Spectral reconstruction with a variational method

Tim Harris, Harvey Meyer, Daniel Robaina

HELMHOLTZ ASSOCIATION
Helmholtz Institute Mainz

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

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Outline

Motivation and introduction

- Observables in a strongly-coupled QCD medium
- Inverse problem for numerical data
- Standard regularizations: linear and non-linear methods

Variational methods

- Variational methods for vacuum correlators
- A variational method in the frequency domain

Preliminary results

- Application to vacuum correlation functions from an EFT

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add data

Observables in a strongly-coupled QCD medium

In-medium observables are related to the spectral function

$$\rho(\omega) \equiv \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}t\, e^{-i\omega t}\left(G^{>}(t) - G^{<}(t)\right), \quad \text{where } G^{\gtrless}(t) \equiv \begin{cases} \mathrm{Tr}\{\hat\rho\, \mathcal{J}(t)\mathcal{J}^\dagger(0)\} \\ \mathrm{Tr}\{\hat\rho\, \mathcal{J}^\dagger(0)\mathcal{J}(t)\} \end{cases}$$

- transport of a conserved charge, e.g.

$$\frac{\rho_L(\omega \boldsymbol{p})}{\omega} = \frac{\chi_s(\boldsymbol{p})}{\pi}\frac{D\omega^2}{\omega^2 + (D\boldsymbol{p}^2)^2}$$

[Aarts et al., 1307.6763]

- thermal production rates, e.g.

$$\left.\frac{d^4 N_{\ell_+\ell_-}}{d\omega\, d^3p}\right|_{\boldsymbol{p}=0} \propto \alpha_{\mathrm{em}}^2 n_{\mathrm{B}}(\omega)\frac{\rho_{\mathrm{V}}(\omega)}{\omega^2}$$

[CMS, 1208.2826]

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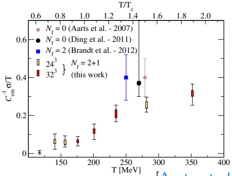

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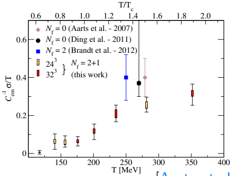

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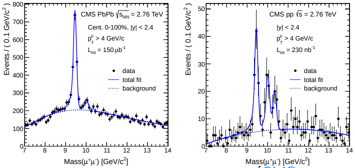

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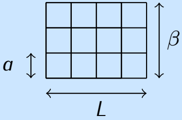
Lattice QCD



Lattice is a non-perturbative regularization of a QFT

Many Euclidean observables can be estimated numerically in a finite time

By importance-sampling the distribution implicitly defined though

$$Z = \int_{U\ \mathrm{per.}\,\beta} \mathcal{D}U\, e^{-S_E[U]},$$

we generate $\{U_i\}_{i=1,\dots,N}$ and calculate stochastic estimator of the Euclidean correlation

$$\langle \hat{\mathcal{O}}(x_1, x_2, \dots, x_n) \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(x_1, \dots, x_n)[U_i] + \mathrm{O}(1/\sqrt{N}).$$

We can access, e.g.

$$G_E(\tau) \equiv G^{>}(-i\tau) = \int_0^\infty d\omega\, K(\tau,\omega)\rho(\omega), \quad \text{where} \quad K(\tau,\omega) = \frac{\cosh(\omega\tau - \omega\beta/2)}{\sinh(\beta\omega/2)}$$

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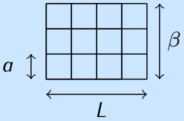

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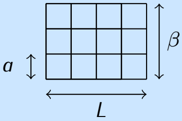

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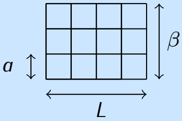

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for $G_E(\tau) = G_E(\beta - \tau)$.

Spectral decomposition in the vacuum

The Euclidean correlation admits the spectral decomposition

$$G_E(\tau) = \sum_{n=0}^{\infty} |Z^n|^2 \, e^{-E_n \tau}$$

in the vacuum, where $Z^{n*} \equiv \langle n|\mathcal{J}^\dagger(x)|0\rangle$ and $H_{\rm QCD}|n\rangle = E_n|n\rangle$.

- Used to extract matrix elements of operators in the ground state in the asymptotic $\tau \to \infty$ limit for vacuum correlators.
- Spectrum is discrete in a finite volume.

Inverse problem for finite noisy numerical data

- $K : X \to Y$ is a compact linear operator between normed linear spaces, X, Y.
- Suppose we have finite noisy data, y^δ, with $\|y - y^\delta\|_Y < \delta$.
- A *regularization strategy* $R_\alpha : D(R_\alpha) \subseteq Y \to X$ where $R_\alpha y \overset{\alpha\to 0}{\to} K^{-1}y = x$.
- $R_{\alpha=0}$ is unbounded $\Rightarrow$ not continuous on Y.

Let $x_\alpha^\delta \equiv R_\alpha y^\delta$, then the approximation error $\|x - x_\alpha^\delta\|_X$ satisfies

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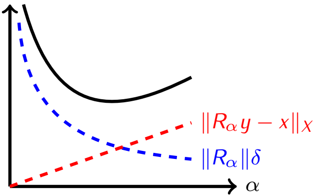

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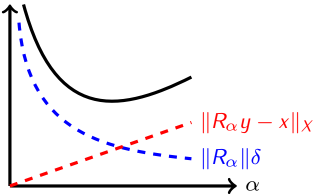

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A linear regularization – Backus-Gilbert method I

$$x_\alpha^\delta \to \hat{\rho}(\bar{\omega}), \qquad \text{and} \qquad y^\delta \to G_E(ia) \equiv G_i,$$

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Choose to optimize a quadratic form in $q_i(\bar{\omega})$:

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where

$$\hat{\delta}(\bar{\omega}, \omega) \equiv \sum_{i=1}^{\beta/a} q_i(\bar{\omega}) K(ia, \omega) = R_\alpha K$$

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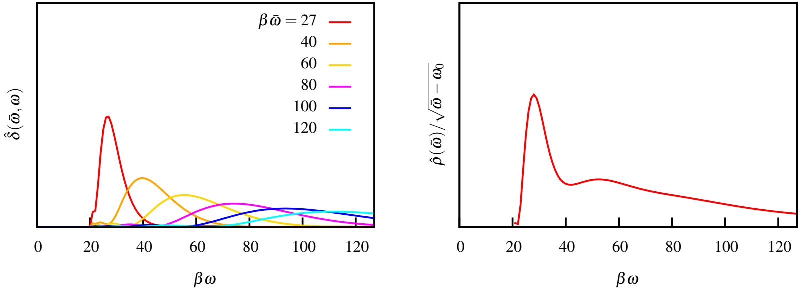

Backus-Gilbert estimator with $\alpha = 10^{-8}$ on a Euclidean correlator $\beta/a = 128$.

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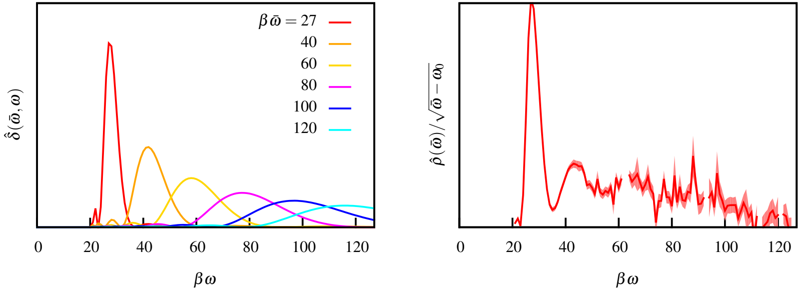

Backus-Gilbert estimator with $\alpha = 10^{-12}$ on lattice correlator $\beta/a = 128$.

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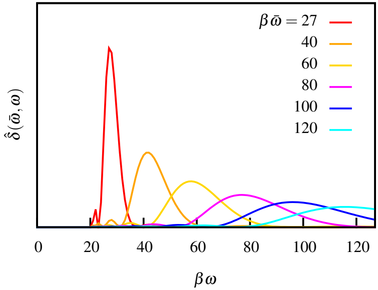

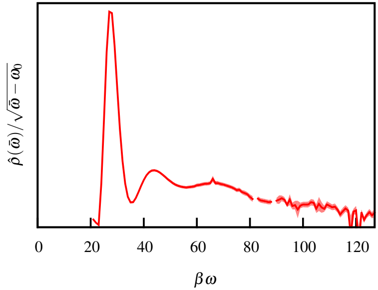

Backus-Gilbert estimator with $\alpha = 10^{-11}$ on lattice correlator $\beta/a = 128$.

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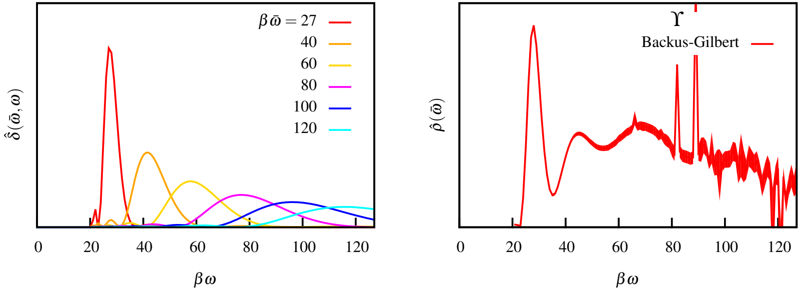

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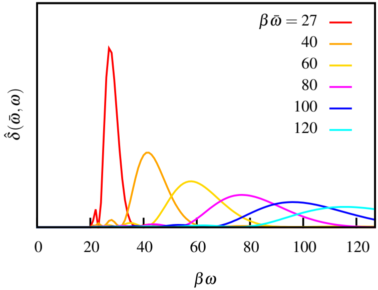

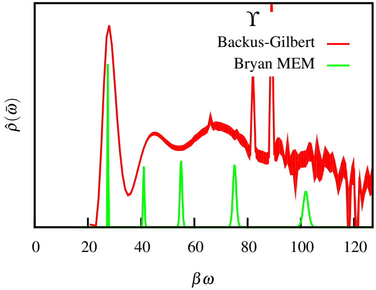

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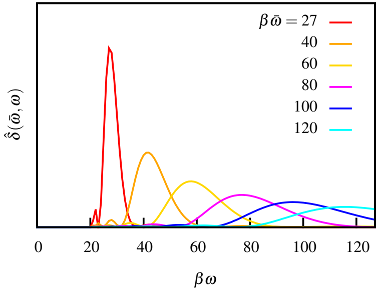

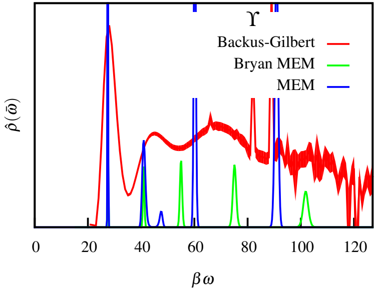

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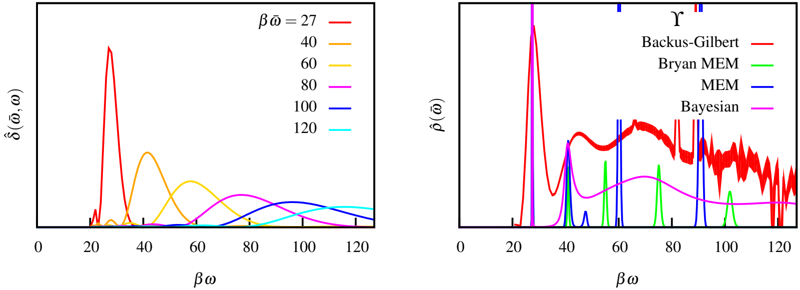

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Non-linear regularization strategies

Regulate the degenerate maximum-likelihood optimization with a convex functional

$$R_\alpha : y^\delta \mapsto x_\alpha^\delta = \underset{x' \in X'}{\operatorname{argmin}} \left\{ \frac{1}{2} \| y^\delta - Kx' \|_Y^2 + \alpha S[x'] \right\}$$

- MEM: family of regularizations indexed by $m \in X$

$$S[x] = \int_0^\infty \mathrm{d}\omega \left[x - x \log\left(\frac{x}{m}\right) - m \right]$$

Bryan's algorithm: the normal equation for the optimization

$$\delta_x S[x] = K^\top \mathrm{Cov}^{-1}(y^\delta - Kx) \Longrightarrow \tilde{x} = V\Xi(\,\cdot\,) \Longrightarrow \tilde{x} \in \mathrm{span}(\mathrm{r.s.v.}K)$$

where $K = U\Xi V^T$ if $\tilde{x} \equiv \delta_x S[x]$ invertible.
The optimization can be given some Bayesian interpretation which also gives a prescription for integrating out α.

Variational method and the GEvP

Consider a *basis* of interpolating operators $\{\mathcal{J}_i(x)\}_{i=1,\dots,N}$ and recall the spectral decomposition

$$G_E(\tau) = \sum_{n=0}^{\infty} |Z^n|^2 \, e^{-E_n \tau}.$$

It can be shown (perturbatively) that the eigenvalues of the GEvP

$$G_{E,ij}(\tau) v_j^n(\tau, \tau_0) = \lambda^n(\tau, \tau_0) G_{E,ij}(\tau_0) v_j^n(\tau, \tau_0)$$

have the asymptotic behaviour

$$\lambda^n(\tau, \tau_0) = e^{-E_n \tau} + \mathrm{O}(e^{-(E_{N+1} - E_n)\tau})$$

and the operators in the *rotated basis* satisfy

$$\mathcal{J}_i v_i^n |0\rangle = |n\rangle + \mathrm{O}(e^{-(E_{N+1} - E_n)\tau_0}).$$

[Blossier *et al.*, 0902.1265]

Note that $G_{E,ij}$ is a Hermitian, positive-definite bilinear form on the space of operators, and *not* an linear map $X \to X$.

Variational method and the GEvP

Consider a *basis* of interpolating operators $\{\mathcal{J}_i(x)\}_{i=1,\dots,N}$ and recall the spectral decomposition

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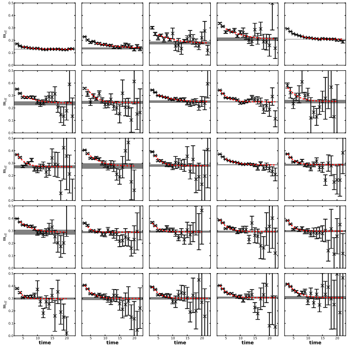
The variational method *in vacuo*

- Used in many studies to *reliably* extract tens of energy levels in single channels.
- $\geq$100 operators used: $(\bar{\Psi}\Gamma\Psi)(x)$, $(\bar{\Psi}\Delta\Psi)(x)$, multiparticle operators with relative momentum, ...



[Morningstar *et al.*, 1510.00371]

A variational method in the frequency domain

- Ohno *et al.* applied the GEvP directly to the correlators at finite β.

[Ohno *et al.*, 1104.3384]

- Filtering with τ, τ_0 may not be possible when β is finite, or when the spectrum becomes dense, e.g. at $T > 0$:

$$\rho(\omega) = \frac{2\sinh(\beta\omega/2)}{Z} \sum_{n,m} |Z^{mn}|^2 e^{-\beta(E_n+E_m)/2} \delta(\omega - E_m + E_n),$$

where $Z^{nm*} = \langle n | J^\dagger(0) | m \rangle$.

solve the GEvP in the spectral function estimator $\hat{\rho}_{ij}(\bar{\omega}) = R_\alpha G_{E,ij}(\tau)$:

$$\hat{\rho}_{ij}(\bar{\omega}) v_j^n(\bar{\omega}, \tau_0) = \lambda^n(\bar{\omega}, \tau_0) G_{E,ij}(\tau_0) v_j^n(\bar{\omega}, \tau_0)$$

- the method asks for operators which couple optimally to a *local* region in freqency domain rather than at large times
- applicable even when $\tau \to \infty$ not available if good resolution is available

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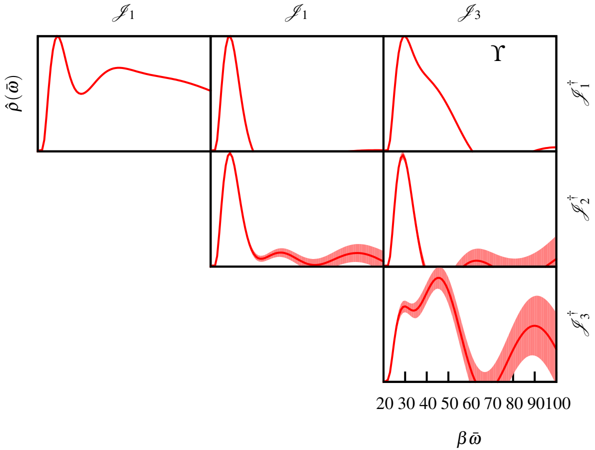
where $Z^{nm*} = \langle n | J^\dagger(0) | m \rangle$.

Backus-Gilbert well-suited because it resolves the spectral function *locally*.

$$\begin{array}{ccc} Y^{N^2} & \xrightarrow{R_\alpha} & X^{N^2} \\ \text{GEvP} \downarrow & & \downarrow \text{GEvP} \\ Y^{N^2} & \xrightarrow{R_\alpha} & X^{N^2} \end{array}$$

Υ channel

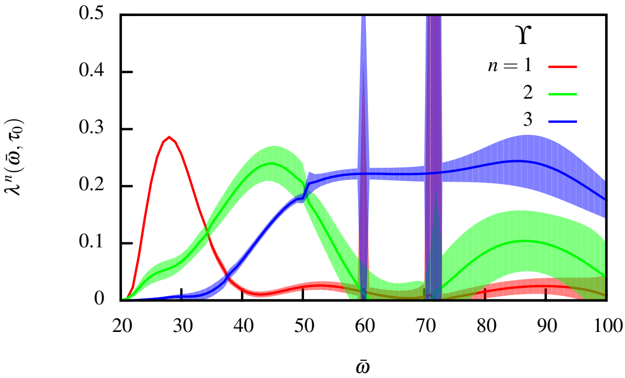


BG matrix estimator $\hat{\rho}_{ij}(\bar{\omega})$ in Υ channel for β/a=128

Operator basis $\{\mathcal{J}_i^\dagger(x) = \sum_k \chi_i^\dagger(x)\sigma_k\psi_i(x)\}$ where $\psi_1(x) \equiv \psi(x)$ and

$$\psi_2(y) \equiv \sum_{\boldsymbol{x}} e^{-(\boldsymbol{x}-\boldsymbol{y})^2/\sigma^2}\psi(x), \qquad \psi_3(y) \equiv \sum_{\boldsymbol{x}}(4\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{\sigma^2} - 3)e^{-(\boldsymbol{x}-\boldsymbol{y})^2/\sigma^2}\psi(x).$$

Υ channel



Eigenvalues in Υ channel for β/a=128

Applications

Note the local *rank-1* (in the operator space) property of the spectral function:

$$\rho_{ij}(\omega) = \frac{2\sinh(\beta\omega/2)}{Z} \sum_{n,m} Z_i^{mn} Z_j^{mn*} e^{-\beta(E_n+E_m)/2} \delta(\omega - E_m + E_n)$$

Identifying quasiparticle poles:

- A large hierarchy between in the eigenvalues is the approximate rank

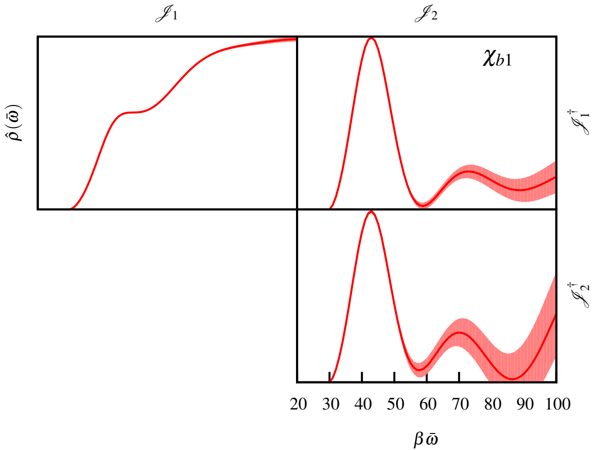
$$\lambda^n(\bar{\omega}, \tau_0) \gg \lambda^{m\neq n}(\bar{\omega}, \tau_0).$$

Estimating the contribution to a spectral function from a local frequency region:

- Define an optimal operator $\sum_i \mathcal{J}(x)_i^\dagger v_i^n(\bar{\omega}^n)$

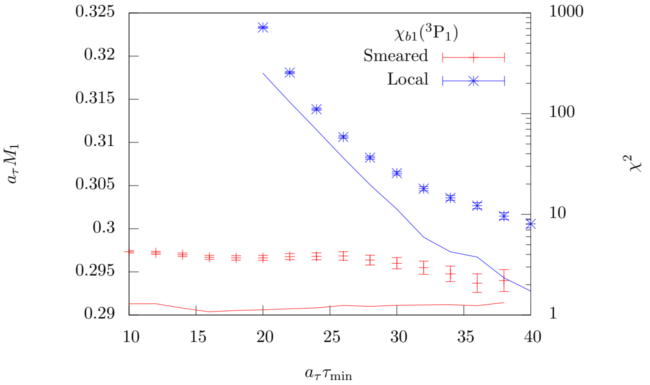
$$\bar{\omega}^n \equiv \underset{\bar{\omega}}{\text{argmax}} \{\lambda^n(\bar{\omega}, \tau_0)\}$$

χ_{b1} channel



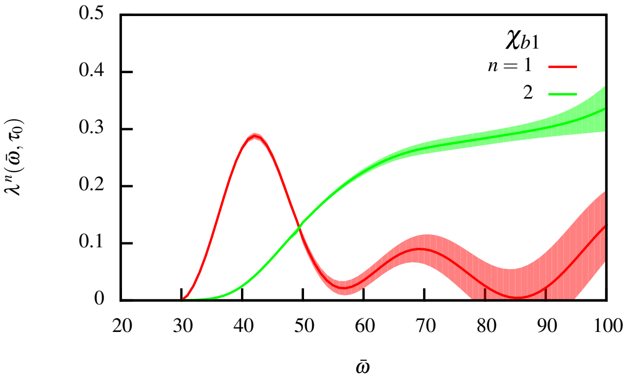
BG matrix estimator $\hat{\rho}_{ij}(\bar{\omega})$ in χ_{b1} channel for β/a=128

χ_{b1} channel



Sliding-window plot for χ_{b1} effective energy showing slow convergence on the ground state.

χ_{b1} channel



Eigenvalues in χ_{b1} channel for β/a=128

Conclusions

- Regulation easily controlled with linear methods

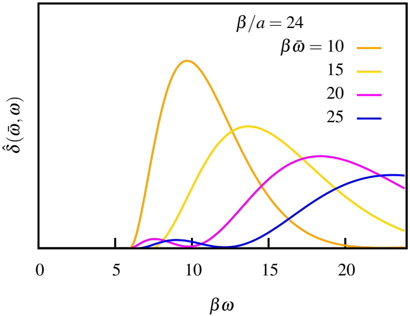
  $\rightsquigarrow$ Backus-Gilbert transforms kernel to a smoothing one

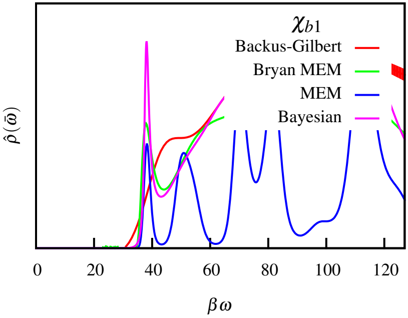
- GEvP can focus on *local* regions of a spectral function

  - ◁ define *optimal* operators with compact-enough support in the frequency domain
  - ▷ identify quasiparticles
  - ▷ applying other (non-linear) inverse methods using *optimal* basis

- Application to finite-β relativistic and non-relativistic channels underway

Conclusions





χ_{b1}
Backus-Gilbert
Bryan MEM
MEM
Bayesian
$\hat{\rho}(\bar{\omega})$
0
20
40
60
80
100
120
$\beta\omega$