
<td>0.00</td>
<td>11.06</td>
<td>0.46</td>
<td>0.19</td>
<td>0.00</td>
</tr>
<tr>
<td>$H_4$</td>
<td>4.37</td>
<td>0.28</td>
<td>0.27</td>
<td>0.83</td>
<td>7.65</td>
<td>0.31</td>
<td>0.22</td>
<td>0.84</td>
<td>10.90</td>
<td>0.37</td>
<td>0.15</td>
<td>0.87</td>
</tr>
<tr>
<td>$H_5$</td>
<td>4.48</td>
<td>0.23</td>
<td>0.33</td>
<td>1.00</td>
<td>7.73</td>
<td>0.25</td>
<td>0.27</td>
<td>1.00</td>
<td>10.95</td>
<td>0.30</td>
<td>0.18</td>
<td>1.00</td>
</tr>
<tr>
<td>$H_6$</td>
<td>4.57</td>
<td>0.22</td>
<td>0.37</td>
<td>0.85</td>
<td>7.82</td>
<td>0.25</td>
<td>0.30</td>
<td>0.87</td>
<td>11.01</td>
<td>0.30</td>
<td>0.20</td>
<td>0.89</td>
</tr>
<tr>
<td>$H_7$</td>
<td>4.67</td>
<td>0.19</td>
<td>0.43</td>
<td>1.00</td>
<td>7.89</td>
<td>0.22</td>
<td>0.35</td>
<td>1.00</td>
<td>11.05</td>
<td>0.26</td>
<td>0.24</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Consistency test:**

1. The multipole expansion requires $\langle 1/r \rangle \gtrsim E_{\text{kin}}$.

**Conclusion:**

- As expected the our approach works better in bottomonium than charmonium.

**Spin symmetry multiplets**

<table>
<thead>
<tr>
<th>$H_i$</th>
<th>Multiplet</th>
<th>$\Sigma_u^-, \Pi_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>${1^{--}, (0, 1, 2)^{++}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_2$</td>
<td>${1^{++}, (0, 1, 2)^{--}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>${0^{++}, 1^{--}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>${2^{++}, (1, 2, 3)^{--}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_5$</td>
<td>${2^{--}, (1, 2, 3)^{++}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_6$</td>
<td>${3^{--}, (2, 3, 4)^{--}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
<tr>
<td>$H_7$</td>
<td>${3^{++}, (2, 3, 4)^{++}}$</td>
<td>$\Sigma_u^-, \Pi_u$</td>
</tr>
</tbody>
</table>
Λ–doubling effect

- In Braaten et al 2014 a similar procedure was followed to obtain the hybrid masses.
- No Λ–doubling effect mixing terms were included, and phenomenological potentials fitting the lattice data.
- We can compare the results to estimate the size of the Λ–doubling effect.

Charmonium sector

<table>
<thead>
<tr>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5</td>
</tr>
<tr>
<td>4.0</td>
</tr>
<tr>
<td>4.5</td>
</tr>
<tr>
<td>5.0</td>
</tr>
</tbody>
</table>

Braaten et al 2014 results plotted in dashed lines.

- The mixing lowers the mass of the $H_1(H_4)$ multiplet with respect to $H_2(H_4)$. 
Identification with experimental states

Most of the candidates have $1^{--}$ or $0^{++}/2^{++}$ since the main observation channels are production by $e^+e^-$ or $\gamma\gamma$ annihilation respectively.

- Charmonium states (Belle, CDF, BESIII, Babar):

  ![Diagram of charmonium states]

  - Bottomonium states: $Y_b(10890)[1^{--}]$, $m = 10.8884 \pm 3.0$ (Belle). Possible $H_1$ candidate, $m_{H_1} = 10.79 \pm 0.15$.

  However, except for $Y(4220)$, all other candidates observed decay modes violate Heavy Quark Spin Symmetry.
Comparison with direct lattice computations

Charmonium sector

- Calculations done by the Hadron Spectrum Collaboration using unquenched lattice QCD with a pion mass of 400 MeV. Liu et al. 2012
- They worked in the constituent gluon picture, which consider the multiplets $H_2$, $H_3$, $H_4$ as part of the same multiplet.
- Their results are given with the $\eta_c$ mass subtracted.

<table>
<thead>
<tr>
<th>Split (GeV)</th>
<th>Liu</th>
<th>$V^{(0.25)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta m_{H_2-H_1}$</td>
<td>0.10</td>
<td>0.13</td>
</tr>
<tr>
<td>$\delta m_{H_4-H_1}$</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>$\delta m_{H_4-H_2}$</td>
<td>0.13</td>
<td>0.09</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_1}$</td>
<td>0.20</td>
<td>0.44</td>
</tr>
<tr>
<td>$\delta m_{H_3-H_2}$</td>
<td>0.09</td>
<td>0.31</td>
</tr>
</tbody>
</table>

- Our masses are $0.1 - 0.14$ GeV lower except the for the $H_3$ multiplet, which is the only one dominated by $\Sigma_u$.
- Good agreement with the mass gaps between multiplets, in particular the $\Lambda$-doubling effect ($\delta m_{H_2-H_1}$).
Conclusions

- We have obtained a Schrödinger equation framework for quarkonium hybrids using pNRQCD.
- The matching between the static energies computed in lattice NRQCD and in weakly–coupled pNRQCD is well established.
- Mixing terms are important due to the short range degeneracy of the static energies.
- Several experimental candidates for charmonium hybrids and one candidate to the bottomonium hybrids.
Thank you for your attention