Renormalization of zero-range effective interactions in finite nuclei
Examples for a simplified Skyrme interaction

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The Nuclear Many-Body Problem

- **Nucleus**: from few to more than 200 strongly interacting and self-bound fermions.
- **Underlying interaction** is not perturbative at the (low)energies of interest for the study of masses, radii, deformation, giant resonances,...
- **Complex systems**: spin, isospin, pairing, deformation, ...
- **Many-body** calculations based on NN scattering data in the vacuum are not conclusive yet:
  - different nuclear interactions in the medium are found depending on the approach
  - EoS and (recently) few groups in the world are able to perform extensive calculations for light and medium mass nuclei

Based on effective interactions, **Self-consistent Mean-Field models** are successful in the description of masses, nuclear sizes, deformations, Giant Resonances,... and can be applied along the whole nuclear chart (expected to be more accurate for heavy than for light systems).

1 Some heavier nuclei (Tin) have been also accurately predicted with the help of many-body (Oxigen) data [A. Ekström et al. Phys. Rev. C 91, 051301(R) (2015)].
Mean field framework is particularly suitable for the study of ground state properties in heavy elements

- The complexities of the system are effectively included into the interaction, simplifying the wave functions using Slater determinants (|φ⟩) ⇒ SCMF
  \[ E = \langle \psi | \hat{H} | \psi \rangle \equiv \langle \phi | \hat{H}_{\text{eff}} | \phi \rangle \]

- Minimizing the energy functional, ground state properties are obtained

- Successful effective interactions: non-relativistic Skyrme (zero-range) or Gogny (zero-range + finite range) and based on relativistic Lagrangians (both types)

- Effective interactions are usually FITTED to bulk properties of a set of nuclei and on empirical properties of nuclear matter
Motivation
Mean Field approach to the nucleus: drawbacks

But note!
- Fitted parameters contain correlations beyond the mean-field
- We know the description of single-particle states and their spectroscopic factors is not good at the Mean-Field level.
- Mean-Field predicts a low density of states around the Fermi energy ($m^*/m$ should $\sim 1$ at the nuclear surface and smaller in the interior)
- Only oscillation frequencies but not widths of Giant Resonances and other excited states are well reproduced.

... possible solution
- Many corrections to the static Mean-Field solution are dynamical correlations arising, for example, from the coupling with collective states (nuclear surface vibrations or phonons) [e.g. C. Mahaux et al., Phys. Rep. 120 (1985)1; P.F. Bortignon et al. Phys. Rep. 30, 305 (1977); M. Baldo et al., JGP 42 085109 (2015)]
- Accounting for the coupling of the vibrational and nucleonic degrees of freedom improves the description of single particle properties and resonance widths.
- Microscopic calculations are now feasible, adding PVC on top of the Mean Field [G. Colo et al. PRC 82 064307 (2010); E. Litvinova et al. PRC 73, 044328 (2006)]
Beyond-mean-field and divergences
Theoretical scheme

On the one side ...

- If beyond MF contributions are computed, correlations are explicitly considered and a refitting of the parameters is needed
- PVC (specific type of BMF) calculations are commonly used with non-refitted interactions in the literature (thought some recipes exist)

On the other side ...

- If a zero-range interaction is used, divergences arise and a renormalization scheme is needed
- Finite range interactions will not suffer from divergences but matrix elements (integrals) are more complicated to calculate.

We consider ...

- the lowest-order approximation BMF [PVC in which the phonon is replaced by a particle-hole (p-h) pair]
- the total energy in both nuclear matter and finite nuclei

Diagrammatic representations of the first-order and second-order total energy (for simplicity only direct terms are displayed)

- simplified version of the Skyrme interaction in the vertex
- cut-off renormalization
Beyond-mean-field and divergences

Skyrme, *NPA* 9 (1959) 615

“... It is generally believed that the most important part of the two-body interaction can be represented by a contact potential ...”

“... this suggests an expansion in powers of $k$ and $k'$ (initial and final relative momentum). If this expansion is stopped at the quadratic terms only a small number of undetermined coefficients occur, and an attempt can be made to determine these by comparison with experimental energies.... ”

“... on the other hand this form is unrealistic for large momentum transfers, so that it is not suitable for the discussion of second-order effects, unless some momentum cut-off is introduced ...”

$$V(r, r', R, R') = \frac{\sqrt{2}}{4} g \left( \frac{R}{\sqrt{2}} \right) \delta(r) \delta(r') \delta(R - R')$$

where $g = t_0 + \frac{t_3}{6} [\rho(R)]^\alpha$

(r-spin-dependent, velocity-dependent, and spin-orbit terms are dropped) $r \equiv (r_1 - r_2)/\sqrt{2}$ and $R \equiv (r_1 + r_2)/\sqrt{2}$

$$v^{\lambda \lambda'}(r', r) = \frac{1}{\Omega} \int d_3 k d_3 k' e^{i k' \cdot r'} v(k', k) \theta(\lambda - k) \theta(\lambda' - k') e^{-i k \cdot r}$$

$$= \frac{1}{4\pi^4} \frac{\lambda^2 \lambda'^2}{rr'} J_1(r \lambda)J_1(r' \lambda') \xrightarrow{\lambda, \lambda' \to \infty} \delta(r) \delta(r')$$
The system is **translationally invariant**
- Wave-functions are **plane waves**
- **Momentum is conserved**
- Transferred momentum \((q = (k' - k)/\sqrt{2})\) is a good quantum number (rel. momentum \(k \equiv (k_1 - k_2)/\sqrt{2}\))

**Potential** energy of the system [interaction strength \(g = t_0 + \frac{t_3}{6} \rho^\alpha\)]

**First order** (HF):
\[
E = \frac{1}{2} \sum \langle ij | \tilde{V} | ij \rangle \rightarrow \frac{E}{A} = \frac{d\Omega^2}{A(2\pi)^6} \int_{k_1, k_2 < k_F} d^3k_1 d^3k_2 v(k_1, k_2) = \frac{3}{8} g \rho
\]

**First order** with the cutoff on relative momentum \(k < \lambda\):
\[
\frac{E}{A} = \frac{3}{8} g \rho \left(8\beta^3 - 9\beta^4 + 2\beta^6\right) \text{ where } \beta \equiv \min\{1, \lambda/(\sqrt{2}k_F)\}
\]

**Second order**: \(\Delta E = \frac{1}{2} \sum \frac{\langle mn | V | ij \rangle \langle ij | \tilde{V} | mn \rangle}{\varepsilon_i + \varepsilon_j - \varepsilon_m - \varepsilon_n} \rightarrow \)
\[
\frac{\Delta E}{A} = \frac{d\Omega^3}{A(2\pi)^9} \int_{k_1, k_2, q} \frac{v^2(k_1, k_2, q) d^3k_1 d^3k_2 d^3q}{\varepsilon_k_1 + \varepsilon_k_2 - \varepsilon_k_1 + q - \varepsilon_k_2 - q} \sim v^2 \int \frac{d^3q}{q^2}
\]

The **integral** can be calculated with the **cutoff in the relative momentum** \(k, k' < \lambda = \lambda'\) and with the cutoff in the **transferred momentum** \(q < \Lambda\): both results coincide for large values of the cutoff \((\lambda > 2\sqrt{2}k_F)\) where \(\lambda = \sqrt{2\Lambda}\)
In NM there exist two well-defined equivalent momenta that can be cut!

1) Initial and final relative momentum \( \mathbf{k} \) between interacting particles OR
2) the momentum transfer \( \mathbf{q} \) from initial to final states. Their relation:

\[
\mathbf{q} = (\mathbf{k}' - \mathbf{k}) / \sqrt{2}
\]

* The relation between cutoffs is still to be fully clarified
* The comparison here is numerical since the analytic expressions are hard to derive
* The non-divergent contributions (analytic) are identical

Which is the physical range for the momentum cutoff? from 0 to ?

- Typical distance between nucleons in the nucleus: \( 2r_0 \sim 2 \times 1.2 \text{ fm} \)
  \[ pc \sim 2\pi \hbar c / 2r_0 \sim 1.3 \text{ fm}^{-1} \]
- Mass of the “Goldstone” boson carrying the interaction: the pion \( \frac{140 \text{ MeV}}{\hbar c} \sim 0.7 \text{ fm}^{-1} \)
- Energy needed to excite the nucleon: \( m_\Delta - m_{\text{nucl}} \sim 1.5 \text{ fm}^{-1} \)
Nuclear matter
Total energy with cutoff $\Lambda$ (before refitting)

In K. Moghrabi et al. PRL 105 262501 (2010): Cutoff on transferred momentum ($q < \Lambda$); $t_0 - t_3$ model; SkP as reference EoS (since $m^*/m = 1$)
Towards finite systems

- Computation of the total energy at second order of a finite nucleus: as a test we choose $^{16}$O
- Simplified model with only $t_0 - t_3$ part of the Skyrme interaction

\[
\Delta E = \frac{1}{2} \sum \frac{\langle mn|V|ij\rangle\langle ij|\overline{V}|mn\rangle}{\varepsilon_i + \varepsilon_j - \varepsilon_m - \varepsilon_n}
\]

The divergence increases linearly with increasing model space (maximum particle state energy). Cutoff not implemented yet.
Towards finite systems
Our theoretical method

- **Use the interactions** $\text{SkP}_\Lambda$ fitted in symmetric nuclear matter for each cutoff to check if the divergence is cured.

- **Absence of translational invariance**
  - No momentum conservation

- The **Skyrme interaction is a low-momentum expansion** of the effective nuclear potential going up to second order in relative momenta $\Rightarrow$ **cutoff on relative momenta** [B. G. Carlsson et al. PRC 87 054303 (2013)]
  - We will use relative and center-of-mass coordinates

- The **cutoff** is included **also at mean-field** level
Towards finite systems
Practical steps

Interaction in terms of the cutoffs $\lambda = \lambda'$ on initial and final relative momenta:

$$v^{\lambda\lambda'}(r', r) = \frac{1}{\Omega} \int d_3 k d_3 k' e^{i k' \cdot r'} v(k', k) \theta(\lambda - k) \theta(\lambda' - k') e^{-i k \cdot r}$$

$$= \frac{1}{4\pi^4} \frac{\lambda^2 \lambda'^2}{rr'} j_1(r\lambda) j_1(r'\lambda')$$

“The interaction acquires a finite range”

To compute the matrix elements of the interaction ...

- ... we expand the sp wave functions on a HO basis and transform initial and final two-particle states to the CM and relative motion coordinates (Brody-Moshinsky)

- Solve HF equations on a HO basis, using the matrix elements of $v^{\lambda\lambda'}$

- Computation of the second order total energy with the same interaction and cutoff used at mean-field level.

- The interaction used is the same at each step and we have no free parameters, but the second order correction is added perturbatively.

- No self-consistency since the equations are not solved iteratively.
The mean-field results

Total energy – $^{16}$O

![Graph showing the mean-field results for the total energy of $^{16}$O. The graph plots the system energy (E$_{HF}$) in MeV against the separation distance (λ) in fm$^{-1}$, with two curves representing different models, SKP and SKP$\Lambda$. The energy values range from -250 to 0 MeV for λ ranging from 0 to 3 fm$^{-1}$.]}
The mean-field results

Total energy – $^{16}$O

Mean-field total energy with the standard SkP interaction, without any cutoff
The mean-field results

Total energy – $^{16}$O

Mean-field total energy with the standard SkP interaction, without any cutoff

Mean-field total energy with the standard SkP interaction including the cutoff
The mean-field results

Total energy \(-^{16}_O\)

Mean-field total energy with the renormalized SkP interaction including the cutoff

Mean-field total energy with the standard SkP interaction including the cutoff

Mean-field total energy with the standard SkP interaction, without any cutoff
The mean-field results

Total energy – $^{16}\text{O}$

The interaction is barely renormalized
The mean-field results

Total energy – $^{16}_{\text{O}}$

The interaction is barely renormalized

The interaction is renormalized a lot to leave enough room for the second order
The mean-field results
Total energy – $^{16}$O

The interaction is barely renormalized

The interaction is renormalized a lot to leave enough room for

We expect that in these regions the problem cannot be treated perturbatively
If the cutoff is small the system resembles a gas of nucleons: the mean square radius when $\Lambda = 0.28 \text{ fm}^{-1}$ is $\approx 10 \text{ fm}$.
The total energy at second order
Refitted interactions $^{16}$O

Total energy constant within $a \sim 10\%$
Mean Field
- provides an approximate realization of an energy density functional
- suitable for the study of masses, radii, deformation, ... along the whole nuclear chart
- sp properties are not within its domain: an extension is required

Beyond Mean Field
- different approaches exist: not a unique and systematic way to improve the method/s and to develop renormalization procedures
- PVC provides a better description of sp and collective phenomena
- we tried to develop a simple model based on the PVC idea but substituting the phonons by $p - h$ pairs. We address the problem of renormalizability of a zero-range interaction in NM and FN.

In more detail, ...
- the second order corrections to the total energy have been included perturbatively
- total energy of $^{16}$O with a simplified Skyrme interaction
- use the interaction fitted in nuclear matter for each cutoff
- cutoff renormalization feasible on the relative momenta
- window in $\lambda$ and $\varepsilon_p^{\text{max}}$ where the total energy is constant within 10-15%
We aim at developing a fully self-consistent method BMF, that can be systematically improved and renormalized, and that allow us to study nuclear phenomena along the whole nuclear chart. Some of the main steps to be done are to ...

- treat the **continuum exactly**, in part also to avoid model dep. on $\varepsilon_p^{\text{max}}$
- include **pairing** in our formalism
- solve **self-consistently the Dyson equation** for the sp propagator up the required order
- **refit** the parameters at the adopted level of app. by using FN data
- include the **full Skyrme** interaction and/or propose a **new interaction**
- explore different **renormalization** schemes such as the dimensional regularization
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